A short course on matching theory, ECNU Shanghai, July 2011.

Sergey Norin

LECTURE 5
Drawing Pfaffian graphs with crossings

5.1. Outline of Lecture

- Drawings in the plane with crossings.
- Cross-cap-odd embeddings in the Klein bottle.
- k-Pfaffian graphs.
- Pfaffian braces.

5.2. Drawing Pfaffian graphs with crossings

In this section we extend Kasteleyn's theorem (Theorem 2 in Lecture 4) to graphs drawn in the plane with crossings.

By *a drawing* Γ of a graph G we mean an immersion of G in the plane such that edges are represented by homeomorphic images of $[0,1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a graph G drawn in the plane let $cr(e,f)$ denote the number of times the edges e and f cross. For a set $J \subseteq E(G)$ let $cr(J,\Gamma)$, or $cr(J)$ if the drawing is understood from context, denote $\sum cr(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in J$.

Theorem 1. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

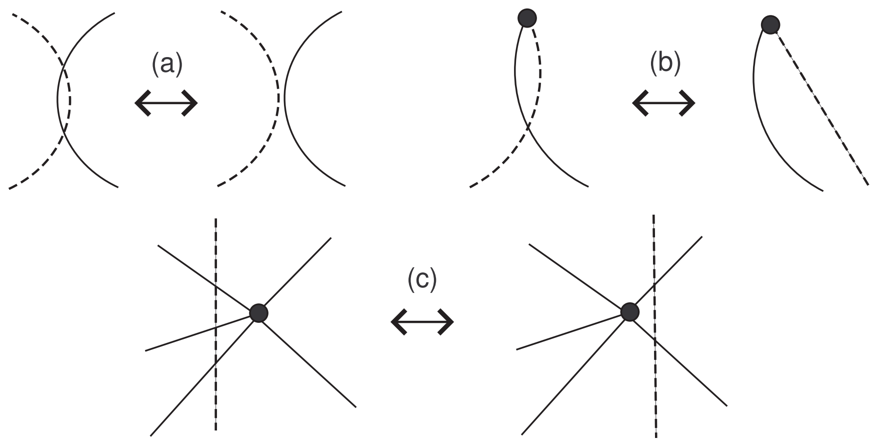

Figure 1. Changing the drawing.

We will derive Theorem 1 from a more general result. To state it we need a definition. Let Γ be a drawing of a graph G in the plane. We say that $S \subseteq E(G)$ is a *marking* of Γ if $cr(M)$ and $|M \cap S|$ have the same parity for every perfect matching M of G.

Theorem 2. *For a graph G the following are equivalent:*
(a) G is Pfaffian;
(b) some drawing of G in the plane has a marking;
(c) every drawing of G in the plane has a marking;
(d) there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.

We say that Γ is *a standard drawing* of a labeled graph G if the vertices of Γ are arranged on a circle in order and every edge of Γ is drawn as a straight line.

Theorem 2 immediately follows from the next two lemmas.

Lemma 1. *There exists a one-to-one correspondence between Pfaffian orientations of a labeled graph G and markings of its standard drawing Γ.*

Proof. Let D be an orientation of G. Let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. The sign of M is the sign of the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Let $i(P)$ denote the number of inversions in P. We have

$$sgn_D(M) = sgn(P) = (-1)^{i(P)} = \prod_{1\le i<j\le 2k} sgn(P(j) - P(i)) =$$
$$= \prod_{1\le i<j\le k} sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i))\times$$
$$\times \prod_{1\le i\le k} sgn(v_i - u_i). \tag{1}$$

In Γ edges u_iv_i and u_jv_j cross if and only if, in the circle containing the vertices of Γ, each of the two arcs with ends u_i and v_i contains one of the vertices u_j and v_j, in other words if and only if

$$sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) = -1.$$

Define $S_D = \{uv \in E(D)|u > v\}$. From (1) we deduce that

$$sgn(M) = (-1)^{cr(M)} \times (-1)^{|M\cap S_D|}.$$

Therefore M has a positive sign if and only if $cr(M)$ and $|M\cap S_D|$ have the same parity. It follows that D is a Pfaffian orientation of G if and only if S_D is a marking of the standard drawing of G. $\square$

Lemma 2. *Let Γ_1 and Γ_2 be two drawings of a labeled graph G in the plane. Then Γ_1 has a marking if and only if Γ_2 has one. If some drawing of a labeled graph G in the plane has a marking then there exists another drawing of G in the plane that has an empty set as a marking.*

Proof. We may assume without loss of generality that the vertices of G are represented by the same points in the plane in both Γ_1 and Γ_2. We transform the drawing Γ_1 into the drawing Γ_2 by smoothly changing the images of edges, one edge at a time. We consider changes in the number of crossings between edges. One can classify events that cause these changes into three types (see Figure 1). We show that none of these events affects the existence of a marking.

In the event of type (a) and (b) the parity of the number of crossings between any two non-adjacent edges remains unchanged. In the event of type (c) the image of an edge e passes through an image of a vertex v, such that e is not incident to v. The number of crossings in any perfect matching containing e changes by one. Therefore one can replace a marking S of a drawing prior to this event by a marking $S\triangle\{e\}$ of a drawing after the event.

The argument above also shows how to obtain a drawing with an empty set as a marking from a drawing with an arbitrary marking. One

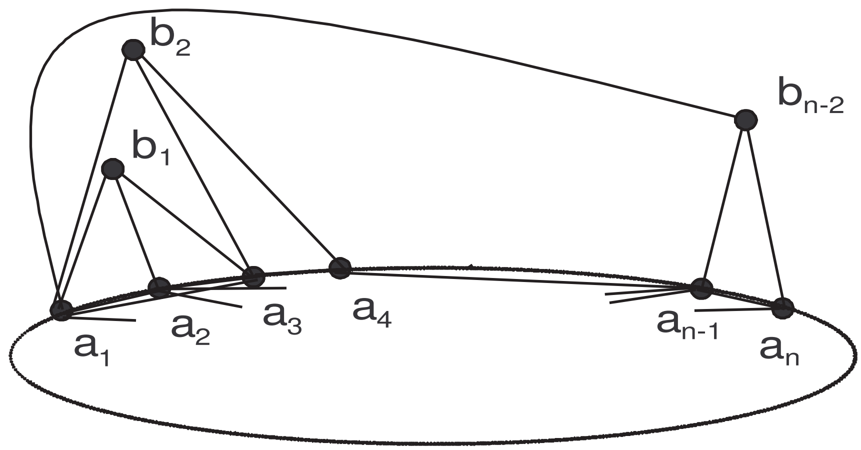

Figure 2. Dense Pfaffian brick

has to transform every edge belonging to the marking so that this edge passes through a single vertex in the course of transformation. □

Exercise 1. a) Show that if $G = K_{3,3}$ or $G = K_5$ then

$$\sum_{\substack{\{e,f\}\subset E(G)\\ e\not\sim f}} cr(e,f)$$

is odd for every drawing of G in the plane, using methods similar to those employed in the proof of Lemma 2. (The summation above is over all pairs of independent edges of G, i.e. over all pairs of edges which do not share an end.)

b) Use the result of part a) and Kuratowski's theorem (every non-planar graph contains a subdivision of $K_{3,3}$ or K_5) to derive the following theorem of Hanani and Tutte:

If a graph G can be drawn in the plane so that every pair of independent edges intersects even number of times then G can be drawn in the plane without crossings.

Exercise 2. For an integer $n \geq 3$ the graph H_n is defined as follows. Let $V(H_n) = \{a_1, a_2, \dots a_n, b_1, b_2, \dots, b_{n-2}\}$. Let the vertices $a_1, a_2, \dots, a_n$ form a clique and let b_i be joined by an edge to a_1, a_{i+1} and a_{i+2} for every $1 \leq i \leq n-2$ (see Figure 5.2).

a) Show that H_n is a brick. **b)** Show that H_n is Pfaffian.

5.3. Cross-cap-odd embeddings in the Klein bottle

In this section we use Theorem 1 to extend Kasteleyn's theorem to certain graphs embedded in the Klein bottle. Let G be a graph embedded on a surface $\mathcal{S}$, which is obtained from a sphere by replacing k disjoint disks with Möbius strips. If $k = 1$ then $\mathcal{S}$ is the projective plane, and if $k = 2$ then $\mathcal{S}$ is the Klein bottle. We say that a cycle C in G is *separating* if cutting $\mathcal{S}$ along C separates the surface, and we say that C is *non-separating* otherwise. Finally, we say that an embedding of G in $\mathcal{S}$ is *cross-cap-odd* if a non-separating cycle C in G is odd if and only if cutting $\mathcal{S}$ along C produces a surface with connected boundary.

Theorem 3. *Every graph that admits a cross-cap-odd embedding in the Klein bottle is Pfaffian.*

Proof. Let G be a graph and let Γ be a cross-cap-odd embedding of G in the Klein bottle. Without loss of generality, we assume that G is matching-covered and connected, and as such it is 2-connected. If G does not contain a non-separating cycle then G is planar, and hence Pfaffian by Kasteleyn's theorem. Therefore we assume that G contains a non-separating cycle.

We claim that every separating cycle is even. We prove the claim by induction on $|E(G)|$. If G contains a vertex of degree two then the claim follows from induction hypothesis by considering the graph obtained from G by contracting one of the edges incident to such a vertex.

Therefore we assume that G has minimum degree three and fix a non-separating cycle C in G. By a standard "ear decomposition" argument, there exists $e = uv \in E(G) - E(C)$ such that $G \setminus e$ is 2-connected. We start by proving that there exists a non-separating cycle containing e in G. Let P_1 and P_2 be two vertex disjoint (possibly trivial) paths with ends u and u', and v and v' respectively, such that $u', v' \in V(C)$, and P_1 and P_2 are otherwise disjoint from C. The vertices u' and v' separate C into two paths Q_1 and Q_2. One of the cycles $P_1 \cup \{e\} \cup P_2 \cup Q_1$ and $P_1 \cup \{e\} \cup P_2 \cup Q_2$ is non-separating.

Suppose now that there exists an odd separating cycle in G. By induction hypothesis applied to $G \setminus e$ every such cycle contains e. We choose a separating cycle C' and a non-separating cycle C'', such that C' is odd, $e \in E(C') \cap E(C'')$ and subject to that $E(C') \cup E(C'')$ is minimal. If $C' \setminus C''$ is a path then the cycle D with edge set $E(C') \triangle E(C'')$ is non-separating and of the same homotopy type as C''. Therefore $|E(D)|$ and $|E(C'')|$ have the same parity, in contradiction with the parity of C'. If $C' \setminus C''$ is not a path then let P be a subpath of C''

with both ends in C' and otherwise disjoint from C'. Let P' be a subpath of C' with the same ends as P, such that $e \in P'$. By the choice of C and C' the cycle $D' = P \cup P'$ is non-separating and even. But then the cycle $D' \triangle C'$ is non-separating, odd and does not contain e, in contradiction with induction hypothesis. This finishes the proof of the claim.

Consider now the standard representation of the Klein bottle as a disk bounded by quadrilateral $ABCD$ with pairs of the quadrilateral's opposite sides identified as follows: AB with DC, and AD with CB. By bisubdividing edges of G if necessary we assume that every edge in $E(G)$ crosses the boundary of the quadrilateral at most once. Let $E_2, E_3 \subseteq E(G)$ be the sets of all edges of G that cross AB and AD, respectively, and let $E_1 = E(G) - E_2 - E_3$. Note that $(V(G), E_1 \cup E_2)$ is bipartite with bipartition (X, Y), and every edge of E_3 joins two vertices of X or two vertices of Y. We may extend Γ to a drawing Γ' of G in the plane such that for $e, f \in E(G)$ we have $cr(e, f) = 1$ if and only if $e \neq f$, $|\{e, f\} \cap E_3| \geq 1$ and $|\{e, f\} \cap E_1| = 0$, and we have $cr(e, f) = 0$, otherwise.

Let $k = (|X| - |Y|)/2$. Let E', E'' be the sets of all edges in E_3 joining two vertices of X and two vertices of Y, respectively. For a perfect matching M of G denote $|M \cap E'|$ by n_M. We have $|M \cap E''| = n_M - k$ and $|M \cap E_3| = 2n_M - k$. Note that

$$cr_{\Gamma'}(M) = \frac{(2n_M + k)(2n_M + k - 1)}{2} + (2n_M + k)|M \cap E_2|.$$

We construct a Pfaffian marking S of Γ'. If k is even then $cr_{\Gamma'}(M) = n_M + k/2$ modulo 2 and therefore $S = E'$ is a Pfaffian marking of Γ' if k is divisible by four and $S = E' \triangle \delta(v)$ is a Pfaffian marking of Γ' for every $v \in V(G)$ otherwise. If k is odd then $cr_{\Gamma'}(M) = n_M + (k-1)/2 + |M \cap E_2|$ modulo 2 and $S = E' \cup E_2$ is a Pfaffian marking of Γ' if $k = 1$ modulo 4 and $S = (E' \cup E_2) \triangle \delta(v)$ is a Pfaffian marking of Γ' for every $v \in V(G)$ otherwise.

It follows from Theorem 1 that G is Pfaffian. □

Note that Theorem 3 can not be extended to graphs that admit a cross-cap-odd embedding on surfaces of higher genus, as $K_{3,3}$ admits a cross-cap-odd embedding on a surface of Euler characteristic -1 (see Figure 5.3). Note also that non-bipartite graphs that admit an embedding in the projective plane with all faces even also admit a cross-cap-odd embedding in the Klein bottle and are therefore Pfaffian.

The class of graph described in Theorem 3 to the best of our knowledge represents the largest topologically defined class of Pfaffian graphs.

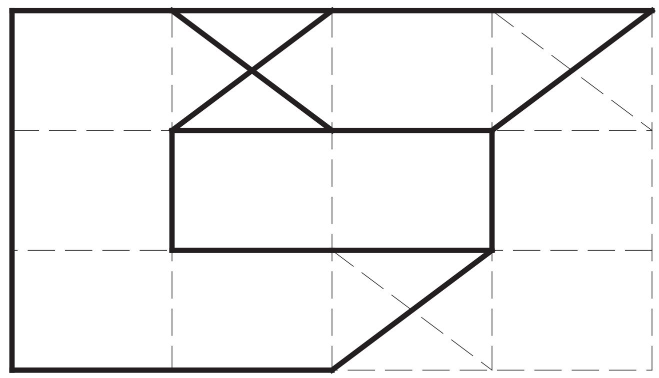

Figure 3. An cross-cap-odd embedding of $K_{3,3}$ on a surface with three "crosscaps".

It is tempting to conjecture that every Pfaffian graph can be decomposed in some way into graphs in this class. An example in Exercise 2 shows that such a decomposition, if it exists, must be "finer" than the tight cut decomposition. A structural decomposition of Pfaffian bipartite graphs into planar pieces is known and is described in a later section.

5.4. k-Pfaffian graphs.

A graph G is *k-Pfaffian* if there exist orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i \mathrm{sgn}_{D_i}(M) = 1$ for every perfect matching M of G. Thus if k is fixed and the the orientations and coefficients as above are given, then the number of perfect matchings of G can be calculated efficiently. The following was noted by Kasteleyn and proved by Galluccio and Loebland Tesler.

Theorem 4. *Every graph that has an embedding in the orientable surface of genus g is 4^g-Pfaffian.*

Exercise 3. Use methods similar to those employed in the proof of Theorem 3 to prove Theorem 4 for $g = 1$.

Does there exist a geometric characterization of k-Pfaffian graphs, similar to the characterization of Pfaffian graphs offered in Theorem 1? In particular, is every k-Pfaffian graph 4^g-Pfaffian for some integer g with $4^g \leq k$. The answer to the second question turns out to be negative in general, but positive for very small values of k. To prove the corresponding result we need an auxiliary lemma.

Lemma 3. *Let G be a labeled graph, let k be an odd integer and let $D_1, D_2, \ldots, D_k$ be orientations of G. Then there exists an orientation D of G such that for every perfect matching M of G we have*

$$\mathrm{sgn}_D(M) = \mathrm{sgn}_{D_1}(M)\,\mathrm{sgn}_{D_2}(M)\ldots\mathrm{sgn}_{D_k}(M).$$

Proof. Define the orientation D of G as follows. For every edge $uv \in E(G)$, let $uv \in E(D)$ if $|\{i \mid 1 \leq i \leq k, uv \in D_i\}|$ is odd and let $vu \in E(D)$ otherwise. Denote by S_i the set of edges on which D differs from D_i. We have

$$\mathrm{sgn}_{D_i}(M) = (-1)^{|M \cap S_i|}\,\mathrm{sgn}_D(M).$$

It follows that

$$\mathrm{sgn}_{D_1}(M)\,\mathrm{sgn}_{D_2}(M)\ldots\mathrm{sgn}_{D_k}(M) = (-1)^{|M\cap S_1|+|M\cap S_2|+\ldots+|M\cap S_k|}\,\mathrm{sgn}_D(M).$$

It remains to note that by definition of D

$$|E \cap S_1| + |E \cap S_2| + \ldots + |E \cap S_k|$$

is even for every $E \subseteq E(G)$. $\square$

Theorem 5. *Every 3-Pfaffian graph is Pfaffian.*

Exercise 4. Prove Theorem 5 using Lemma 3 and the following observation.

Suppose that D_1, D_2 and D_3 are orientations of the graph G so that

$$\alpha_1\,\mathrm{sgn}_{D_1}(M) + \alpha_2\,\mathrm{sgn}_{D_2}(M) + \alpha_3\,\mathrm{sgn}_{D_3}(M) = 1$$

for all $M \in \mathcal{M}(G)$ and $\alpha_1, \alpha_2, \alpha_3 \neq 0$. Then for all matchings $M_1, M_2 \in \mathcal{M}(G)$ either $\mathrm{sgn}_{D_i}(M_1) = \mathrm{sgn}_{D_i}(M_2)$ for $i = 1, 2, 3$, or the signs of M_1 and M_2 differ in exactly two of these orientations.

Miranda and Lucchesi exhibited an example of a 6-Pfaffian graph which is not 5-Pfaffian. They stated the following problem.

Problem 1. Is it true that for every even $k \geq 4$ there exist k-Pfaffian graphs, which are not $(k-1)$-Pfaffian? Do there exist k-Pfaffian graphs, which are not $(k-1)$-Pfaffian for any odd $k > 1$?

5.5. Pfaffian braces.

In this section we give (without proof) a characterization of Pfaffian braces due to Robertson, Seymour and Thomas and, independently, to McCuaig. We also describe two problems closely connected to the problem of characterizing bipartite Pfaffian graphs.

We start with the characterization. Let G_0 be a graph, let C be a central cycle of G_0 of length four, and let G_1, G_2, G_3 be three subgraphs

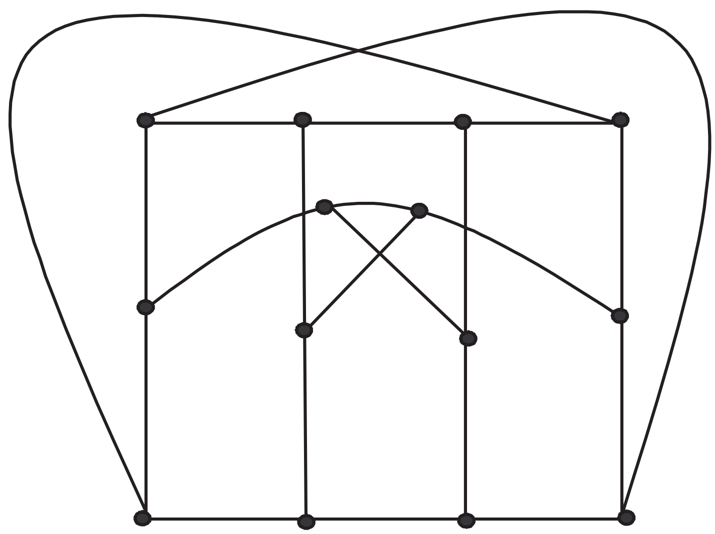

Figure 4. The Heawood graph

of G_0 such that $G_1 \cup G_2 \cup G_3 = G_0$, and for distinct integers $i, j \in \{1, 2, 3\}$, $G_i \cap G_j = C$ and $V(G_i) - V(C) \neq \emptyset$. Let G be obtained from G_0 by deleting some (possibly none) of the edges of C. In these circumstances we say that G is a *trisum* of G_1, G_2 and G_3. The *Heawood graph* is the bipartite graph associated with the incidence matrix of the Fano plane (see Figure 4).

Theorem 6. *A brace has a Pfaffian orientation if and only if either it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated application of the trisum operation.*

Theorem 6 allows for a polynomial time algorithm for recognizing Pfaffian bipartite graphs. No such algorithm is known for general graphs. There exist various obstructions to obtaining an exact analogue of Theorem 6. In particular, the brick described in Exercise 2 has $2n-2$ vertices, $(n^2+5n-12)/2$ edges and has K_n as a subgraph. Meanwhile, the following exercise implies that every Pfaffian brace is sparse.

Exercise 5. Derive from Theorem 6 that every Pfaffian brace on n vertices has at most $2n-4$ edges and has no $K_{2,3}$ as a subgraph.

A directed graph D is *even* if for every weight function $w : E(D) \to \{0, 1\}$ there there exists a cycle in D of even total weight. It is known that and is not difficult to see that testing evenness of a digraph is polynomial-time equivalent to testing whether a digraph has an even directed cycle. Let G be a bipartite graph with bipartition (A, B), and let Mbe a perfect matching in G. The digraph $D(G, M)$ is obtained from G by directing every edge from A to B, and contracting every edge of M. The following connection between even digraphs and Pfaffian orientations is due to Little.

Exercise 6. Let G be a bipartite graph, and let M be a perfect matching in G. Then G has a Pfaffian orientation if and only if $D(G, M)$ is not even.

The final problem connected to Pfaffian orientations of bipartite graphs which we describe concerns hypergraph coloring. A hypergraph H is a pair $(V(H), E(H))$, where $V(H)$ is a finite set and $E(H)$ is a collection of distinct nonempty subsets of $V(H)$. The hypergraph H is 2-*colorable* if $V(H)$ can be colored using two colors so that no edge is monochromatic. A hypergraph H with no isolated vertices is *minimally non-2-colorable* if H is not 2-colorable and $H - e$ is 2-colorable for every $e \in E(H)$. The next theorem is due to Seymour.

Theorem 7. *Let Hbe a hypergraph with no isolated vertices and $|E(H)| = |V(H)|$, let D be the digraph with bipartition $(V(H), E(H))$ so that D has an edge directed from $v \in V(H)$ to $E \in E(H)$ if and only if v is contained in e and let G be the underlying undirected graph of D. Then H is minimally non-2-colorable if and only if G is matching-covered and D is a Pfaffian orientation of G.*