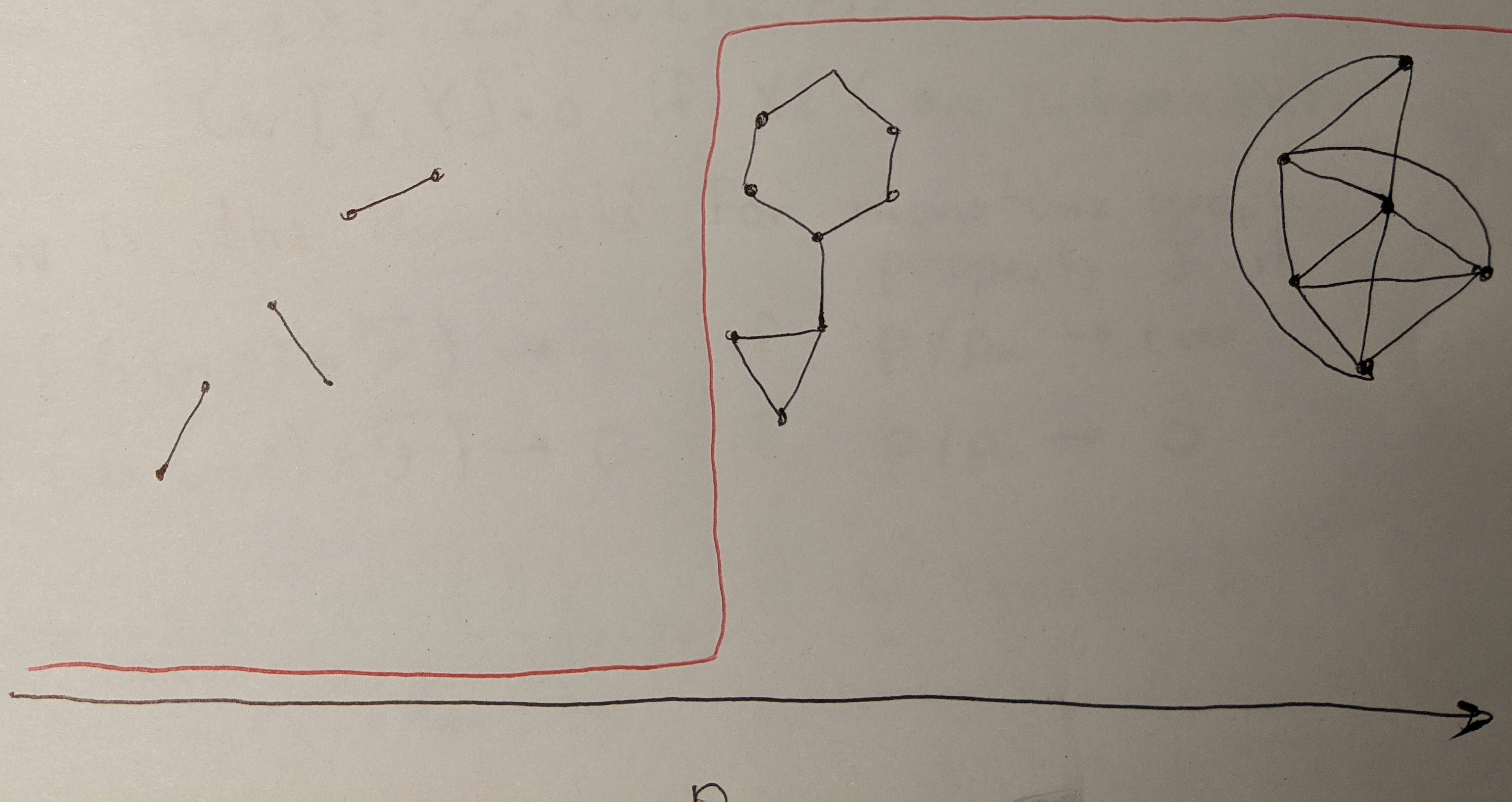
Lecture 7: Thresholds



Reminders:

1. Chebyshev's inequality implies $X \geq 0$

If $\mathrm{Var}[X] = o((\mathbb{E}[X])^2)$

then with high probability

$\boxed{X = (1 + o(1))\mathbb{E}[X]}$ & $X > 0$.

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2. If $X = X_1 + \dots + X_n$

then $\mathrm{Var}[X] = \sum_{i,j} \mathrm{Cov}[X_i, X_j]$

$\mathrm{Cov}[X, Y] = 0$ if X & Y are independent.

— $(p_n)_{n \in \mathbb{N}}$ is **the threshold** for monotone graph property $\mathcal{F}$ if

$\mathbb{P}(G(n,p) \in \mathcal{F}) \to 1$ if $p/p_n \to +\infty$

$\mathbb{P}(G(n,p) \in \mathcal{F}) \to 0$ if $p/p_n \to 0$

— Theorem 4.2: $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is the threshold for the appearance of K_3 subgraph.

Want analogue for any H instead of K_3.

Let $X = X_1 + X_2 + \dots + X_n$

where $X_i = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{if } A_i \text{ does not occur.} \end{cases}$

Let $i \sim j$ if A_i and A_j are not independent,

$$\mathrm{Var}[X] = \sum_{\substack{i=j, \\ \text{or } i\sim j}} \mathrm{Cov}[X_i, X_j] \le \sum_{\substack{i=j, \\ i\sim j}} \mathbb{P}(A_i \wedge A_j)$$

$$\mathrm{Cov}[X_i, X_j] \le \cancel{\mathbb{E}[X_i]\,\mathbb{E}[X_j]}\ \ \mathbb{E}[X_i X_j] = \mathbb{P}(A_i \wedge A_j)$$

$$= \sum_i \mathbb{E}[X_i] + \boxed{\sum_{\substack{i\sim j, \\ i\ne j}} \mathbb{P}(A_i \wedge A_j)} = \mathbb{E}[X] + \boxed{\Delta}$$

If $\mathbb{E}[X] \to \infty$, need $\Delta = o\left((\mathbb{E}[X])^2\right)$

$$\Delta = \sum_i \sum_{\substack{j\sim i \\ j\ne i}} \mathbb{P}(A_i)\,\mathbb{P}(A_j | A_i) = \sum_i \mathbb{P}(A_i) \underbrace{\sum_{\substack{j\sim i \\ j\ne i}} \mathbb{P}(A_j | A_i)}$$

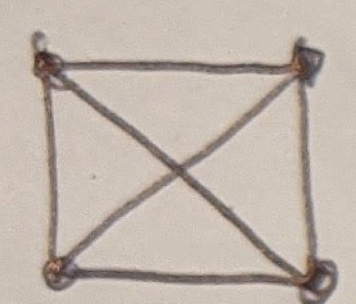
If events $A_1, \dots, A_n$ are symmetric then

$$\boxed{\Delta^* = \sum_{\substack{j\sim i \\ j\ne i}} \mathbb{P}(A_j | A_i)}$$ is the same for every i

$$\& \quad \Delta = \sum_i \mathbb{P}(A_i)\,\Delta^* = \underbrace{\mathbb{E}[X]\,\underbrace{\Delta^*}}\ \overset{?}{=}\ o\left(\underbrace{(\mathbb{E}[X])^2}\right)$$

Lemma 4.3: Let X be as above. If
$\mathbb{E}[X] \to \infty$ and $\Delta^* = o(\mathbb{E}[X])$
then $X = (1+o(1))\,\mathbb{E}[X]$ with high probability.

Theorem 4.4 $n^{-2/3}$ is the threshold for containing a K_4 subgraph



$G(n,p)$ contains K_4.

$\mathbb{E}[\#\text{ of } K_4]$
$= \binom{n}{4} p^6$
$\sim c\, n^4 p^6 \to \infty$
$p^6 >> n^{-4}$
$p >> n^{-2/3}$.

Proof: $p << n^{-2/3}$ then $\mathbb{P}(G(n,p) \text{ containing } K_4) \to 0$

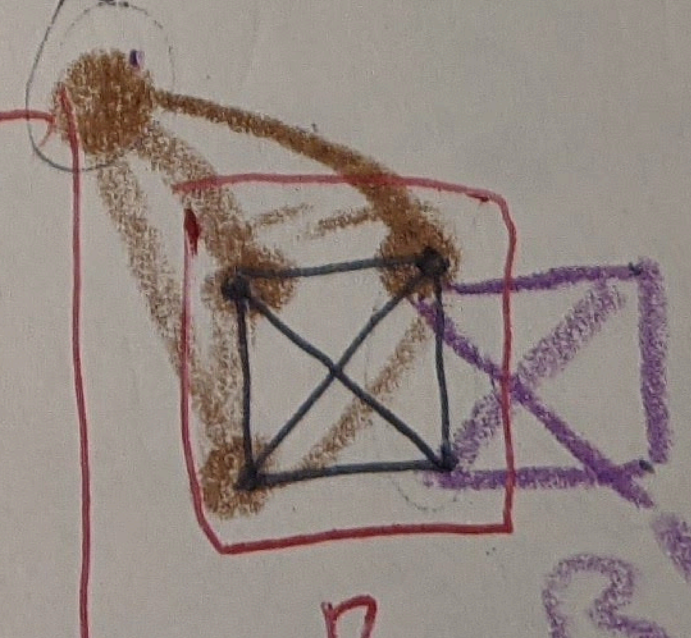
True because $\mathbb{E}[\#K_4 \text{ in } G(n,p)] \sim n^4 p^6 \to 0$.

It remains to show that if $p >> n^{-2/3}$ then $G(n,p)$ contains K_4 w.h.p.

Let $X = \sum_{\substack{B \subseteq [n] \\ |B| = 4}} X_B$. Want $\Delta^* = o(\mathbb{E}[X])$
$= o(n^4 p^6)$ p^5

X_B: B induces $K_4 \to$ event A_B

$\Delta^* = \sum_{B' \sim B} \mathbb{P}(A_{B'} | A_B) = \sum_{|B' \cap B| = 2} \boxed{\mathbb{P}(A_{B'} | A_B)}$ p^5
$+ \sum_{|B' \cap B| = 3} \mathbb{P}(A_{B'} | A_B)$ p^3



$\leq \binom{4}{2}\binom{n-4}{2} p^5 + 3(n-4)\, p^3$

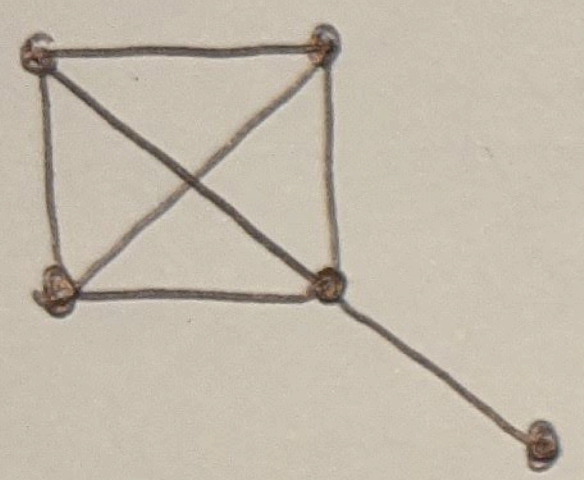
$f(n) >> g(n)$ if $g(n) = o(f(n))$ or $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

$\Delta^* \sim n^2 p^5 + n p^3 \overset{?}{=} o(n^4 p^6)$ if $p >> n^{-2/3}$

$n^2 p^5 = o(n^4 p^6) \Leftrightarrow 1 << n^2 p \leftrightarrow p >> n^{-2}$ ✓

$n p^3 = o(n^4 p^6) \Leftrightarrow p >> 1/n$ ✓

What is the threshold for H in $G(n,p)$?



$H = K_4$ + leaf

$\underbrace{\mathbb{E}(\# H) \sim n^5 p^7}$ $\to \infty$ if $p >> n^{-5/7}$.

$5/7 > 2/3$

$n^{-5/7} << n^{-2/3}$

But for $n^{-2/3} >> p >> n^{-5/7}$ almost surely there are no K_4 so no H.

In general $\mathbb{E}$(number of H subgraphs in $G(n,p)$)

$\sim C_H \, n^{|V(H)|} \, p^{|E(H)|}$.

So $G(n,p)$ w.h.p has no H

if $p << \underbrace{n^{-\frac{|V(H)|}{|E(H)|}}}$.

$G(n,p)$ has no H

if $p \ll n^{-\frac{|V(H')|}{|E(H')|}}$

for some subgraph H' of H.

$$m(H) = \max_{H' \subseteq H} \frac{|E(H')|}{|V(H')|}$$ ~ maximum density of a subgraph of H.

So $\boxed{\underline{G(n,p) \text{ has no } H \text{ if } p \ll n^{-1/m(H)}}}$.

<u>Theorem 4.5</u>: (Bollobas 1981) $n^{-1/m(H)}$ is the threshold for $G(n,p)$ containing a subgraph isomorphic to H.

<u>Proof</u>: Remains to show that for $p \gg n^{-1/m(H)}$ $G(n,p)$ contains H w.h.p.

Consider all injections $\varphi : V(H) \to [n]$ (~~of these~~ $n(n-1)(n-2)\dots(n-|V(H)|+1) \sim n^{|V(H)|}$ such maps)

Let X_φ be the indicator r.v for event A_φ: φ maps all edges of H to edges of $G(n,p)$

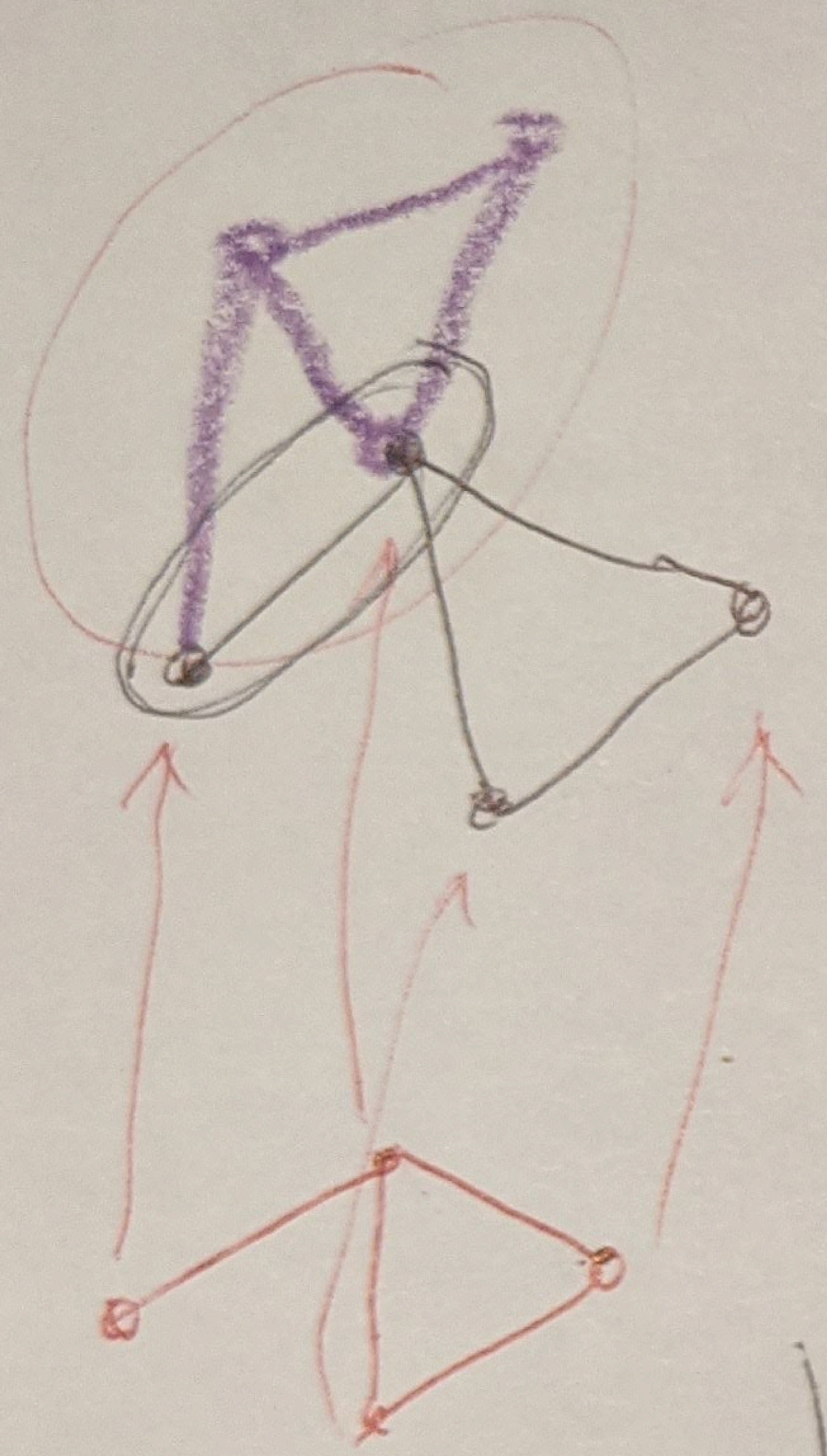
$$\mathbb{P}(A_\varphi) = p^{|E(H)|}.$$

Let $X = \sum X_\varphi$ ~~$\approx n^{|V(H)|}$~~ $\mathbb{E}[X] \sim n^{|V(H)|} p^{|E(H)|} \to \infty$

$m(H) \geq \frac{|E(H)|}{|V(H)|}$

Want $\underline{\Delta^* = o(\mathbb{E}[X])}$

$$\Delta^* = \sum_{\varphi' \sim \varphi} \mathbb{P}(A_{\varphi'} | A_\varphi) = \sum_{H' \subseteq H} \sum_{\substack{\varphi' \text{ s.t.} \\ Im(\varphi') \cap Im(\varphi) = H'}} \mathbb{P}(A_{\varphi'} | A_\varphi)$$



$$\leq \sum_{\substack{H' \subseteq H \\ E(H') \neq \phi}} C_{H'} \, n^{|V(H)| - |V(H')|} \, p^{|E(H)| - |E(H')|}$$

Want: $n^{|V(H)| - |V(H')|} p^{|E(H)| - |E(H')|} << n^{|V(H)|} p^{|E(H)|}$

$$1 << p^{|E(H')|} n^{|V(H')|}$$

$$n^{-\frac{1}{m(H)}} \leq n^{-\frac{|V(H')|}{|E(H')|}} << p$$

✓

We will show that every (non-trivial) monotone graph property has a threshold.

- More generally we say that $\mathcal{F} \subseteq \mathcal{P}([n])$ (a collection of subsets of $[n]$) is an up-set is monotone. if for any $A \in \mathcal{F}$ and $A \subseteq B \subseteq [n]$ we have $B \in \mathcal{F}$.

- Let $[n]_p$ be a probabilistic distribution on subsets of $[n]$ generated by selecting any $i \in [n]$ to be in our set independently with probability p.

$$\mathbb{P}(A = [n]_p) = p^{|A|}(1-p)^{n-|A|}.$$

$\mathcal{F}$ is non-trivial, if $\emptyset \notin \mathcal{F}$, $[n] \in \mathcal{F}$.

Lemma 5.6: Let $\mathcal{F}$ be a non-trivial monotone collection of subsets of $[n]$ then $f(p) = \mathbb{P}([n]_p \in \mathcal{F})$ is continuous and increasing. (for fixed n).