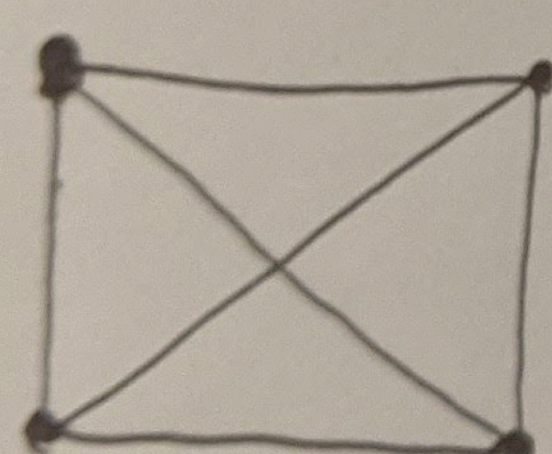
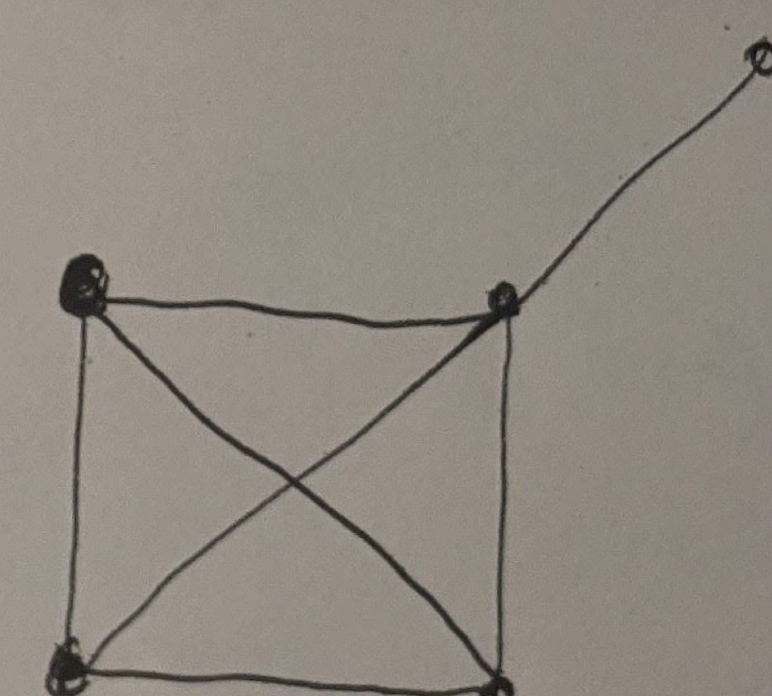
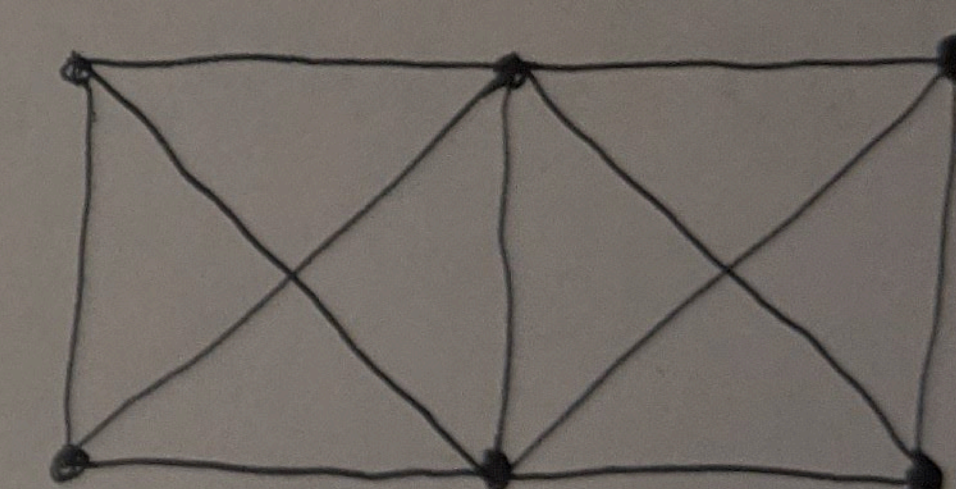
# Lecture 6: The Second Moment Method



$\sim p^6 n^4$



$\sim p^7 n^5$



$\sim p^{11} n^6$

Last time:

Erdös (1962): $\forall k\ \exists \varepsilon > 0\ \forall n \geq k$
$\exists$ graph $G$ with $n$ vertices, $\chi(G) \geq k$
s.t. every subgraph of $G$ on $\leq \varepsilon n$ vertices
is 3-colorable.

Yaqiao: can we make it 2-colorable? No

Kierstead, Szemeredi & Trotter (1984): $\forall k\ \exists c_k$.
Every graph $G$ with $\chi(G) \geq k+2$.
and $n$ vertices contains an odd cycle
with $\leq c_k n^{1/k}$ vertices
(tight up to value of $c_k$).

# 4. The Second moment method.

$X \geq 0$ random variable.

$\boxed{\text{How can we show that } X > 0 \text{ with high probability?}}$

prob $\to 1$ as $n \to \infty$.

Markov's inequality: $\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$.

if $X$ is integral and $\mathbb{E}[X] < 1$ then $X = 0$ with positive prob.

$\mathbb{E}[X] = o(1)$ then $X = 0$ with prob $\to 1$.

If $\mathbb{E}[X] >> 1$ does this imply that
$\mathbb{E}[X] \to \infty$ $X > 0$ w.h.p.? Not always.

Variance $\quad \mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}X)^2]$

$= \mathbb{E}[X^2] - [\mathbb{E}X]^2$.

Theorem 4.1: (Chebyshev's) inequality
concentration inequality.

Let $X$ be a r.v., let $\mu = \mathbb{E}X$,
let $\sigma^2 = \mathrm{Var}[X]$ $\boxed{\sigma \text{ - standard deviation}}$

Then For any $\lambda > 0$

$$P(|X - \mu| \geq \lambda\sigma) \leq \lambda^{-2}.$$

Proof:

$$P(|X-\mu| \geq \lambda\sigma) = P\Big(\underbrace{(X-\mu)^2}\geq \lambda^2 E\big((X-\mu)^2\big)\Big)$$

$$\overset{\text{Markov}}{\leq} \frac{E\big((X-\mu)^2\big)}{\lambda^2 E[(X-\mu)^2]} = \frac{1}{\lambda^2}.$$

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If $\sigma = o(\mu)$ (or $\boxed{Var[X] = o\big((E[X])^2\big)}$)

then $P(|X-\mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} = o(1)$

$P(X=0) = o(1).$

In fact with high probability $\boxed{\underbrace{X = (1+o(1))\mu}}$, if $\sigma = o(\mu)$. $\rightarrow$ $\boxed{X \text{ concentrates around } \mu}$

- $Var[X] = E[(X-EX)(X-EX)] \geq 0$

Covariance of $X$ & $Y$

$Cov[X,Y] = E[(X-EX)(Y-EY)] = E[XY] - E[X]E[Y].$

$Var[X] = Cov[X,X].$

If $X$ & $Y$ are independent then $E[XY] = E[X]E[Y]$

$Cov[X,Y] = 0.$

Covariance is <u>bilinear</u>

$$\mathrm{Cov}[(X_1+X_2),Y] = \mathrm{Cov}[X_1,Y] + \mathrm{Cov}[X_2,Y]$$
$$\mathrm{Cov}[\alpha X, Y] = \alpha\,\mathrm{Cov}[X,Y].$$

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$X = X_1 + X_2 + \dots + X_n$ then

$$\mathrm{Var}[X] = \mathrm{Cov}[X_1+X_2+\dots+X_n, X_1+X_2+\dots+X_n]$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{\mathrm{Cov}[X_i,X_j]}.$$

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Let $X_i$ be the Bernoulli random variable with mean $p$.

$$P(X_i=1)=p \qquad P(X_i=0)=1-p.$$

$X = \sum_{i=1}^{n} X_i$. <u>When is $X$ concentrated around $E[X]$?</u>

$X_1, X_2, \dots, X_n$ are pairwise independent. $E[X]=pn$.

$$\mathrm{Var}[X] = \sum_{(i,j)\in[n]^2} \underbrace{\mathrm{Cov}[X_i,X_j]}_{\substack{\shortparallel \\ 0 \text{ if } i\neq j}} = n\,\mathrm{Var}[X_i] = np(1-p)$$

$$\mathrm{Var}[X_i] = E[X_i^2] - [E X_i]^2 = p - p^2$$
$$= p(1-p)$$

If $\underbrace{np(1-p) = o(p^2 n^2)}_{pn \to +\infty}$ then X "concentrates".

$\mathbb{E}[X] \to \infty$ then

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Let $X$ be the number of triangles in the random graph $G(n,p)$.

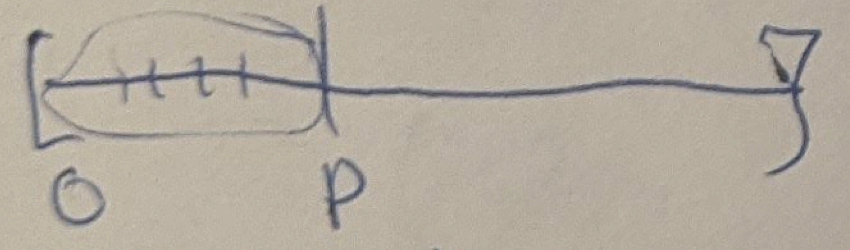
$$\mathbb{E}[X] = \binom{n}{3} p^3 \sim \frac{1}{6} n^3 p^3$$

By Markov's inequality if $p \ll \frac{1}{n}$ $\left(p = o\left(\frac{1}{n}\right)\right)$ then $X = 0$ with high probability.

$p \gg \frac{1}{n}$ then $\mathbb{E}[X]$ is <u>large</u>

Does it mean that $X > 0$ with high probability?

. For each $\{i,j\} \subseteq [n]$ choose $X_{ij} \in [0,1]$ inde~~dentl~~pendently at random.




$$G(n,p) \longrightarrow E(G(n,p)) = \{\{i,j\},\ X_{ij} \leq p\}.$$

(If we add edges one by one at which point do we start seeing triangles?)

Theorem 4.2: If $p >> \frac{1}{n}$ then $G(n,p)$ contains a $K_3$ subgraph with high probability.

( If $(p_n)_{n \in \mathbb{N}}$ is such that $\lim_{n \to \infty} p_n n = +\infty$.

then $\lim_{n\to\infty} P(G(n,p_n) \text{ contains } K_3) = 1$ )

Proof: $\cancel{\square}$ Let X ~~be~~ be the number of $K_3$ subgraph in $G(n,p)$.

$$\mathbb{E}[X] \sim \frac{1}{6} n^3 p^3$$

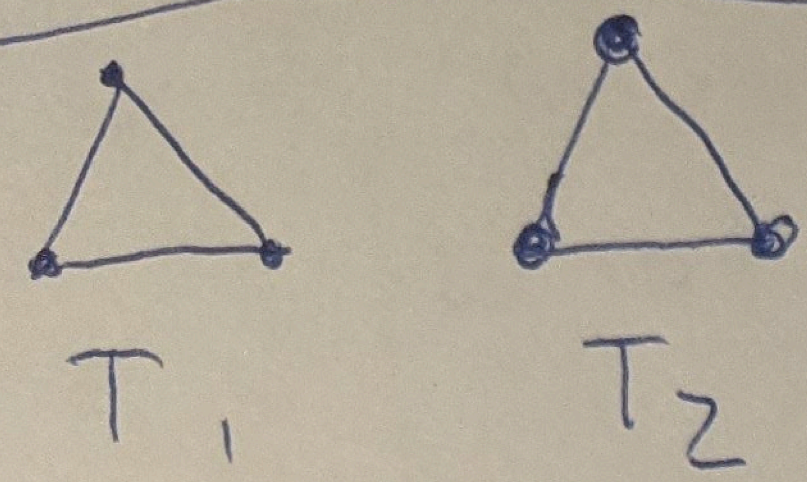
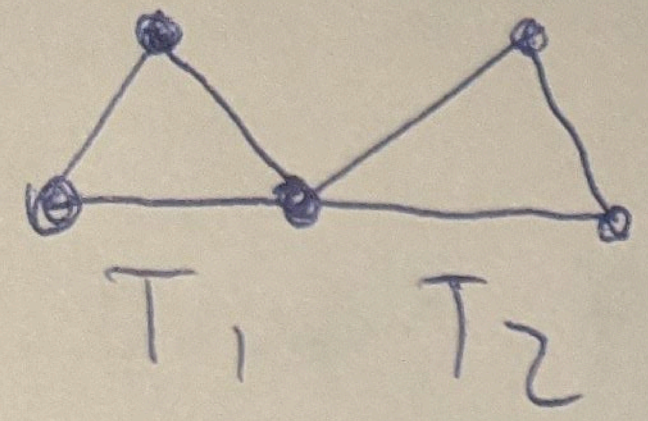
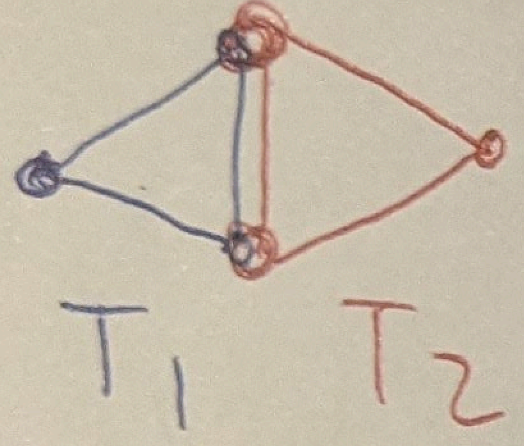
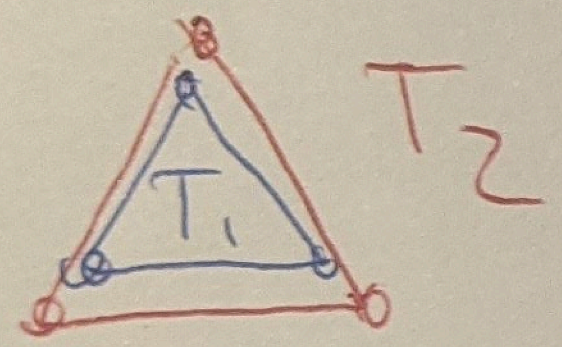
Goal: $Var[X] = o(n^6 p^6) = o(\mathbb{E}[X])^2$

Let $X_T = \begin{cases} 1 & \text{if } T \subseteq [n], |T| = 3, T \text{ forms a } K_3 \\ 0, & \text{otherwise} \end{cases}$

Then $X = \sum_T X_T$

$$Var[X] = \sum_{\substack{(T_1,T_2) \\ T_1,T_2 \in [n]^{(3)}}} \boxed{Cov[T_1, T_2]}$$

understand this

| | $\mathrm{Cov}(T_1, T_2)$ | Contribution to $\mathrm{Var}[X]$ |
|---|---|---|
|   | $0$ | |
|   | $0$ | |
|   | $\mathbb{E}[X_{T_1} X_{T_2}] - p^6$ <br> $\mathbb{E}[X_{T_1} X_{T_2}] = p^5$ | $\underbrace{\binom{n}{2}(n-2)(n-3)}(p^5 - p^6)$ <br> $\sim \frac{1}{2} n^4 p^5$ |
|   | $p^3 - p^6$ | $\frac{1}{6} \underbrace{n^3 p^3}$ |

$$\mathrm{Var}[X] \sim \frac{1}{2} n^4 p^5 + \frac{1}{6} n^3 p^3$$

$$(\mathbb{E}[X])^2 \sim \frac{1}{36} n^6 p^6$$

Need

$$(n^4 p^5 + n^3 p^3) = o(n^6 p^6)$$

$$n^4 p^5 = o(n^6 p^6) \Leftrightarrow n^2 p \to \infty$$

$$n^3 p^3 = o(n^6 p^6) \Leftrightarrow n^3 p^3 \to \infty$$

Both are true if

$$np \to \infty.$$

We will answer for every $H$:

what is the "point" at which $G(n,p)$ is likely to contain $H$?

A graph property $\mathcal{F}$ is <u>monotone</u> if adding edges to a graph in $\mathcal{F}$ yields a graph in $\mathcal{F}$.

- containing $H$ subgraph
- non-planar
- non-bipartite
- Hamiltonian.
- being connected.

A sequence $(p_n)_{n \in \mathbb{N}}$ is the <u>threshold</u> for $\mathcal{F}$:

if

$$P(G(n,p) \in \mathcal{F}) \to 1 \quad \text{for all } p \text{ s.t. } \frac{p}{p_n} \to +\infty$$

$$P(G(n,p) \in \mathcal{F}) \to 0 \quad \text{for all } p \text{ s.t. } \frac{p}{p_n} \to 0.$$

<u>Example</u>: 4.2 says that $\frac{1}{n}$ is the threshold for having $K_3$ subgraph.