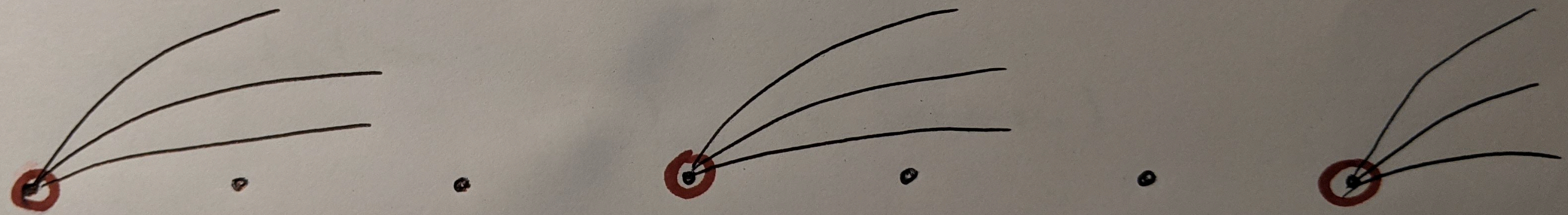
Lecture 3: Linearity of Expectation



$\frac{1}{\deg(v_1)+1}$ $\qquad$ $\frac{1}{\deg(v_i)+1}$

$X = c_1 X_1 + c_2 X_2 + \dots + c_n X_n$

$X, X_1, \dots, X_n$ are real valued (or vector space over reals valued)

$c_1, c_2, \dots, c_n$ are real

$E[X] = c_1 E[X_1] + c_2 E[X_2] + \dots + c_n E[X_n]$.

In applications, ideally $X_1, \dots, X_n$ are simple,

e.g. indicator r. v.

We use bounds on $E[X]$ to conclude

$$\left.\begin{array}{l} X \geq E[X] \\ X \leq E[X] \end{array}\right\} \text{~~for~~ in some instance}$$

Random permutation:

$\sigma : [n] \to [n]$ (bijection)

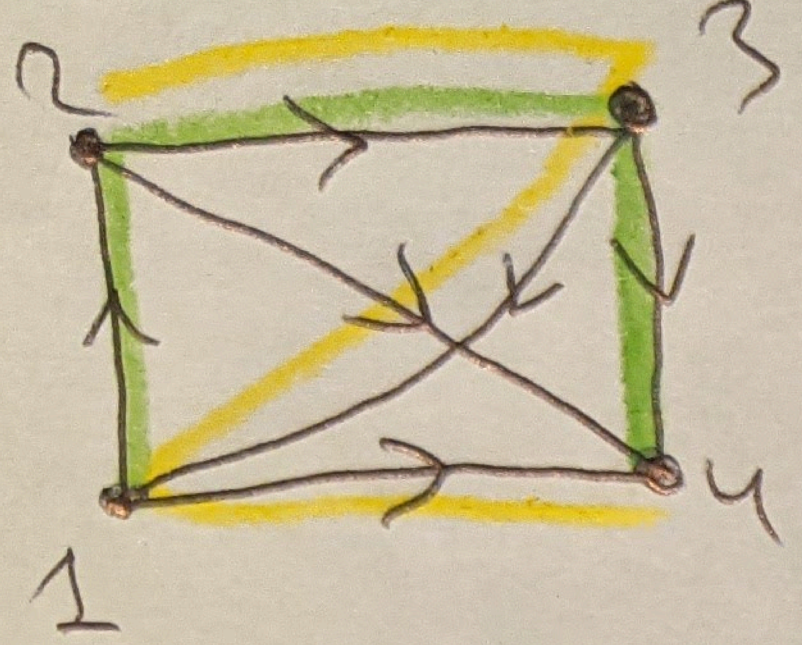
Compute $E[X]$

$X = \#$ of fixed points of $\sigma = \#\, i : \sigma(i) = i$

$$X_i = \begin{cases} 1 & \text{if } \sigma(i) = i \\ 0 & \text{otherwise} \end{cases} \qquad E[X_i] = p(X_i = 1) = p(\sigma(i) = i) = \boxed{1/n}$$

$$E[X] = \sum_i E[X_i] = n \cdot \frac{1}{n} = 1.$$

Tournaments: Complete directed graph



1234
is Ham. path

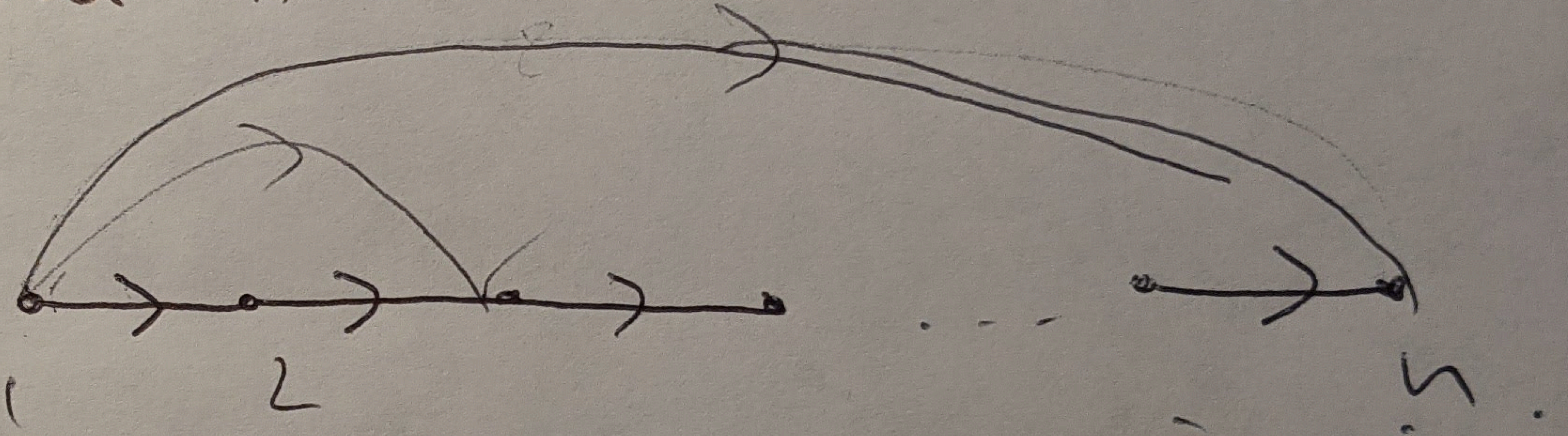
Hamiltonian path in a tournament (with vertex set $[n]$) is a permutation σ:
$$\sigma(1) \to \sigma(2) \to \sigma(3) \to \dots \to \sigma(n).$$

Exercise: Every tournament has a Hamiltonian path.

What is the minimum # of Hamiltonian paths?

$\boxed{\geq 1}$.

$= 1$



if all edges are $i \to j$ $i < j$.

What is the maximum # of Hamiltonian paths?

average $\boxed{\begin{array}{l}\#\text{ of Hamiltonian paths} \\ \shortparallel \\ X\end{array}}$?

For each permutation σ

let $X_\sigma = 1$ if σ is a permutation giving a Ham path

0 otherwise

$E[X_\sigma] = P(X_\sigma = 1) = \left(\frac{1}{2}\right)^{n-1}$

(we are considering a tournament where each edge independently directed each way with prob $= 1/2$)

$$E[X] = \sum_{\sigma} E[X_\sigma] = n! \left(\frac{1}{2}\right)^{n-1}$$

Theorem (Szele, 1943) 2.1: There exists for each n a tournament with n vertices & at least $n! \left(\frac{1}{2}\right)^{n-1}$ Ham. paths.

Alon 1990: maximum $\leq n! \left(\frac{1}{2} + o(1)\right)^n$

Turán's theorem:

What is the maximum number of edges in a graph with n vertices & no complete subgraph on r+1 vertices?

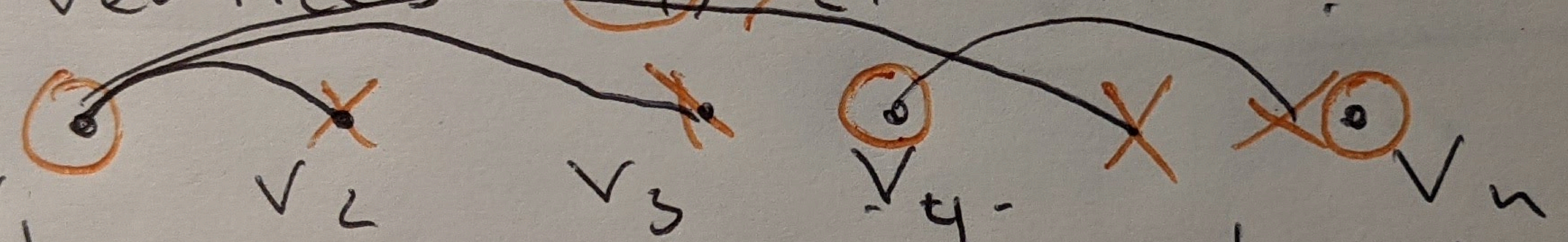
Theorem 2.2 (Caro 1979, Wei 1981): Every graph G contains an independent set on at least

$$\sum_{v \in V(G)} \frac{1}{\deg(v)+1}$$

vertices.

Independent set : collection of vertices s.t. no two are adjacent.

Proof: Order vertices $v_1, \cancel{v_2}, \dots, v_n$. One by one add



the vertex with lowest index still in our list to the independent set, throw away all of its neighbors.

What is the size of the resulting set for a random order.

$$X = \sum_{v \in V(G)} X_v \qquad E[X] = \sum_v E[X_v]$$

$$X_v = \begin{cases} 1 & \text{if } v \text{ is in the set} \\ 0, & \text{otherwise.} \end{cases}$$

$$\boxed{\Pr(v \text{ is the set}) \geq \frac{1}{\deg(v)+1}}$$

↓

implies the theorem

If v precedes all of its neighbors then v gets picked in the independent set, with probability $\dfrac{1}{\deg(v)+1}$

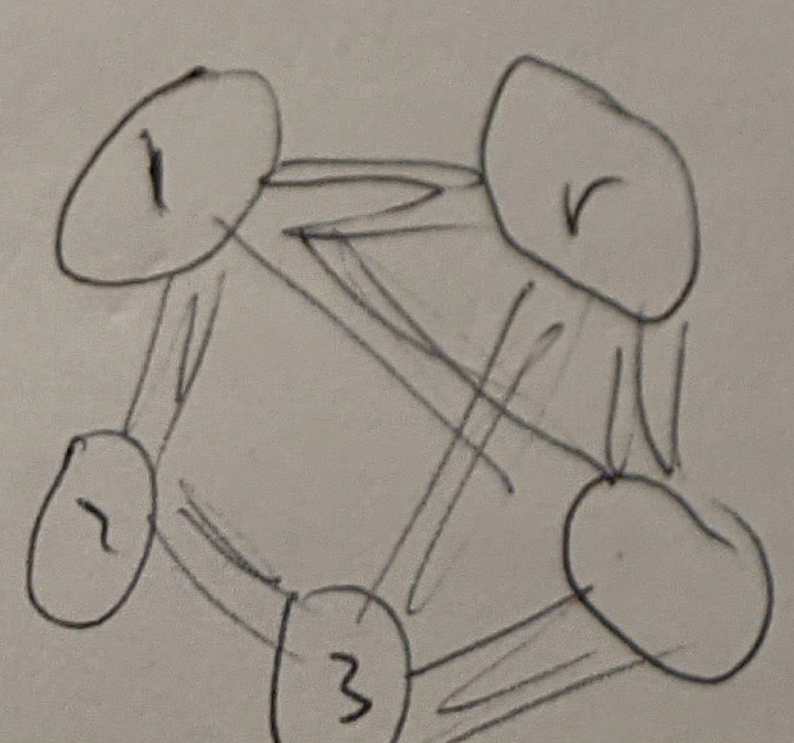
Corollary 2.3: Every graph G with n vertices & m edges contains an independent set of size $\geq n \cdot \dfrac{1}{\frac{2m}{n}+1}$.

Proof: By convexity of $\frac{1}{x}$

$$\frac{\sum_{v \in V(G)} \frac{1}{\deg(v)+1}}{n} \geq \frac{1}{\frac{\sum_{v \in V(G)} \deg(v)}{n} + 1} = \frac{1}{\frac{2m}{n}+1}.$$

Theorem 2.4 (Turán 1941): Let G be a graph on n vertices and no K_{r+1} subgraph then

$$\underbrace{|E(G)| \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}}.$$



– almost tight example.

Proof: By 2.3 applied to the complement of G

G contains a complete subgraph of size

$$\geq n \frac{1}{\frac{2\left(\binom{n}{2}-m\right)}{n}+1} \qquad (m = |E(G)|)$$

So

$$r \geq n \cdot \frac{n}{2\left(\binom{n}{2}-m\right)+n} = \frac{n^2}{n(n-1)-2m+n} = \frac{n^2}{n^2-2m} = \frac{1}{1-\frac{2m}{n^2}}$$

$$1-\frac{2m}{n^2} \geq \frac{1}{r}$$

$$\frac{2m}{n^2} \leq 1-\frac{1}{r}$$

$$m \leq \left(1-\frac{1}{r}\right)\frac{n^2}{2}.$$

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average degree d, n vertices

$\Rightarrow$ independent set $\geq \frac{n}{d}(1+o(1))$

(tight if G is a union of complete graphs)

Shearer 1982

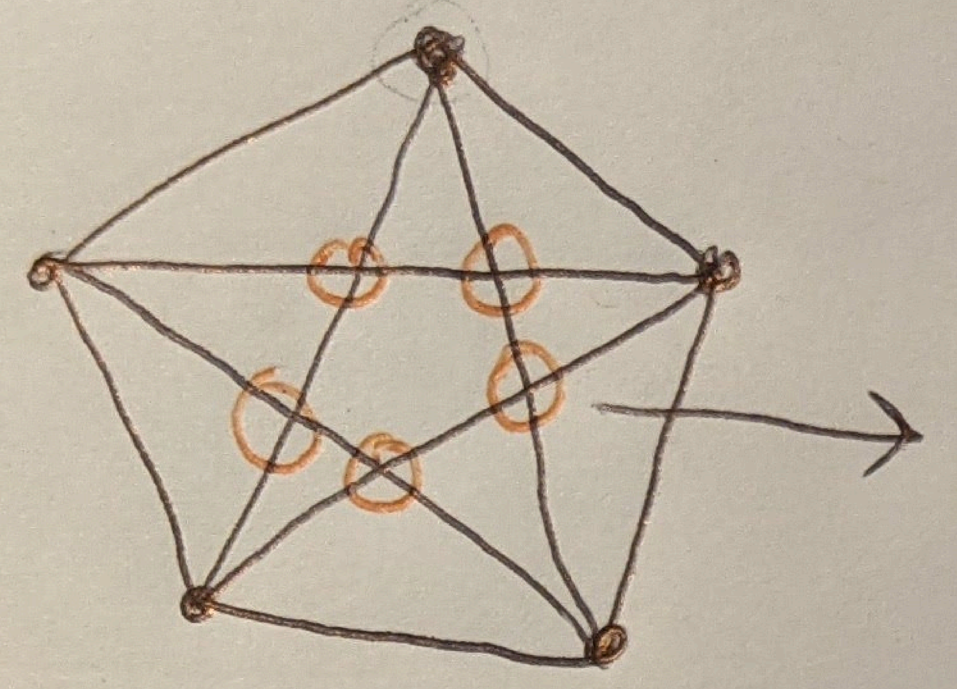
if G has no K_3 subgraphs $\Rightarrow$ independent set

$$\geq \frac{n \log d}{d}(1+o(1))$$

(almost tight)

Crossing Lemma

Drawing graphs in the plane with crossings

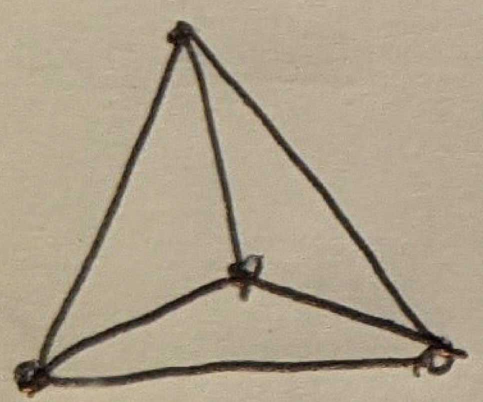


edges are represented by continuous curves joining corresponding vertices

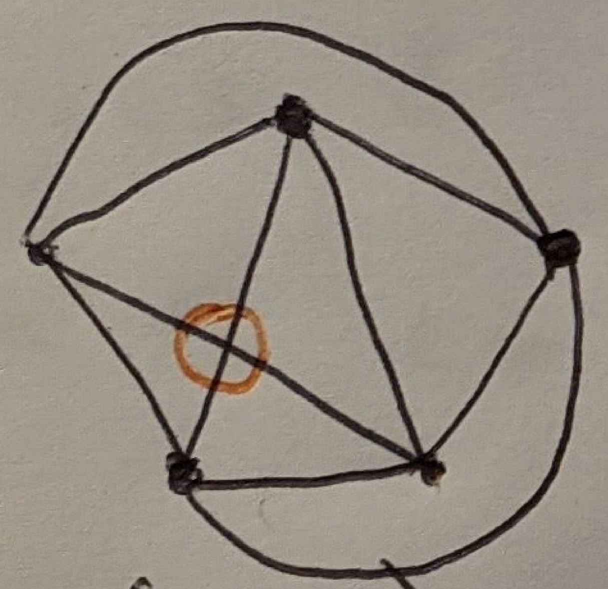
A **crossing** is intersection of edges, other than the common end.

A planar drawing has no crossings.

$cr(G)$ – the crossing number minimum number of crossings in a drawing of G.



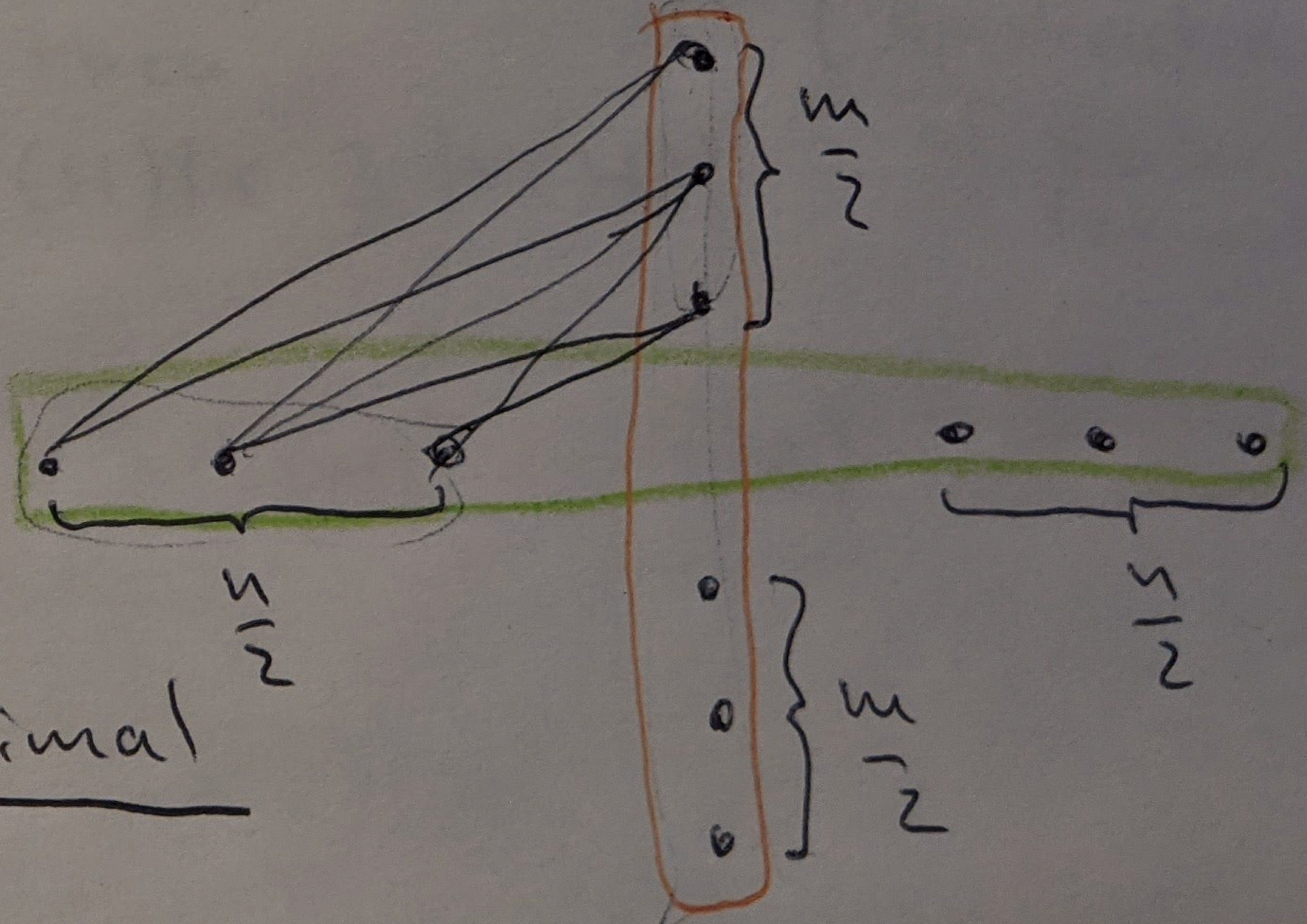
$cr(K_4) = 0$



$cr(K_5) = 1$

Turán brickyard problem:

$$cr(K_{m,n}) \overset{?}{=} (1+o(1)) \frac{1}{16} \frac{m^2 n^2}{2}$$



optimal

$\boxed{\text{Open}}$

$cr(K_n) \to$ is not known

$cr(K_n) = \Omega(n^4)$. ?

$cr(K_n) \geq \frac{n^4}{10000}$?

$X = cr(K_n)$ In a random K_5 chosen from K_n each crossing is present with probability $\frac{c}{n^4}$

In every K_5 there is ≥ 1 crossing.

Theorem 2.5 (Crossing lemma): Let G be a graph with n vertices & m edges, s.t. $m \geq 4n$.

then $cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$.

Proof: If H is a planar graph then $|E(H)| \leq 3|V(H)|$. (Application of Euler's formula)

This implies

$$cr(H) \geq |E(H)| - 3|V(H)|. \quad (*)$$

(remove one edge from each crossing)

To obtain the theorem from (*)

select a random subgraph H of G

by choosing every vertex independently at random with probability p, and keeping all the edges between chosen vertices.

$$\underbrace{\mathbb{E}[cr(H)] \geq \mathbb{E}[|E(H)|] - 3\,\mathbb{E}[|V(H)|]}$$

calculate the terms & optimize over p

(next time)