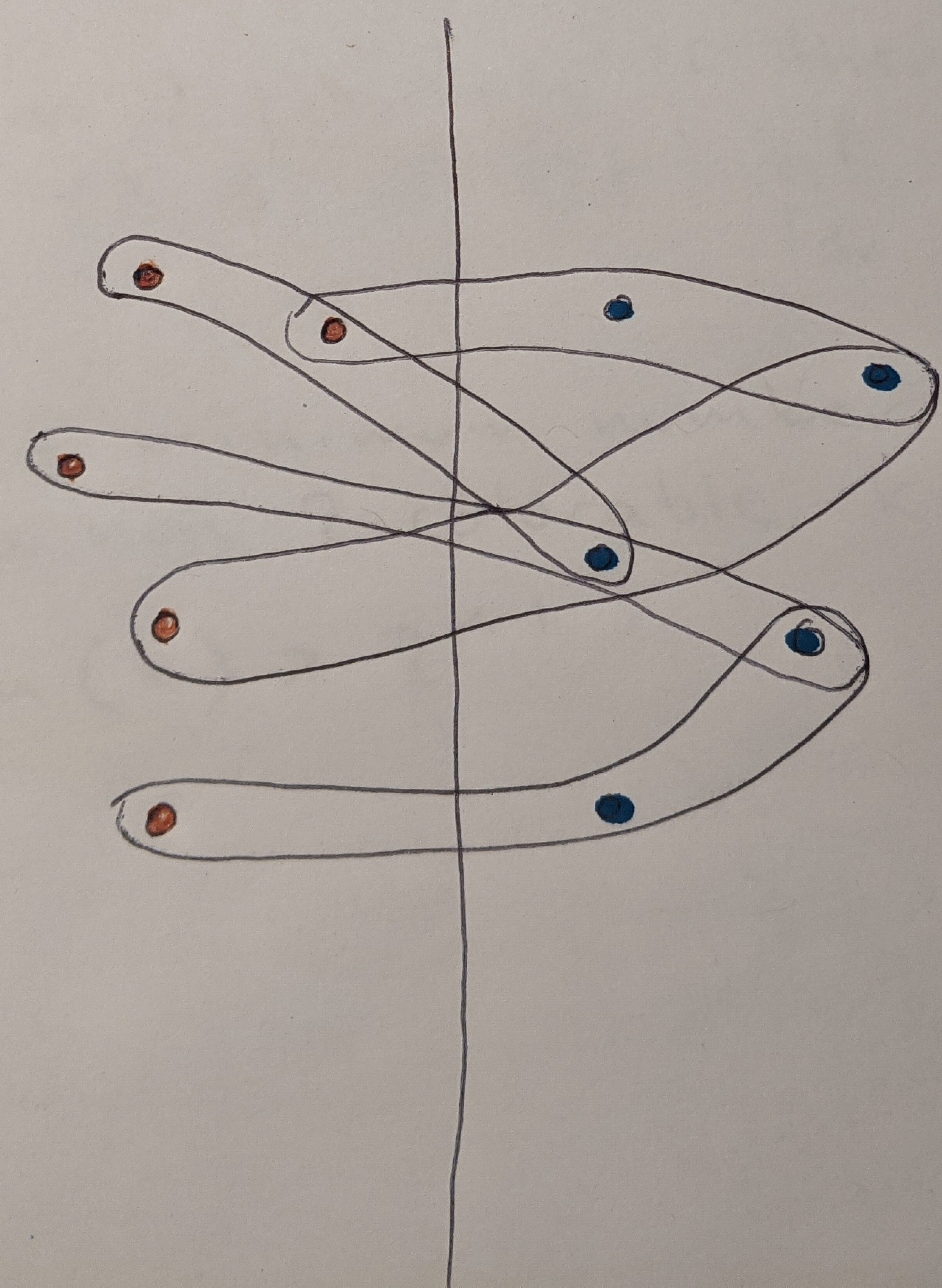
Lecture 2: Introduction Part II.



Notation and Reminders.

X finite set. $\boxed{X^{(k)}}$ - collection of all subsets of X of size k

$$\left(X^{(k)} = \{ A \subseteq X : |A| = k \} \right)$$

A k-uniform hypergraph or k-graph H with vertex set $V(H)$ has edge set $E(H) \subseteq (V(H))^{(k)}$

(edges are k-tuples of vertices.)

A hypergraph H is 2-colorable if there exists a partition (A, B) of $V(H)$ s.t. $e \cap A \neq \emptyset$, $e \cap B \neq \emptyset$ for every $e \in E(H)$.

$m(k)$ - minimum number of edges in a non 2-colorable k-graph.

Lemma 1.4: $m(k) > 2^{k-1}$.

Theorem 1.5: $m(k) = O(k^2 2^k)$.

(There exists $C>0$ s.t. $m(k) \le Ck^2 2^k$ for every $k \in \mathbb{N}$.
I.e. there exists a k-uniform non-2-colorable hypergraph with at most $Ck^2 2^k$ edges).

Proof: Let $n \in \mathbb{N}$ $n = n(k)$ be chosen later.
Let H be generated at random by selecting m subsets of $[n] = \{1, 2, \dots, n\}$. uniformly, independently at random. (edges could be repeated)

Let (A, B) be a partition of $[n]$.
Then $\forall$

$$P(e \subseteq A \text{ or } e \subseteq B) = \frac{\binom{|A|}{k} + \binom{|B|}{k}}{\binom{n}{k}} = \frac{\binom{a}{k} + \binom{n-a}{k}}{\binom{n}{k}} \geq \frac{2\binom{n/2}{k}}{\binom{n}{k}}$$

$|A| = a$

assume n is even

$$= 2 \cdot \frac{n/2}{n} \cdot \frac{\left(\frac{n}{2}-1\right)}{n-1} \cdot \ldots \boxed{\frac{\left(\frac{n}{2}-k+1\right)}{(n-k+1)}} \overset{?}{\geq} 2 \cdot \left(\frac{\frac{n}{2}-k+1}{n-k+1}\right)^k$$

$$= 2^{-k+1} \cdot \left(\frac{n-2(k-1)}{n-(k-1)}\right)^k =$$

$$= 2^{-k+1} \left(1 - \underbrace{\frac{k-1}{n-(k-1)}}\right)^k$$

$\boxed{n = (k-1)k}$

$\boxed{\varepsilon \sim \frac{1}{2e}}$

$$= 2^{-k+1} \left(1 - \frac{k-1}{(k-1)^2}\right)^k = 2^{-k+1}\left(1 - \frac{1}{k-1}\right)^k$$

$$\geq \boxed{\varepsilon}\, 2^{-k}.$$

$p($ ~~…~~ (A,B) is a 2-coloring of $H)$

$$\leq (1-\varepsilon 2^{-k})^m \leq e^{-\varepsilon 2^{-k} m}.$$

because edges are selected independently.

$$\boxed{1+x \leq e^x}$$

$$p(H \text{ is } 2\text{-colorable}) \leq \underbrace{2^n}_{\#\text{ of partitions}} \cdot e^{-\varepsilon 2^{-k} m} \boxed{<} 1$$

we want m to satisfy this.

$$\log 2 \cdot k(k-1) - \varepsilon 2^{-k} m < 0$$

$$\boxed{m > \frac{\log 2\, k^2\, 2^k}{\varepsilon}} = c k^2 2^k \checkmark.$$

$$2^{k-1} \leq m(k) \leq O(k^2 2^k).$$

Graph G (2-graph) is <u>k-colorable</u> if there exists
$c: V(G) \to \{1, 2, \dots, k\}$. s.t. $c(u) \neq c(v)$ for every $\{u,v\} \in E(G)$.

$\chi(G)$ - minimum k s.t. G is k-colorable.
chromatic number

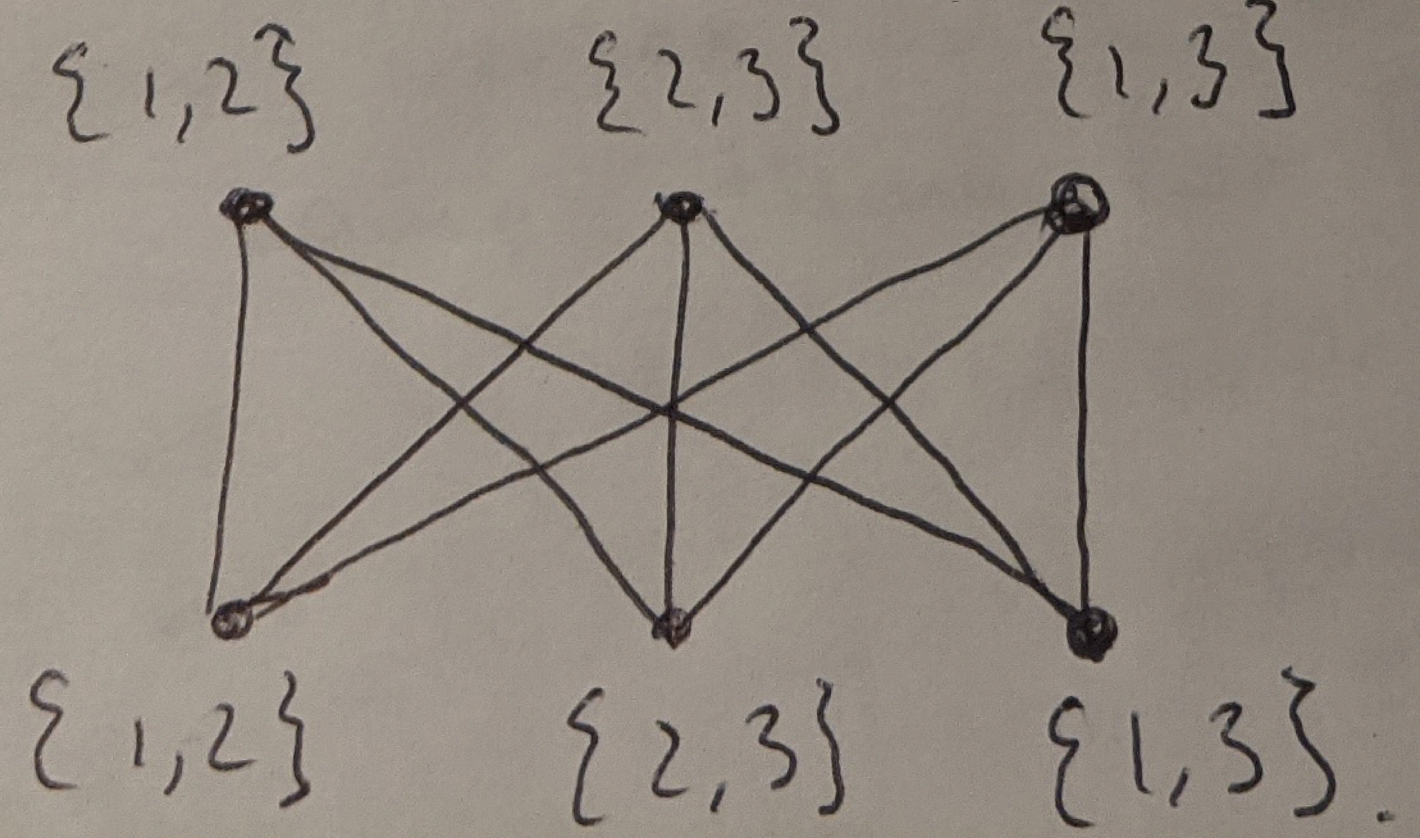
<u>List coloring</u>

Graph G is <u>k-list-colorable</u> if for every assignment of lists $\{L(v)\}_{v \in V(G)}$ s.t. $|L(v)| = k$.
there exists $\{c(v)\}_{v \in V(G)}$ s.t.

- $c(v) \in L(v)$ for every $v \in V(G)$ &
- $c(u) \neq c(v)$ for every $\{u,v\} \in E(G)$.

k-list-colorable $\overset{\not\Leftarrow}{\Rightarrow}$ k-colorable
$\underline{L(v) = \{1, 2, \dots, k\}}$ $\forall v \in V(G)$

$K_{3,3}$:



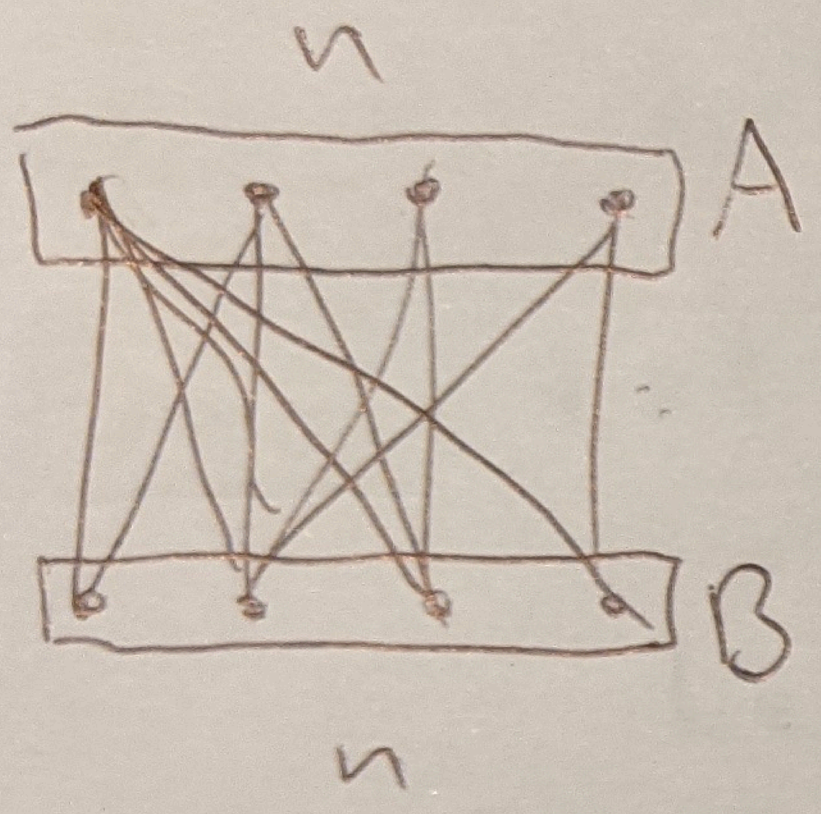
2-colorable

& not 2-list-colorable.

$\chi_L(G)$ — minimum k s.t. G is k-list-colorable

list-chromatic number

(?) Asymptotic behaviour of $\chi_L(K_{n,n})$.



$\boxed{\textbf{Lemma 1.6: } \text{If } n < 2^{k-1} \text{ then } K_{n,n} \text{ is } k\text{-list colorable.}}$

$(\chi_L(K_{n,n}) \le \log_2 n + 2).$

$\chi_L(K_{n,n}) = (1+o(1)) \log_2 n$

<u>Proof:</u> Independently at random for each color in $\bigcup_{v \in V} L(v)$ assign it to $L(A)$ or to $L(B)$

$L(A) \downarrow$ colors allowed on A

$L(B) \downarrow$ color allowed on B.

(A, B) is the bipartition of $K_{n,n}$.

Valid coloring exists as long as "allowed" every vertex in A has a color in L(A) in its list, same thing for B.

$P[v$ has no allowed colors in its list$] = \left(\frac{1}{2}\right)^k$

$E[\#$ vertices with no allowed colors$] = 2n \left(\frac{1}{2}\right)^k < 1.$

There exist a choice of valid coloring.

Theorem 1.7: Let $n = m(k)$
then $K_{n,n}$ is not k-list colorable.

$$(\chi_l(K_{Ck^2 2^k, Ck^2 2^k}) \geq k)$$

$$\boxed{\log_2 n + 2 \geq \chi_l(K_{n,n}) \geq (1 - o(1)) \log_2 n}$$

Proof: Let H be a non-2-colorable k-uniform hypergraph with n edges.
Let $L_1, L_2, \dots, L_n \subseteq \mathbb{N}$ be the edges of H
$|L_i| = k$.

Let $\{v_1, v_2, \dots, v_n\}$ v.s. $\{u_1, u_2, \dots, u_n\}$ be the bipartition of $K_{n,n}$
and let $L(v_i) = L(u_i) = L_i$

Suppose that there exists a ~~choice~~ list coloring from these list. $c(v_1), \dots, c(v_n)$ $c(u_1), \dots, c(u_n)$

Then $\underbrace{\{c(v_1), \dots, c(v_n)\}}_{C_A} \cap \underbrace{\{c(u_1), \dots, c(u_n)\}}_{C_B} = \emptyset$.

C_A — sets of vertices of H — C_B.

$C_A \cap L_i \neq \emptyset \neq C_B \cap L_i$ for every i.

contradicts the choice of H.

Let G be a graph.

A <u>dominating set</u> $X \subseteq V(G)$ s.t. for every $v \in V(G)$

$v \in X$ or v is adjacent to a vertex of X.

Want: a <u>small</u> dominating set.

in terms of $n = |V(G)|$.

<u>Additional assumption</u>: Every vertex has d neighbors.

At least $n/d+1$ vertices are needed.

<u>Theorem 1.8</u>: Let G be a graph s.t. every vertex has at least d neighbors with n vertices

Then G contains a ~~independent~~ dominating set of size

$$\leq n \cdot \frac{1+\ln(d+1)}{d+1}.$$

<u>Proof</u>: Let $X \subseteq V(G)$ be obtained by adding every vertex $v \in V(G)$ to X independently at random with probability p (to be decided later.)

$\cancel{\underbrace{n(1-p)^{d+1}}_{<1}}$

$$P[u \notin X \text{ \& } u \text{ has no neighbors in } X] \leq (1-p)^{d+1}.$$

Let Y consist of all such vertices ("alteration")

Then $X \cup Y$ is dominating.

$$E[X \cup Y] = E[X] + E[Y] \leq pn + n(1-p)^{d+1}$$

$$= n\,(p + (1-p)^{d+1})$$

$$p + \underbrace{(1-p)}^{d+1} \leq p + e^{-p(d+1)} = \frac{1+\ln(d+1)}{d+1}$$

optimal $\quad p = \dfrac{\ln(d+1)}{d+1}$

$$E[X \cup Y] \leq n\left(\frac{1+\ln(d+1)}{d+1}\right)$$ → there exists a dominating set of at most this size.