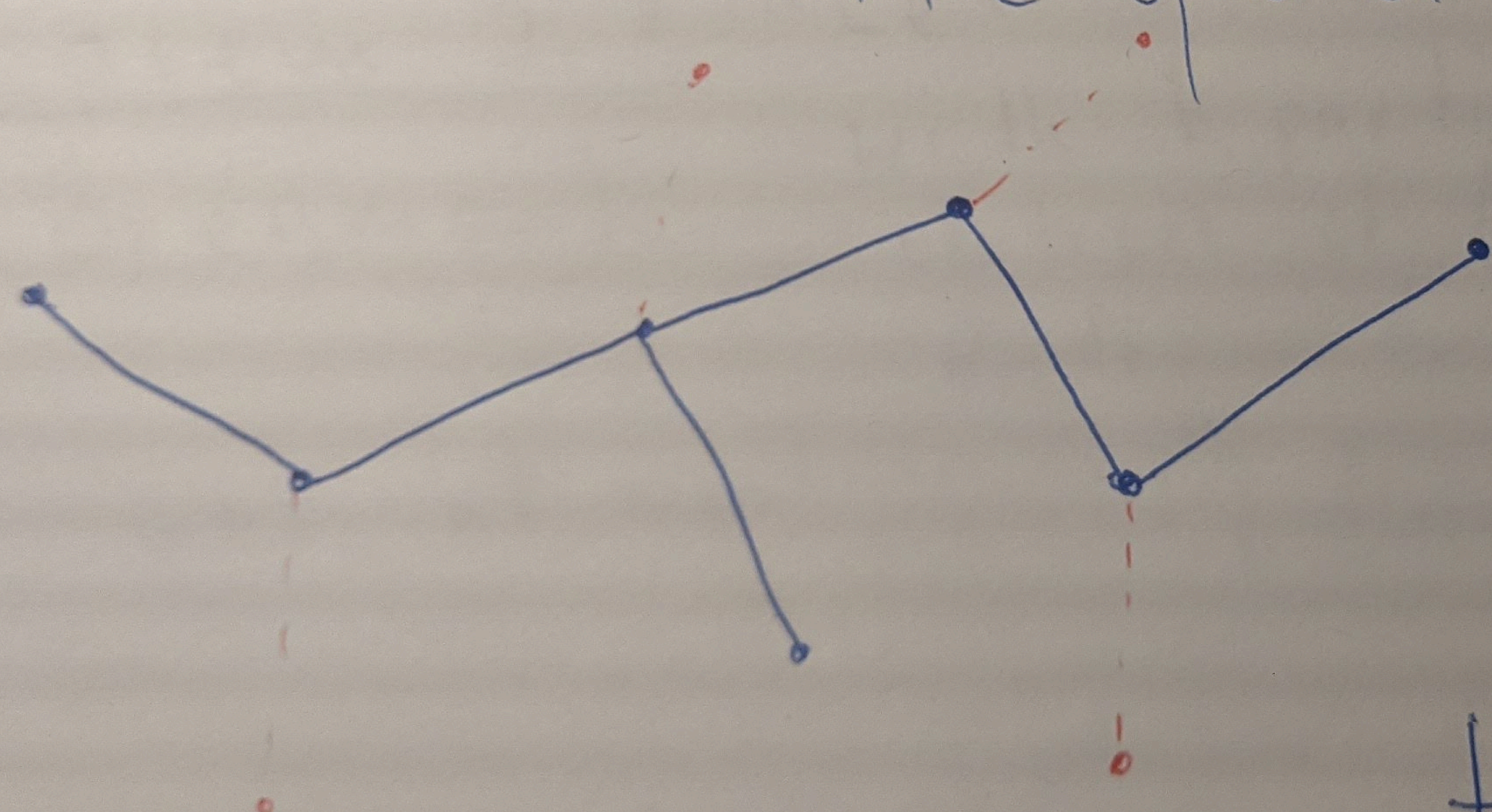
Lecture 17:

Brégman & Sidorenko inequalities



$$t(H,G) \geq t(K_2,G)^{|E(H)|}.$$

Entropy: $H(X) = \sum -p(x) \log_2 p(x)$

9.1: $H(X) \leq \log_2 |\text{support}(X)|$.

$H(Y|X) = H(X,Y) - H(X)$

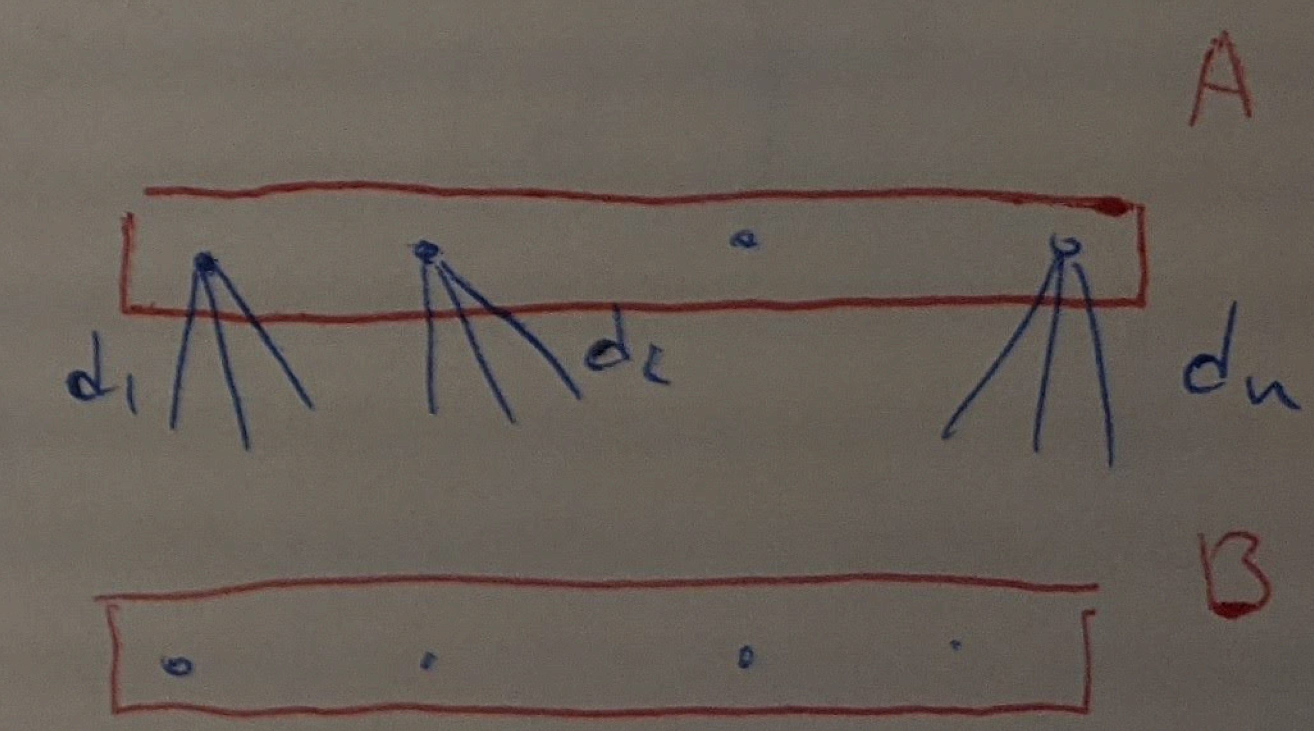
9.2, 9.3: $H(X) + H(Y) \geq H(X,Y) \geq H(X)$

9.5. $H(Y|X) \leq H(Y)$

$H(X|Y,Z) \leq H(X|Z)$

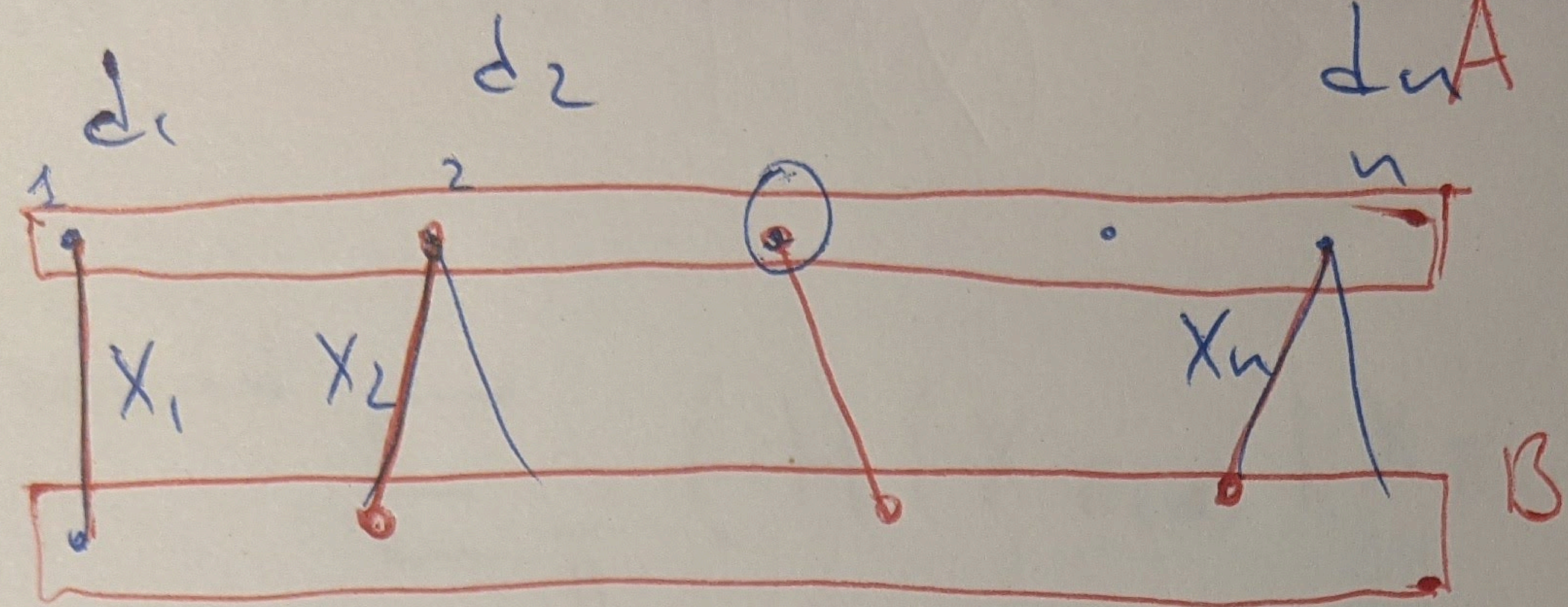
Theorem 9.7 (Brégman): Let G be a bipartite graph with bipartition (A,B) s.t. vertices of A have degrees $d_1, d_2, \ldots, d_n$ then

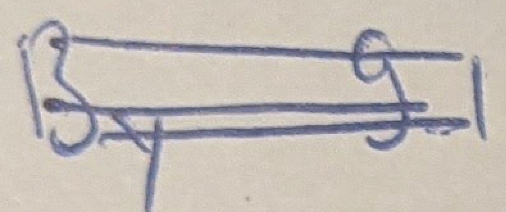
G has at most $\prod_{i=1}^{n} (d_i!)^{1/d_i}$ perfect matchings.



Tight when $G = K_{d_1,d_1} \sqcup K_{d_2,d_2} \sqcup \ldots$

Proof:





Consider uniform distribution on perfect matching of G

$$H(X_1, X_2, \ldots, X_n) = \log_2(\#\text{ of perfect matchings}).$$

Attempt # 1:

$$H(X_1, X_2, \ldots, X_n) \le \sum_{i=1}^{n} H(X_i) \overset{\text{g.1}}{\le} \sum_{i=1}^{n} \log_2 \underbrace{|\text{support}(X_i)|}_{\shortparallel}$$

$$\le \sum_{i=1}^{n} \log_2 d_i$$

$$\underbrace{\text{Implies } \# \text{ of p.m.} \le \prod_{i=1}^{n} d_i}\quad \underline{\text{easy}} \qquad (d_i!)^{1/d_i}$$

- **Trick:** Order the vertices uniformly at random.

$\pi : [n] \to [n]$

$$H(X_1, X_2, \ldots, X_n) = H(X_{\pi(1)}, \ldots, X_{\pi(n)})$$

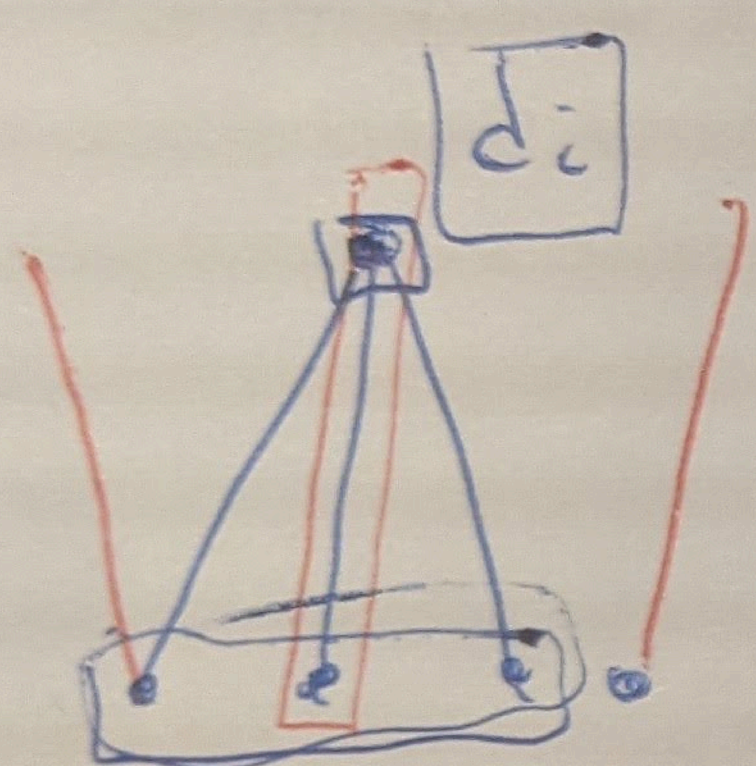
$$= H(X_{\pi(1)}) + H(X_{\pi(2)} \mid X_{\pi(1)}) + \cdots + \underbrace{H(X_{\pi(k)} \mid X_{\pi(1)}, \ldots, X_{\pi(k-1)})}.$$

$$+ \cdots$$

To bound our entropy consider k s.t. $\underbrace{\pi(k)=i}_{\text{For given } i}$.

and bound $\mathbb{E}_{\pi}\left(H(X_{\pi(k)} \mid X_{\pi(1)}, \dots, X_{\pi(k-1)})\right)$.

$$\le \mathbb{E}_{\pi}\left[\log_2 \left|\mathrm{supp}(X_{\pi(k)} \mid X_{\pi(1)}, \dots, X_{\pi(k-1)})\right|\right]$$



$|\mathrm{supp}(X_{\pi(k)} \mid X_{\pi(1)} \dots)|$

s.t.

$\pi(k)=i$

is uniformly distributed on $[d_i]$

$$= \frac{1}{d_i}\left(\log_2 1 + \log_2 2 + \dots + \log_2 d_i\right) = \frac{1}{d_i}\log_2(d_i!)$$

In total

$$\log_2(\#\text{p.m.}) = H(X_1, \dots, X_n) \le \sum_{i=1}^{n} \frac{1}{d_i} \log_2 (d_i)!$$

Exponentiating we get 9.7.

Corollary 9.8: (Kahn & Lovász) Let G be a graph then G has at most

$$\prod_{v \in V(G)} \left((\deg(v))!\right)^{\frac{1}{2\deg(v)}}$$

perfect matchings.

Proof sketch: If G is bipartite, then

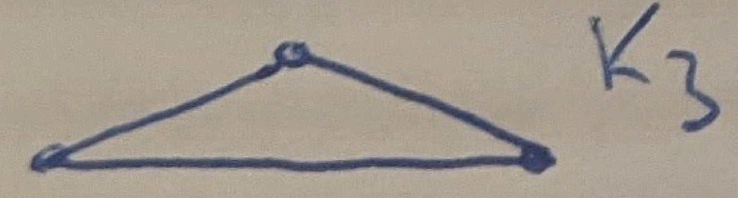
$$\#\text{ of p.m.} \le \prod_{v \in A} \left((\deg(v))!\right)^{\frac{1}{\deg(v)}}$$

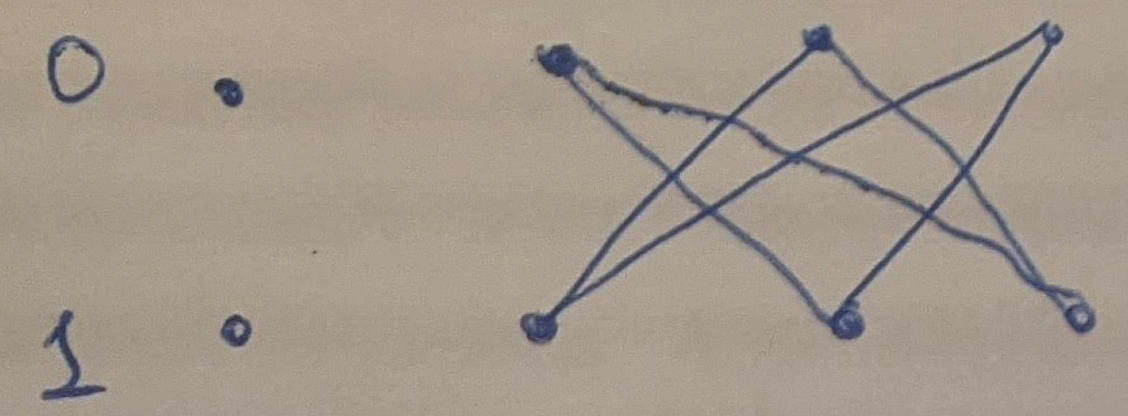
$$\le \prod_{v \in B} \left((\deg(v))!\right)^{\frac{1}{\deg(v)}}$$

we get corollary by taking geometric mean.

If G is not bipartite then one can consider a bipartite graph $\underbrace{G \times K_2} = G'$

$V(G') = V(G) \times \{0,1\}$ and (v,i) is adjacent to (u,j) if and only if v is adjacent to u and $i \neq j$.





$K_3 \times K_2$.

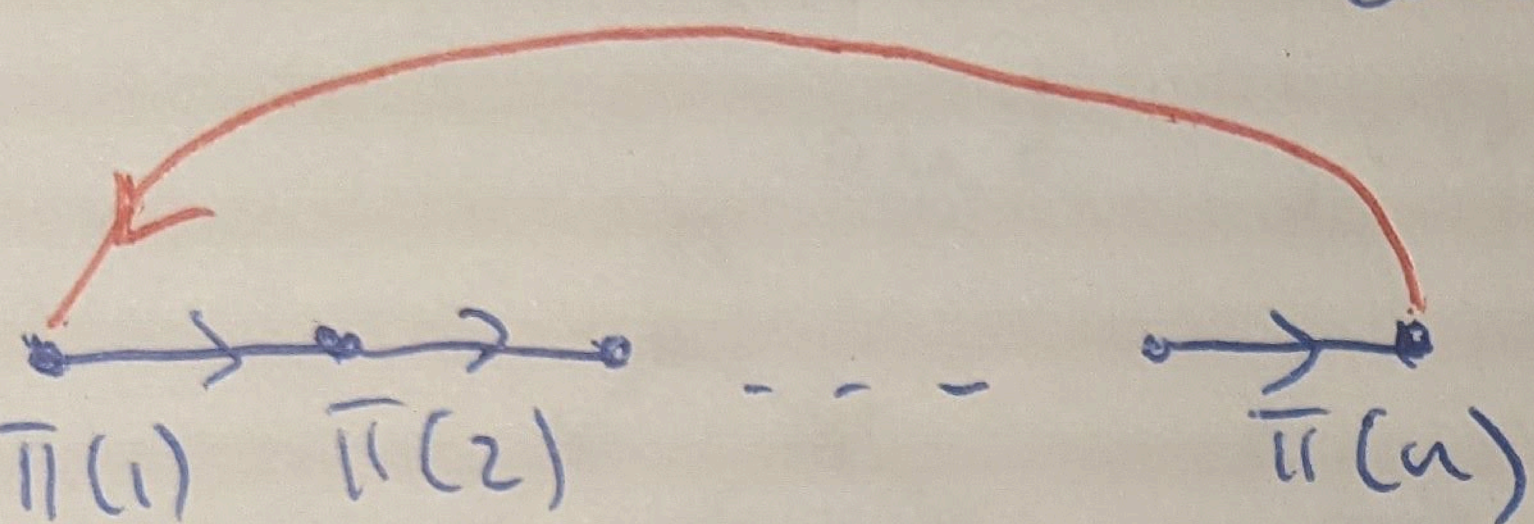
not ~~obvio~~ immediate, but true.

Then $\boxed{\#\text{ of p.m. of } G \times K_2 \ge (\#\text{ of p.m of } G)^2}$

applying bipartite result to $G \times K_2$ yields corollary.

What is the maximum number of Hamilton paths in an n-vertex tournament?

complete directed graph



Hamilton cycle

We proved that there are tournaments with

$$\geq n! \left(\frac{1}{2}\right)^{n-1}$$ Hamilton paths.

Theorem 9.9 (Alon): Every n-vertex tournament has ~~at most~~

$$O\left(n^{3/2} \cdot \frac{n!}{2^n}\right)$$ Hamilton paths.

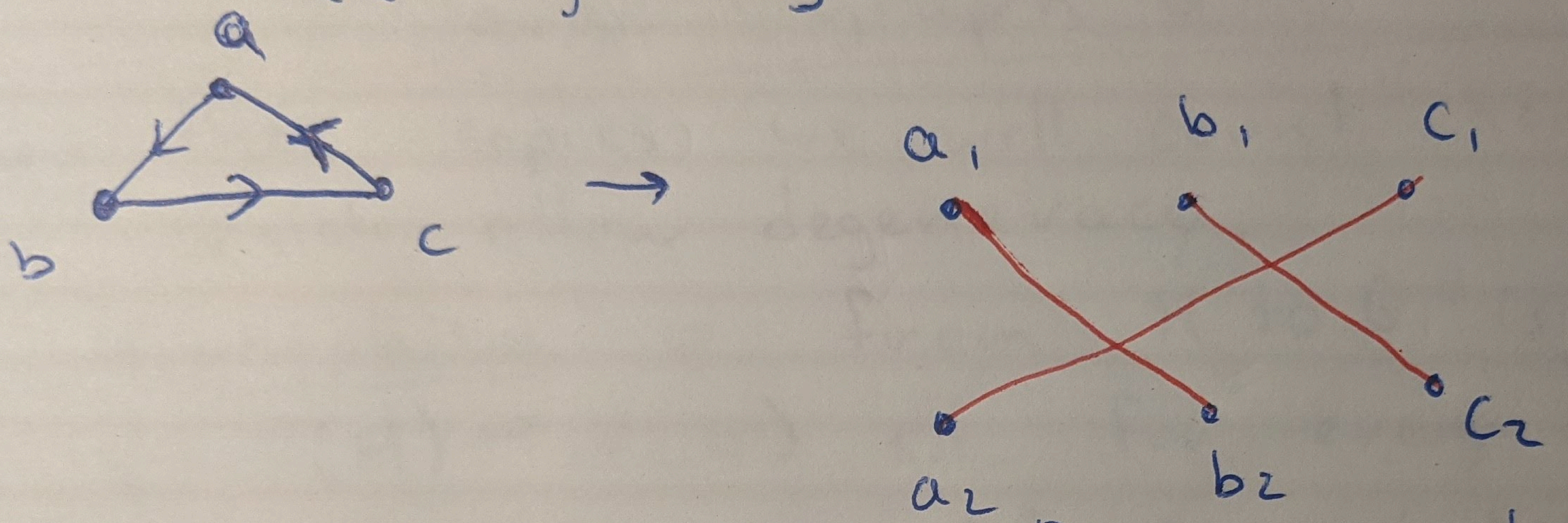
$\Downarrow$ → by removing an edge from every cycle

$\Uparrow$ add an extra vertex and randomly direct edges

[Every n-vertex tournament has

$$O\left(n^{1/2} \frac{n!}{2^n}\right)$$ Hamilton cycles.

Proof: Given a tournament D construct a bipartite graph from D by replacing each $v \in V(D)$ by a pair of vertices v_1 and v_2 and joining u_1 to v_2 for every directed edge $uv \in E(D)$



Hamilton cycles in D correspond to perfect matchings in the resulting graph.

$$\sum d_i = \binom{n}{2}$$

by 9.7 there are

$$\leq \prod_{i=1}^{n} (d_i!)^{1/d_i} \quad (*)$$

Ham. cycles in D where d_i is the <u>outdegree</u> of a vertex i.

$$d! \sim \left(\frac{d}{e}\right)^d$$

Let $f(d) = (d!)^{1/d} \sim \frac{d}{e}$

then $f(d)$ is log-concave

$$f^2(d) \geq f(d-1)\, f(d+1). \leftarrow \text{(check)}$$

if n is odd $\quad (*) \leq \left(\left(\frac{n-1}{2}\right)!\right)^{\frac{2 \cdot n}{n-1}} = \underbrace{O\left(\sqrt{n} \cdot \frac{n!}{2^n}\right)}$

Sidorenko's inequality

$\boxed{\text{What is the minimum number of copies of a "small fixed" graph } H \text{ in a "large" graph } G\text{?}}$

Rather than copies we will count homomorphisms which allow degeneracy.

A homomorphism φ from H to G is a map.
$\varphi : V(H) \to V(G)$ s.t. for every $uv \in E(H)$
we have $\varphi(u)\varphi(v) \in E(G)$.

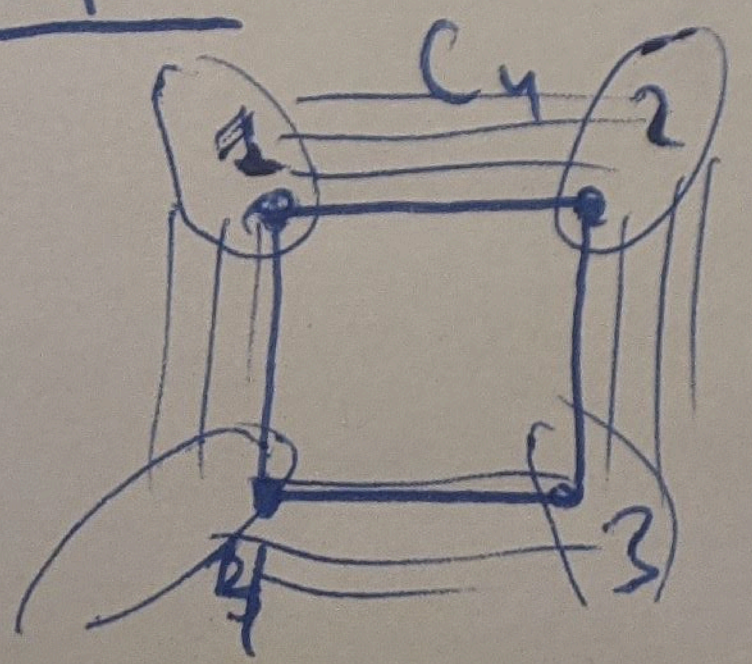
Let $\hom(H,G) = \#$ of homomorphisms H to G.

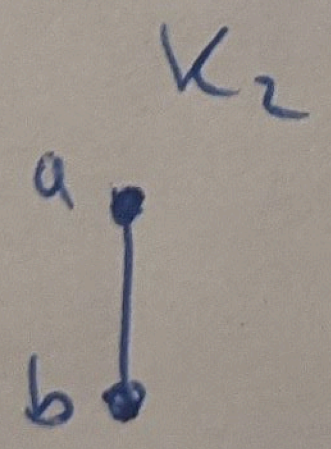
Let $t(H,G) = \dfrac{\hom(H,G)}{|V(G)|^{|V(H)|}}$.

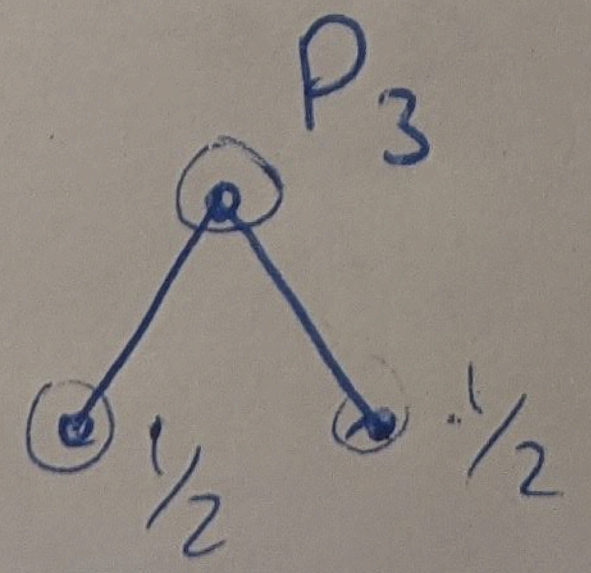
$\downarrow$ # of maps $V(H) \to V(G)$.

be the homomorphism density of H in G
$\to$ probability that a random map $V(H) \to V(G)$ is a homomorphism.

Examples:







$t(K_2, C_4) = \frac{8}{16} = \frac{1}{2}$

$\hom(K_2, C_4) = 8$ $\qquad t(K_3, C_4) = 0$

$t(P_3, C_4) = \frac{1}{4}$.

What is $\underline{\text{minimum}}$ $t(H,G)$ given that $t(K_2,G)=p$?

$$\frac{2|E(G)|}{|V(G)|^2}$$

If $p=0$ → answer $=0$.

$p \to 1$ → answer $\to 1$ for any fixed H.

$$t(H,G) \geq 1 - |E(H)|(1-p)$$

union bound.

$H=K_2$ → get a linear function

$H=P_3$ → $\underbrace{\min t(P_3,G)}_{t(K_2,G)=p} \leq \boxed{p^2}$ → $\underline{p^2 \text{ is tight}}$

$G=G(n,p)$ $\quad n\to\infty$ $\quad t(K_2,G)\to p$.

$$\boxed{\begin{aligned} &t(P_3,G)\to p^2 \\ &t(H,G)\to \boxed{p^{|E(H)|}} \end{aligned}}$$

For what graphs H do we have

$$t(H,G) \geq p^{|E(H)|} \text{ for every } G \text{ with } t(K_2,G)=p \text{ ?}$$

i.e. for what H

$$t(H,G) \geq \left(t(K_2,G)\right)^{|E(H)|} \quad \text{for every } G.$$

(true for $H=K_2$, true for P_3).

If H is not bipartite then $t(H,G)=0$ for any bipartite G, so the above is false.

Conjecture: (Sidorenko, 93)

$$t(H,G) \geq \left(t(K_2,G)\right)^{|E(H)|}.$$

for any bipartite H.

(random graphs minimize number of copies of H among graphs of given density).