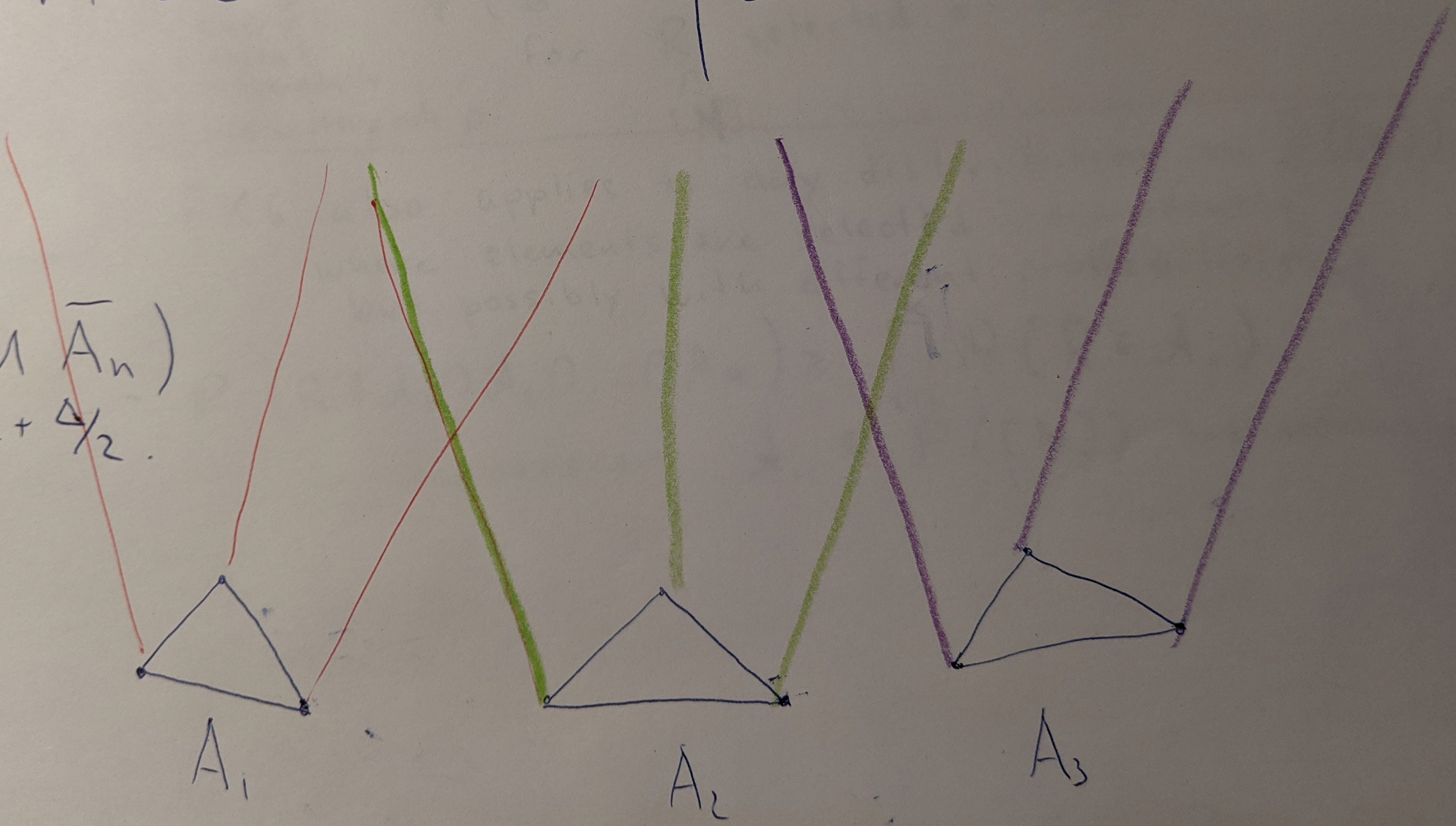
Lecture 14:

Janson inequalities

$$\mathbb{P}(\bar{A}_1 \wedge \ldots \wedge \bar{A}_n) \le e^{-\mu + \Delta/2}.$$



Recall: The FKG inequality:

$f, g : \mathcal{P}([n]) \to \mathbb{R}_+$ increasing

then $\mathbb{E}_\mu[fg] \geq \mathbb{E}_\mu[f] \cdot \mathbb{E}_\mu[g]$

for any log-supermodular prob. distribution μ on $\mathcal{P}([n])$

In particular, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([N])$ are increasing families and $\mu = [N]_p$ then

$$\mathbb{P}(R \in \mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(R \in \mathcal{A})\,\mathbb{P}(R \in \mathcal{B})$$

for $R \subseteq [N]$ selected randomly according to $[N]_p$

$[N]_p$: every element is independently selected with prob. p.

Remark:

- FKG also applies to any distribution on $\mathcal{P}([N])$ where elements are selected independently but possibly with different probabilities.
- $\mathbb{P}(R \in \mathcal{A}_1 \cap \mathcal{A}_2 \cap \ldots \cap \mathcal{A}_k) \geq \prod_{i=1}^{k} \mathbb{P}(R \in \mathcal{A}_i)$ whenever $\mathcal{A}_i \subseteq \mathcal{P}([N])$ are increasing.

8. Janson inequalities.

[What is the probability that $G(n,p)$ contains no K_3 subgraph?]

Setup of the results in this section

Let $R \in \mathcal{P}([N])$ selected by independently selecting its elements

Let $S_1, S_2, \dots S_k \subseteq [N]$

We want to upper bound the probability that R contains none of S_i.

$N = \binom{n}{2}$

$|S_i| = 3 \quad k = \binom{n}{3}$

S_i correspond to Δ's in K_n.

Let A_i be the event $\{S_i \subseteq R\}$.

[$A_i = \{R : S_i \subseteq R\}$

A_i is increasing

and let

$X = \sum \mathbb{1}_{A_i}$

= # of events A_i that occur

= # of sets S_i that are in R.

We are interested in $\mathbb{P}(X=0) = \mathbb{P}(R \notin A_1 \cup A_2 \cup \dots \cup A_k)$

$= \mathbb{P}(R \in \bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_k)$

Let $\mu = \mathbb{E}(X)$.

$$\mathbb{P}(X=0) = \mathbb{P}(\bar{A}_1 \wedge \bar{A}_2 \wedge \dots \wedge \bar{A}_k) \geq \prod_{i=1}^{k} \mathbb{P}(\bar{A}_i) =$$

$\bar{A}_i$ are decreasing

$$= \prod_{i=1}^{k} (1 - \underbrace{\mathbb{P}(A_i)}) \qquad \mathbb{P}(A_i) = o(1)$$

$$\simeq \prod_{i=1}^{k} e^{-\mathbb{P}(A_i)\cdot(1-o(1))}$$

$$= e^{-(1-o(1)) \sum \underbrace{\mathbb{P}(A_i)}}$$

$$= e^{-(1-o(1))\mu}$$

Lemma 8.1: If $\mathbb{P}(A_i) = o(1)$ then $\mathbb{P}(X=0) \geq e^{-(1-o(1))\mu}$. $\boxed{e^{-\mu}}$

Let

$$\Delta = \sum_{\substack{i\sim j \\ \underbrace{(i,j)}}} \mathbb{P}(A_i \wedge A_j)$$

where $i \sim j$ if $S_i \cap S_j \neq \emptyset$

Note that A_i if independent of $\{A_j\}_{\substack{j\neq i \\ j \nsim i}}$.

Recall that Δ appeared in Chebyshev's inequality which implied that if $\mu \to \infty$ $\Delta = o(\mu^2)$ then $\mathbb{P}(X=0) = o(1)$.

Theorem 8.2: In the setting above

$$\mathbb{P}(X=0) \le e^{-\mu + \frac{\Delta}{2}}.$$

First Janson inequality

Combining 8.1 & 8.2. we get that if $\mathbb{P}(A_i) = o(1)$ & $\Delta = O(\mu)$

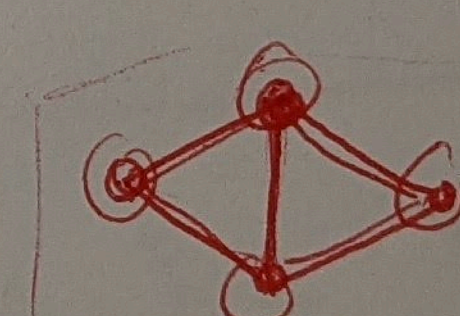
then $\mathbb{P}(X=0) = e^{-(1-o(1))\mu}$.

For the problem of estimating prob. that $G(n,p)$ has no K_3.

$A_1, A_2, \ldots, A_{\binom{n}{3}}$ each corresponding to presense of a particular Δ in $G(n,p)$

$$\mu = p^3 \binom{n}{3} \sim \frac{p^3 n^3}{6} \qquad \Delta \approx \Theta(n^4 p^5) \qquad \cancel{n^4 p^5} << n^3 p^3$$

$$np^2 \le o(1)$$



n^4

So $\Delta = O(\mu)$ when $p = o(1/\sqrt{n})$.

Corollary 8.3: For $\boxed{p = o(1/\sqrt{n})}$

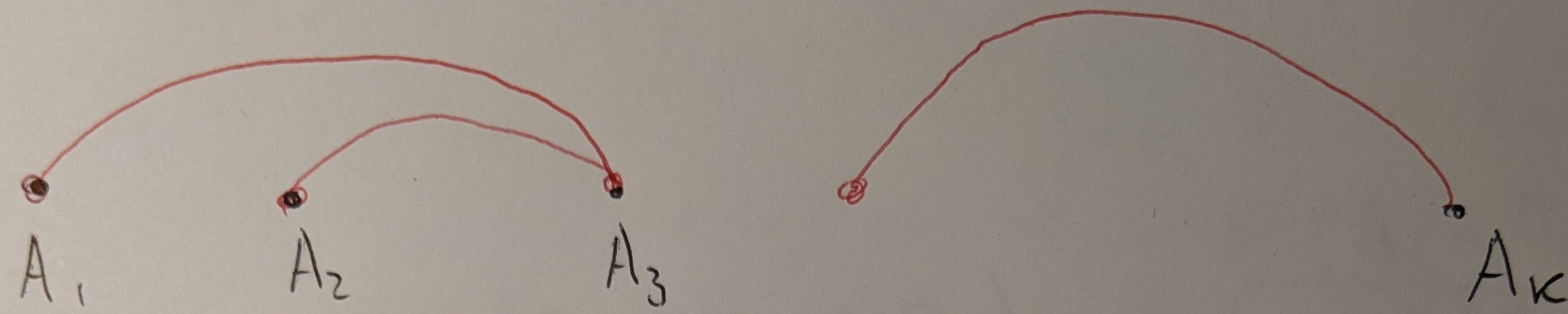
$$\mathbb{P}(G(n,p) \text{ has no } K_3) = e^{-(1-o(1))\frac{p^3 n^3}{6}}.$$

If $p = \frac{c}{n}$ $\quad \mathbb{P}(G(n,\frac{c}{n}) \text{ has no } K_3) = e^{-(1-o(1))\frac{c^3}{6}}$

has no sharp

Proof:

Lutz Warnke via Yufei Zhao's notes.



Let $r_i = \mathbb{P}(A_i \mid \bar{A}_1 \wedge \bar{A}_2 \wedge \dots \wedge \bar{A}_{i-1})$

$$\mathbb{P}(X=0) = \mathbb{P}(\bar{A}_1 \wedge \bar{A}_2 \wedge \dots \wedge \bar{A}_k)$$

$$= \mathbb{P}(\bar{A}_1)\,\mathbb{P}(\bar{A}_2 \mid \bar{A}_1)\,\mathbb{P}(\bar{A}_3 \mid \bar{A}_2 \wedge \bar{A}_1) \dots \mathbb{P}(\bar{A}_k \mid \bar{A}_1 \wedge \dots \wedge \bar{A}_{k-1})$$

$$= (1-r_1)(1-r_2)\dots(1-r_k)$$

$$\le e^{-r_1-r_2-\dots-r_k}.$$

Need to show $\sum r_i \ge \mu - \frac{\Delta}{2} = \sum \mathbb{P}(A_i) - \sum_{\substack{\{i,j\} \\ j\sim i}} \mathbb{P}(A_i \wedge A_j)$

We will show $\boxed{r_i \ge \mathbb{P}(A_i) - \sum_{\substack{j<i \\ j\sim i}} \mathbb{P}(A_i \wedge A_j)}$

It implies the theorem.

Let $D_0 = \bigwedge_{\substack{j<i \\ j \not\sim i}} \bar{A}_j$, $D_1 = \bigwedge_{\substack{j<i \\ j \sim i}} \bar{A}_j$.

D_1 is decreasing

Then $r_i = \boxed{\mathbb{P}(A_i \mid D_0 \wedge D_1)} = \dfrac{\mathbb{P}(A_i \wedge D_0 \wedge D_1)}{\mathbb{P}(D_0 \wedge D_1)}$

$\geq \mathbb{P}(A_i \wedge D_1 \mid D_0 \wedge D_1) \qquad \geq \dfrac{\mathbb{P}(A_i \wedge D_0 \wedge D_1)}{\mathbb{P}(D_0)}$

$\geq \mathbb{P}(A_i \wedge D_1 \mid D_0) =$

increasing increasing

$= \mathbb{P}(\underbrace{A_i \mid D_0}_{A_i \text{ and } D_0 \text{ are independent}}) - \boxed{\mathbb{P}(A_i \wedge \bar{D}_1 \mid D_0)}$ decreasing

$\leq$ → FKG

$\geq \mathbb{P}(A_i) - \boxed{\mathbb{P}(A_i \wedge \bar{D}_1)}$

Remains to check that

$$\mathbb{P}(A_i \wedge \bar{D}_1) \leq \sum_{\substack{j<i \\ j \sim i}} \mathbb{P}(A_i \wedge A_j)$$

$$\mathbb{P}\Big(A_i \wedge \big(\bigvee_{\substack{j<i \\ j \sim i}} A_j\big)\Big) \leq \sum_{\substack{j<i \\ j \sim i}} \mathbb{P}(A_i \wedge A_j) \quad \checkmark$$

union bound.

$$= \bigvee_{\substack{j<i \\ j \sim i}} (A_i \wedge A_j)$$

What about setting when $\mu < \Delta$

can we give meaningful upper bounds on $\mathbb{P}(X=0)$.

Theorem 8.3: In our setting if $\Delta \geq \mu$

Second Janson inequality

$$\mathbb{P}(X=0) \leq e^{-\frac{\mu^2}{2\Delta}}$$

The proof parallels the bootstraping method used in crossing lemma.

Select random subset of events with certain probability & apply 8.2.

- this will allow us to estimate the prob $G(n,p)$ is Δ-free for remaining regime $p = \Omega(1/\sqrt{n})$

and to prove that $\underbrace{\chi(G(n, 1/2)) = (1+o(1)) \frac{n}{2\log_2 n}}$: