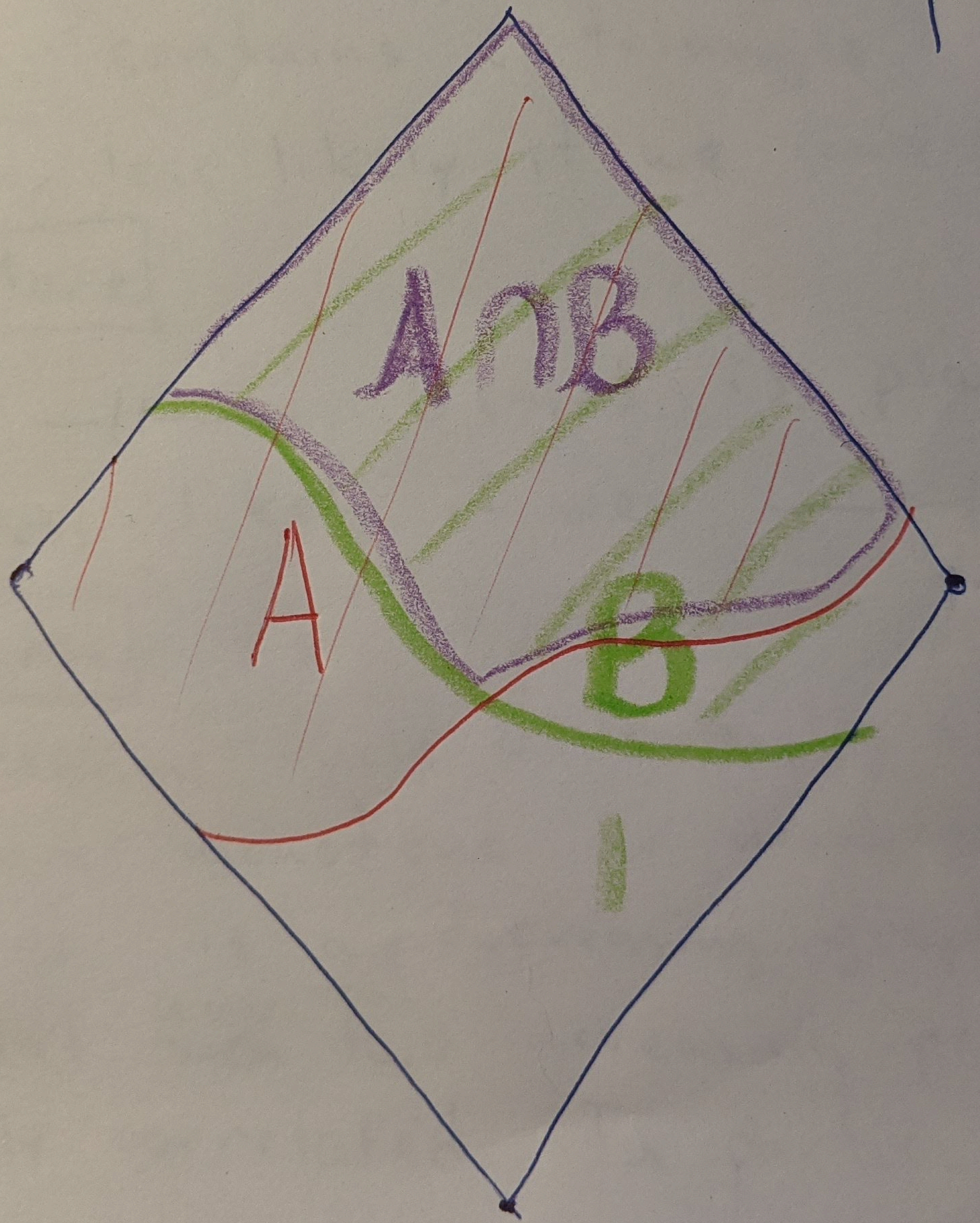
Lecture 13:

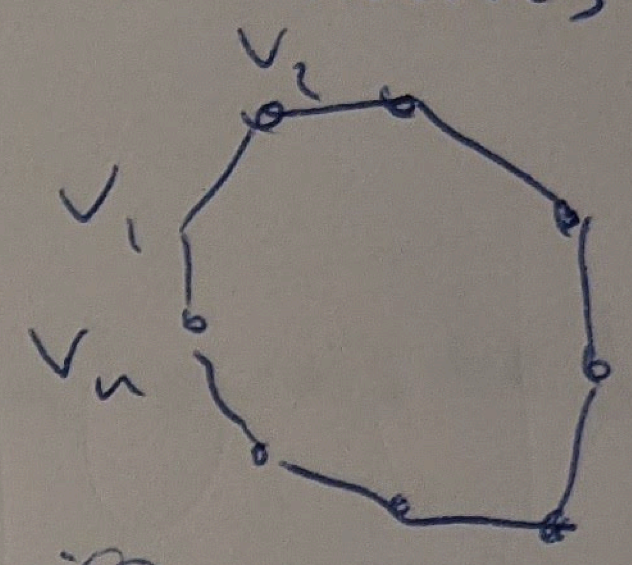
Correlation inequalities



7. Correlation inequalities.

$G(n,p)$

Let A be the event that $G(n,p)$ contains a Hamiltonian cycle.



B be the event that $G(n,p)$ ~~is~~ contains a triangle.

Is A more or less likely if we condition on B?

$\boxed{\text{More}}$

C is —||— $G(n,p)$ is planar

Is A more/less likely if we —||— on C?

$\boxed{\text{Less}}$

~~A~~ Hamiltonian cycle & containing a triangle are monotone or monotonely increasing.

Planar is a decreasing graph property.

We will show that ~~an~~ any two increasing properties are positively correlated. In particular, the above answers are correct

The four functions theorem

Let $\mathcal{P}(X)$ denote the set of all subsets of X

We will work with $\mathcal{P}([n])$, and functions

$$\varphi : \mathcal{P}([n]) \to \mathbb{R}_+^{\circ}.$$

(Examples of such functions we are interested in) are $\varphi_{\mathcal{A}}(A) = \begin{cases} 1 & \text{if } A \in \mathcal{A} \\ 0 & \text{if } A \notin \mathcal{A}. \end{cases}$

$$\varphi(\mathcal{A}) = \sum_{A \in \mathcal{A}} \varphi(A).$$

Theorem 7.1 (Ahlswede, Daykin, 78): Let $\alpha, \beta, \gamma, \delta : \mathcal{P}([n]) \to \mathbb{R}_+$.

such that

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\,\delta(A \cap B).$$

Then for any families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([n])$

$$\alpha(\mathcal{A})\,\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \overset{*}{\cup} \mathcal{B})\,\delta(\mathcal{A} \overset{*}{\cap} \mathcal{B}).$$

$\mathcal{A} \overset{*}{\cup} \mathcal{B} := \underbrace{\{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}}_{\text{(not a multiset)}}$, $\mathcal{A} \overset{*}{\cap} \mathcal{B}$ is defined similarly.

Corollary 7.2: For any $\mathcal{A}, \mathcal{B} \subseteq P([n])$

$$|\mathcal{A}||\mathcal{B}| \leq |\mathcal{A} \overset{*}{\cup} \mathcal{B}||\mathcal{A} \overset{*}{\cap} \mathcal{B}|.$$

Corollary 7.3: If $\mathcal{A} \subseteq P([n])$ then
(Marica, Schonheim)
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$$|\mathcal{A}| \leq |\mathcal{A} - \mathcal{A}|.$$

where $\mathcal{A} - \mathcal{A} = \{ A \setminus B : A, B \in \mathcal{A} \}$.

Proof: Let $\mathcal{B} = \{ [n] - A : A \in \mathcal{A} \}$.

$|\mathcal{A}| = |\mathcal{B}|$

$\mathcal{A} \overset{*}{\cap} \mathcal{B} = \mathcal{A} - \mathcal{A}$.

$\mathcal{A} \overset{*}{\cup} \mathcal{B}$ is set of complements of $\mathcal{A} - \mathcal{A}$

$|\mathcal{A} \overset{*}{\cup} \mathcal{B}| = |\mathcal{A} - \mathcal{A}|$.

So by 7.2. $|\mathcal{A}|^2 \leq |\mathcal{A} - \mathcal{A}|^2$

Proof: of 7.1: Let $P = \mathcal{P}([n])$

It suffices to show that

$$\boxed{\alpha(P)\,\beta(P) \le \gamma(P)\,\delta(P)} \quad (*)$$

by replacing each ~~$\varphi \in \{\alpha, \beta\}$~~ γ,

α with α' s.t. $\alpha'(A) = 0$ for $A \notin \mathcal{A}$

$\alpha'(A) = \alpha(A)$ for $A \in \mathcal{A}$

and replacing β, γ, δ similarly.

– We prove $(*)$ by induction on $[n]$.

The base case $n = 1$ contains all the technical work.

$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$. Let $\varphi(\emptyset) = \varphi_0$, $\varphi(\{1\}) = \varphi_1$ for $\varphi \in \{\alpha, \beta, \gamma, \delta\}$.

Given $\alpha_0\beta_0 \le \boxed{\gamma_0}\delta_0$, $\alpha_1\beta_0 \le \gamma_1\delta_0$, $\alpha_0\beta_1 \le \gamma_1\delta_0$, $\alpha_1\beta_1 \le \gamma_1\boxed{\delta_1}$.

need to show $(\alpha_0+\alpha_1)(\beta_0+\beta_1) \le (\gamma_0+\gamma_1)(\delta_0+\delta_1)$. $(**)$

If $\gamma_1 = 0$ or $\delta_0 = 0$ inequality is easy

otherwise $\gamma_0 \ge \alpha_0\beta_0/\delta_0$ $\quad \delta_1 \ge \dfrac{\alpha_1\beta_1}{\gamma_1}$

Substituting and multiplying by $\delta_0\gamma_1$ in $(**)$

we get

$$(\alpha_0\beta_1 + \gamma_1\delta_0)(\delta_0\gamma_1 + \alpha_1\beta_1) \ge (\alpha_0+\alpha_1)(\beta_0+\beta_1)\,\gamma_1\delta_0.$$

Expanding & rearanging we get

$$[(\gamma_1\delta_0 - \alpha_0\beta_1)(\gamma_1\delta_0 - \alpha_1\beta_0) \geq 0.$$ which is true. ✓

<u>Induction step:</u> If (*) holds for $\mathcal{P}([n-1])$ we want to show it for $\boxed{\mathcal{P}([n])}$.

~~Let~~ For $X \subseteq [n-1]$ Let

$$\varphi_*(X) = \varphi(X) + \varphi(X \cup \{n\})$$

For $\varphi \in \{\alpha, \beta, \gamma, \delta.\}$

(*) is equivalent to $\alpha_*(P')\beta_*(P') \leq \gamma_*(P')\delta_*(P')$

where $P' = \mathcal{P}([n-1])$

It suffices to show by ind hypothesis that

$$\underbrace{\alpha_*(A)\beta_*(B) \leq \gamma_*(A \cup B)\delta_*(A \cap B)}$$

this is true by the base case for appropriate choice of functions.

<u>Exercise</u>: Deduce that if $\mathcal{A}, \mathcal{B}$ are monotone increasing graph properties

then $\mathbb{P}(G(n,\tfrac{1}{2}) \in \mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(G(n,\tfrac{1}{2}) \in \mathcal{A}) \cdot \mathbb{P}(G(n,\tfrac{1}{2}) \in \mathcal{B})$.

<u>Proof</u>: $\alpha, \beta, \gamma, \delta : \mathcal{P}(\{\binom{n}{2}\}) \to \mathbb{R}_+$

equivalently $\alpha, \beta, \gamma, \delta$ are functions on n vertex graphs.

$$\alpha(G) = \begin{cases} 1, & G \in \mathcal{A} \\ 0, & G \notin \mathcal{A} \end{cases}, \quad \beta(G) = \begin{cases} 1, & G \in \mathcal{B} \\ 0, & \text{otherwise} \end{cases},$$

$$\gamma(G) = \begin{cases} 1, & G \in \mathcal{A} \cap \mathcal{B} \\ 0, & \text{otherwise} \end{cases}, \quad \delta(G) \equiv 1.$$

$$\alpha(G_1)\beta(G_2) \leq \gamma(G_1 \cup G_2) \checkmark$$

By <u>7.1.</u> $\quad |\mathcal{A}||\mathcal{B}| \leq |\mathcal{A} \cap \mathcal{B}| \cdot 2^{\binom{n}{2}}$

Dividing by $\left(2^{\binom{n}{2}}\right)^2$ we get the result.

Let μ be a probability distribution on $\mathcal{P}([n])$.

μ is log super modular if

$$\mu(A)\mu(B) \leq \mu(A \cup B)\mu(A \cap B)$$

for any $A, B \subseteq [n]$.

Theorem 7.2: (The FKG inequality) Fortuin, Kasteleyn, Ginibre, 1971

If μ is log-supermodular ~~func~~ prob. distribution on $\mathcal{P}([n])$ and $f, g : \mathcal{P}([n]) \to \mathbb{R}_+$ increasing.

($f(A) \leq f(B)$ whenever $A \subseteq B$)

then $\mathbb{E}_\mu[fg] \geq \mathbb{E}_\mu[f]\,\mathbb{E}_\mu[g]$.

Proof: We apply 7.1 to

$\cancel{\alpha(A)\ f(A)\ \mu(A)}$

$\cancel{\beta(A) = f}$

$$\alpha(X) = f(X)\mu(X)$$
$$\beta(X) = g(X)\mu(X)$$
$$\gamma(X) = f(X)g(X)\mu(X)$$
$$\delta(X) = \mu(X).$$

By 7.1. it suffices to check

$$\boxed{f(A)}\,\boxed{\mu(A)} \cdot \boxed{g(B)}\,\boxed{\mu(B)} \leq \boxed{f(A \cup B)}\,\boxed{g(A \cup B)}\,\boxed{\mu(A \cup B)\,\mu(A \cap B)}.$$

Theorem 7.3: Let $p \in (0,1)$, and let $\mathcal{A}, \mathcal{B}$ be graph properties then

1. If $\mathcal{A}$ and $\mathcal{B}$ are increasing then
$$\mathbb{P}(G(n,p) \in \mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(G(n,p) \in \mathcal{A})\, \mathbb{P}(G(n,p) \in \mathcal{B}).$$

2. If $\mathcal{A}$ is decreasing, $\mathcal{B}$ is increasing then
$$\text{——}\,\|\,\text{——} \quad \leq \quad \text{——}\,\|\,\text{——}$$

3. If $\mathcal{A}$ and $\mathcal{B}$ are both decreasing
$$\text{——}\,\|\,\text{——} \quad \geq \quad \text{——}\,\|\,\text{——}.$$

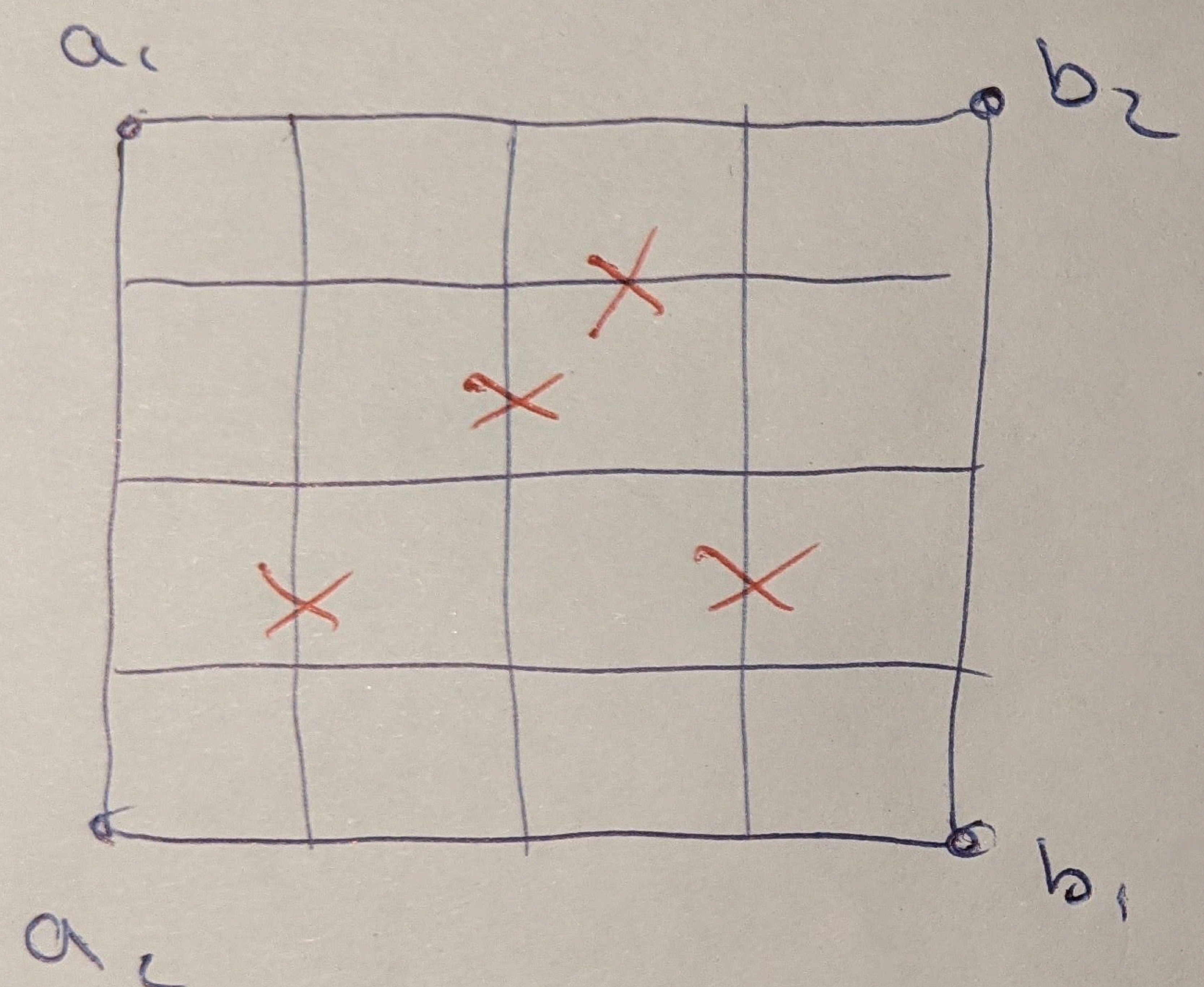
Proof of 1: Let $f(X) = \begin{cases} 1, & X \in \mathcal{A} \\ 0, & X \notin \mathcal{A} \end{cases}$ $\quad g(X) = \begin{cases} 1, & X \in \mathcal{B} \\ 0, & X \notin \mathcal{B} \end{cases}$

By 7.2. it suffices that
$$\mu(X) = p^{|X|} \cdot (1-p)^{n-|X|}$$ is log super modular.

which is easy to check.

To prove 2 & 3 consider complementary properties to $\mathcal{A}$ (and $\mathcal{A}$ & $\mathcal{B}$) respectively.

<u>Percolation</u> Delete edges of a lattice with some probability independently at random.



Physicists care about existence of infinite component containg 0.

FKG implies that

$\mathbb{P}$(a_1 and b_1 are in the same component & a_2 and b_2 are in the same component)

$\geq \mathbb{P}$(a_1 and b_1 —"—) $\cdot$ $\mathbb{P}$(a_2 and b_2 —"—)