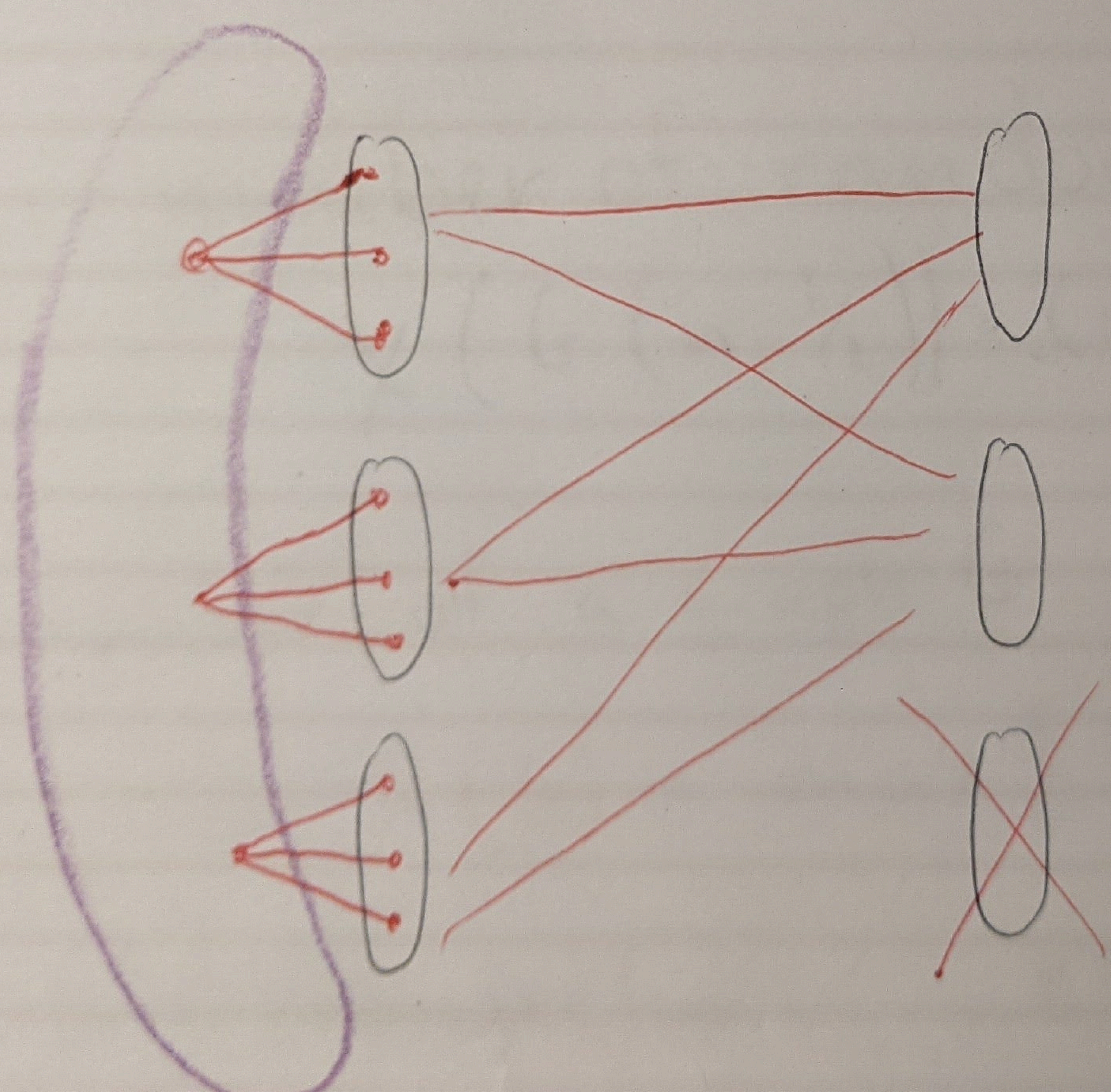
Lecture 10:

Graph minors and probabilistic method



A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

- $\boxed{\text{Hadwiger's conjecture}}$:

If G is a graph with no K_{t+1} minor then G is t-colorable ($\chi(G) \leq t$).

- Chromatic number of random graphs:

$$\chi(G(n, \tfrac{1}{2})) = (1+o(1)) \frac{n}{2\log_2 n} \quad \text{w.h.p.}$$

Hadwiger number of a graph G

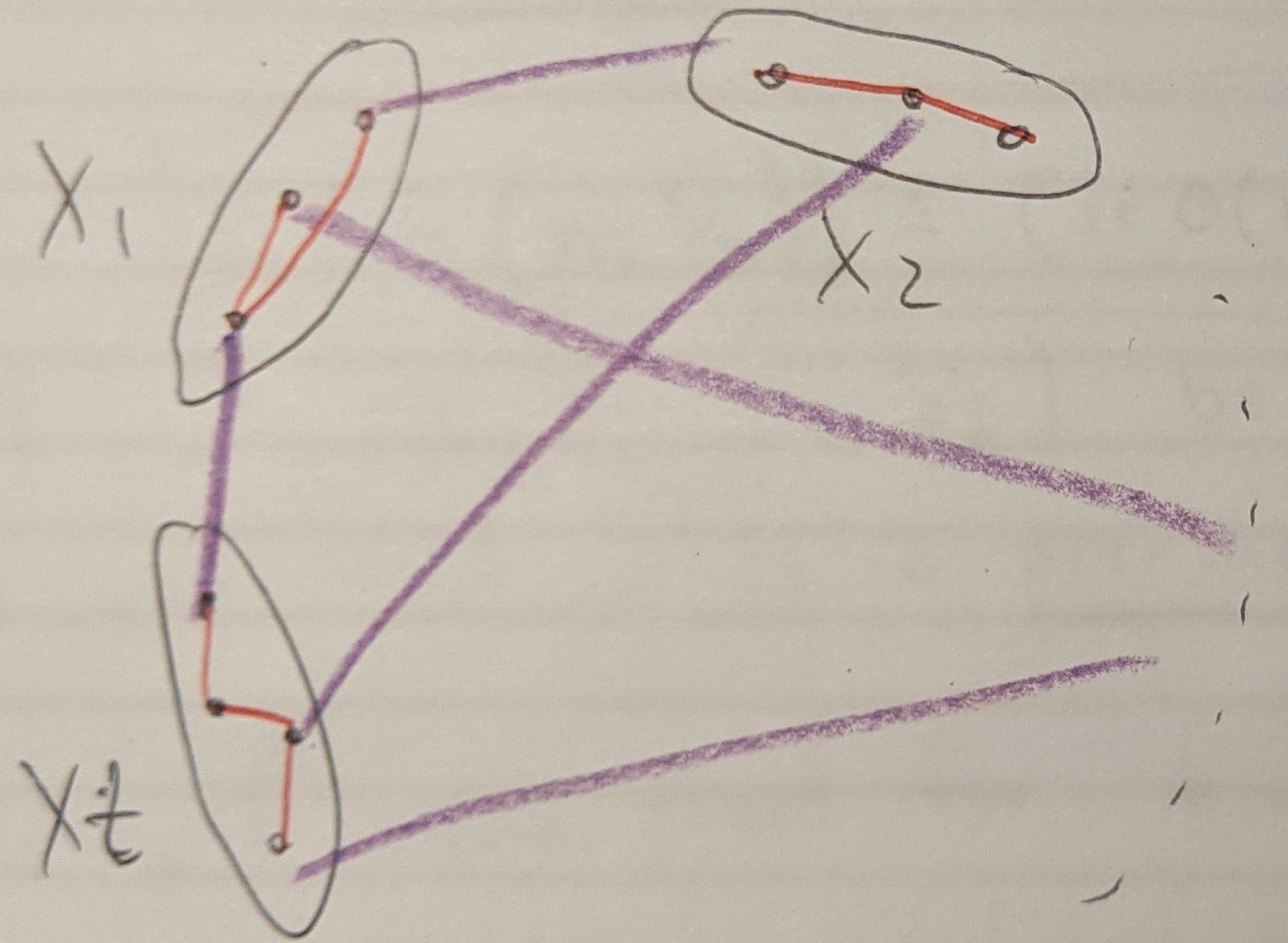
$$h(G) = \max\{ t : K_t \text{ is a minor of } G \}$$

(Hadwiger's conjecture: $h(G) \geq \chi(G)$).

Goal: Determine $h(G(n, \tfrac{1}{2}))$.

A model of K_t in a graph G is a collection $(X_1, X_2, \dots, X_t)$ of subsets of $V(G)$, pairwise disjoint s.t.

- the subgraph of G induced by X_i is connected for every i
- For all $1 \le i < j \le t$ there an edge of G with one end in X_i and another in X_j



Observation:
A graph G contains a model of K_t if and only if K_t is a minor of G.

Proof: Contracting edges within each X_i we can contract it to a single vertex, the edges between X_i's yield a complete graph. Therefore, model $\Rightarrow$ minor.
In the other direction, X_i's will correspond to sets contracted

__Theorem 5.5:__ With high probability

(Bollobás, Catlin, Erdős) 1980

$$h(G(n, 1/2)) = (1+o(1)) \frac{n}{\sqrt{\log_2 n}}$$

(Hadwiger's conjecture "comfortably" holds for almost every graph.

__Comment:__ ~~We can~~ $h(G(n,p)) \approx \frac{n}{\sqrt{\log_{1/p} n}}$. for fixed p.

__Proof:__ $\boxed{h(G(n, 1/2)) \leq (1+o(1)) \frac{n}{\sqrt{\log_2 n}}}$

Let $\mathcal{P} = (X_1, X_2, \ldots, X_t)$ be a partition of $[n] = V(G(n, 1/2))$.

We say that $\mathcal{P}$ is a __pre-model__ if for all $i<j$ there is an edge from X_i to X_j

We will show that for $t = (1+o(1)) \frac{n}{\sqrt{\log_2 n}}$ w.h.p. there is no premodel of size t in $G(n, 1/2)$ (for the right choice of $o(1)$ term)

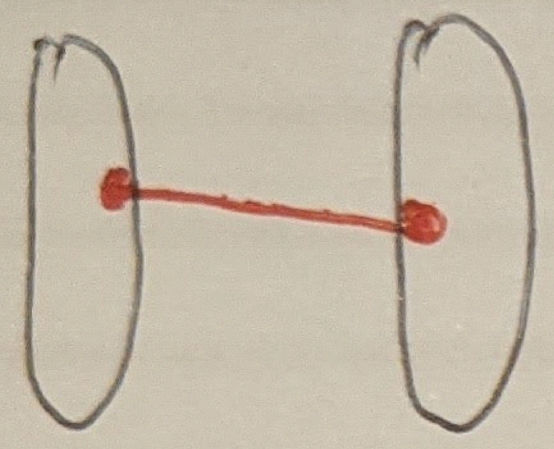
$$\mathbb{P}(\rho \text{ is a pre-model}) = \prod_{i<j}\left(1 - 2^{-n_i n_j}\right) \le \prod_{i<j} e^{-2^{-n_i n_j}} = \exp\Big(-\underbrace{\sum_{i<j} 2^{-n_i n_j}}\Big) \quad \cancel{\le \exp\left(2^{-n^2/t^2}\right)}$$

$$\boxed{\exp\left(\binom{t}{2} 2^{-n^2/t^2}\right)}$$

$\rho = (\underbrace{X_1, X_2, \dots, X_t})$

$|X_i| = n_i$.

$\boxed{1 - x \le e^{-x}}$



$n_1 + n_2 + \dots + n_t = n$.

$n_i \quad n_j$

$X_i \quad X_j$

(?) probability there is an edge from X_i to X_j

$$\frac{\sum_{i<j} 2^{-n_i n_j}}{\boxed{\binom{t}{2}}} \ge \left(\prod_{i<j} 2^{-n_i n_j}\right)^{\frac{1}{\binom{t}{2}}}$$

$\boxed{\text{AM-GM inequality}}$

$$= 2^{-\sum_{i<j} \frac{n_i n_j}{\binom{t}{2}}} \ge 2^{-\frac{\frac{t-1}{2t} n^2}{\frac{t(t-1)}{2}}} = \boxed{2^{-\frac{n^2}{t^2}}}$$

$$2\sum_{i<j} n_i n_j = \left(\sum n_i\right)^2 - \underbrace{\sum n_i^2}_{\ge\, t\left(\frac{n}{t}\right)^2} \le \frac{t-1}{t} n^2$$

$$\left(\sum n_i\right)^2 = n^2$$

$$\mathbb{P}(\text{there exists a pre-model of size } t) \le t^n e^{-2^{\underbrace{(-n^2/t^2)}} \cdot \binom{t}{2}} = o(1)$$

$$e^{\underbrace{2^{-n^2/t^2} \cdot \binom{t}{2}}}$$

$$t^n \ll e^{2}$$

$$\cancel{n}\log t \ll \frac{t^2}{2}\, 2^{-n^2/t^2}$$

$$\underbrace{n = t\left(\sqrt{\log_2 t} - \gamma\right)}$$

$$n^2/t^2 \lesssim \log_2 t - \gamma\sqrt{\log_2 t}$$

$$\log n \sim \log t$$

$$\cancel{n}\log t \ll \frac{t^2}{2}\, 2^{\underbrace{-\log_2 t} + \gamma\sqrt{\log_2 t}}$$

$$\cancel{n}\log t \ll \frac{\cancel{t}}{2}\, 2^{\gamma\sqrt{\log_2 t}}$$

$$\underbrace{(\log t)^{3/2} \ll 2^{\gamma\sqrt{\log_2 t}}}$$

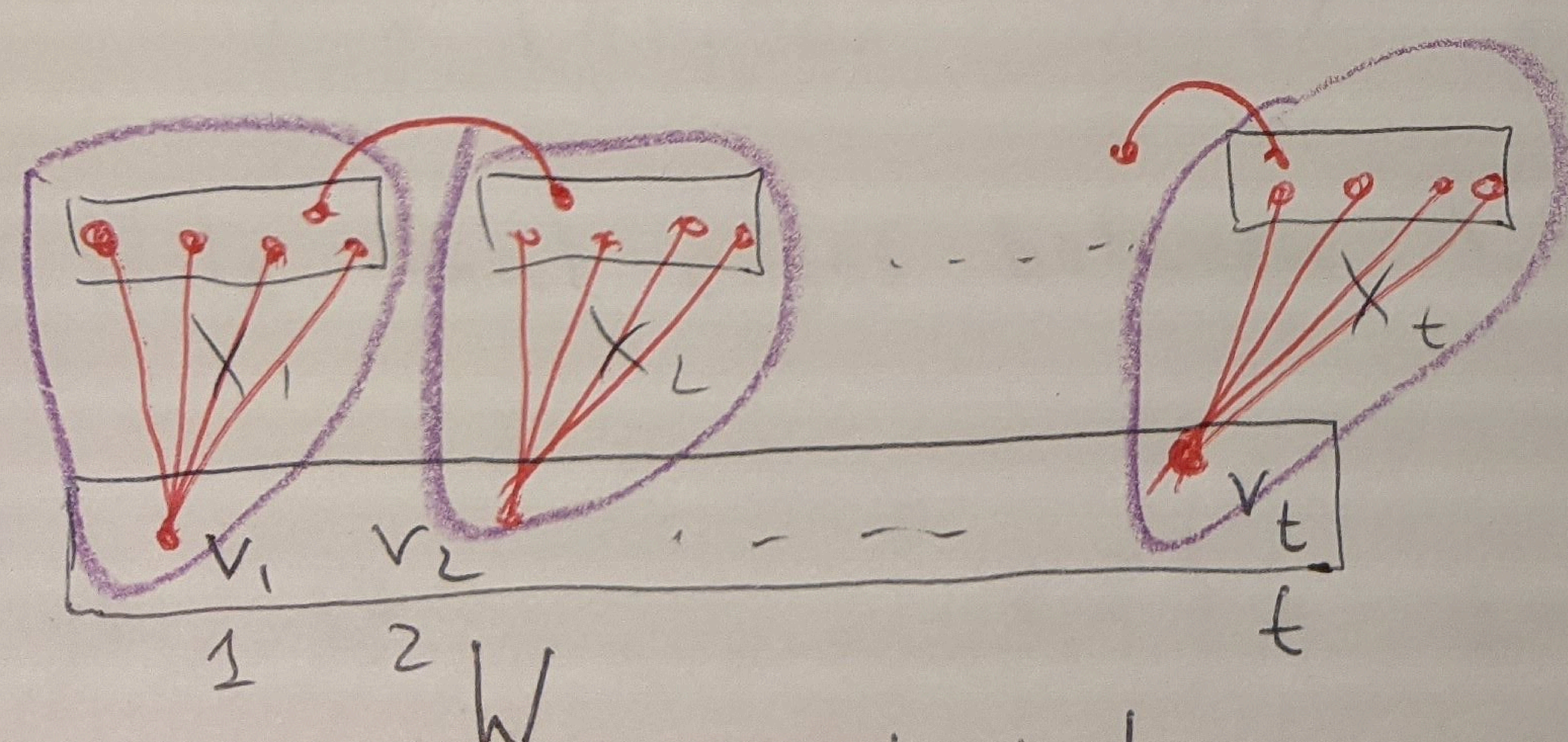
$\gamma = 1$ already suffices.

The other direction:

$$h(G(n, 1/2)) \geq \underbrace{(1-o(1))\frac{n}{\sqrt{\log_2 n}}}$$

$$\boxed{\text{We prove } \geq (1-o(1))\frac{n}{\sqrt{2\log_2 n}}}$$

For every $\varepsilon > 0$ if $\underbrace{t < (1-\varepsilon)\frac{n}{\sqrt{\log_2 n}}}$ and n is large enough then there is a model of K_t in $G(n, 1/2)$ w.h.p.



$|W| = t$

Let $W = [t] \subseteq V(G(n, 1/2))$.

Let $\cancel{k = \sqrt{\log_2 n} + \gamma}$.

$$\underbrace{k = \sqrt{2\log_2 n} + \gamma}$$

W.h.p. there exist $X_1, \dots, X_t \subseteq \underbrace{[n] - W}$ pairwise disjoint
$|X_i| = k$ s.t. every vertex of X_i
is adjacent to $V_i = \{i\}$.

<u>Sketch</u>: We select $X_1, X_2, \dots, X_t$ one by one.
~~After~~ At any step at least $\boxed{\frac{\varepsilon}{2} n}$ vertices
are not selected yet, and the previous selection is independent on edges from current vertex $\boxed{V_i}$.
By Chernoff bound with probability exponentially close to 1 V_i will have $\geq k$ neighbors not yet selected, which will give X_i.

It now suffices to show that with high probability
$P = (\underbrace{X_1}, \underbrace{X_2}, \dots, X_t)$ Form a premodel of K_t.

Note that edges between X_i's are selected independently of selection of X_i

$\mathbb{P}(P \text{ is a premodel}) \geq 1 - \underbrace{\binom{t}{2}} \mathbb{P}[X_i \text{ and } X_j \text{ are not joined by an edge}]$
$= 1 - \binom{t}{2} 2^{-k^2}$.

We need $\binom{t}{2} 2^{-k^2} = o(1) \to$ this doesn't work

$$\binom{t}{2} 2^{-k^2} \leq \underbrace{\frac{t^2}{2} 2^{\overbrace{2\log_2 n - \gamma\sqrt{\log_2 n}}}} = \frac{t^2}{2n^2} 2^{-\gamma\sqrt{\log_2 n}} = o(1).$$

To get $h(G(n,\tfrac{1}{2})) \geq (1-o(1)) \frac{n}{\sqrt{\log_2 n}}$

By the previous calculation the number of edges missing from the premodel is $o(t)$, so by removing $o(t)$ vertices we get a ~~pre~~model of $K_{t-o(t)}$.

By 5.5. there exist graphs with no K_t minor and average degree $\sim \dfrac{t\sqrt{\log_2 t}}{2}$.

One can do slightly better by considering $G(n,p)$ for $p \neq 1/2$ and optimizing.

getting average degree $\boxed{\dfrac{1-e^{-\alpha}}{\sqrt{\alpha}}\, t\sqrt{\log t}}$ for $p = 1-e^{-\alpha}$

Let $\lambda = \max\limits_{\alpha}\left(\dfrac{1-e^{-\alpha}}{\sqrt{\alpha}}\right)$

<u>Theorem 5.6:</u> (Thomason, 2001) Every graph with average degree $\geq (\lambda + o(1))\, t\sqrt{\log t}$ contains a K_t minor.

(This bound is optimum by the above).

<u>Corollary:</u> If G has no K_t minor then

$$\chi(G) \leq (\boxed{\lambda} + o(1))\, t\sqrt{\log t}$$

Hadwiger's conjecture holds with t replaced by