Problem Seminar.
Probability

Classical results.

1. **Monty Hall problem.** Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 (but the door is not opened), and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

2. What is the probability that three randomly chosen points on a circle form an acute triangle?

3. If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?

Problems.

1. **Putnam 1968. B1.** The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:

   (i) $P(x^\circ = 70^\circ)$, the probability that the temperature in Chicago is 70°,

   (ii) $P(y^\circ = 70^\circ)$, and

   (iii) $P(\max(x^\circ, y^\circ) = 70^\circ)$.

   Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

2. **Putnam 1961. B2.** Two points are selected independently and at random from a segment length α. What is the probability that they are at least distance $\beta (< \alpha)$ apart?

3. **Putnam 1958. A3.** Real numbers are chosen at random from the interval $[0, 1]$. Suppose that after choosing the n-th number the sum of the numbers so chosen first exceeds 1. Show that the expected value of n is e.

4. **Putnam 2014. A4.** Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

5. **Putnam 2002. B4.** An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you *must* guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

6. **Putnam 2017. A5.** Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order $A, B, C, A, \ldots$ choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

   Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

7. **Putnam 1999. A6.** Suppose that each of n people writes down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

8. **Putnam 2011. A6.** Let G be an abelian group with n elements, and let

$$\{g_1 = e, g_2, \ldots, g_k\} \subsetneq G$$

   be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, ..., g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$. Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m \to \infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\text{Prob}(g = x) - \frac{1}{n} \right)^2$$

   is positive and finite.