Problem Solving Seminar. Fall 2019.

Problem Set 2. Number theory.

Classical results.

1. **Euler.** For a positive integer n and any integer a relatively prime to n one has
$$a^{\phi(n)} \equiv 1 \pmod{n},$$
where $\phi(n)$ is the number of positive integers between 1 and n relatively prime to n.

2. **Polignac's formula.** If p is a prime number and n a positive integer, then the exponent of p in $n!$ is
$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots.$$

3. **Wilson.**
$$(p-1)! \equiv -1 (\operatorname{mod} p)$$
for any prime p.

4. **Chinese Remainder theorem.** Let $m_1, m_2, \dots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \dots, a_k$ the system of congruences
$$\begin{aligned} x &\equiv a_1 \quad (\operatorname{mod} m_1), \\ x &\equiv a_2 \quad (\operatorname{mod} m_2), \\ &\dots \\ x &\equiv a_k \quad (\operatorname{mod} m_k). \end{aligned}$$
has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \dots m_k$.

Problems.

1. Prove that $n!$ is not divisible by 2^n for any positive integer n.

2. Prove that for every n, there exist n consecutive integers each of which is divisible by two different primes.

3. **Putnam 1956. A2.** Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

4. **Put 1993. B1.** Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which
$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

5. **Putnam 2000. A2.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

6. **Putnam 2017. B2.** Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

7. **IMO 2011.** Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

8. **Put 1996. A6.** The sequence a_n is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

Show that, for all n, a_n is an integer multiple of n.