# Problem Solving Seminar. Fall 2019.
# Problem Set 8. Miscellaneous I

1. **Putnam 2014. A1.** Prove that every nonzero coefficient of the Taylor series of
$$(1 - x + x^2)e^x$$
about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

2. **Putnam 2014. A2.** Let $A$ be the $n \times n$ matrix whose entry in the $i$-th row and $j$-th column is
$$\frac{1}{\min(i, j)}$$
for $1 \leq i, j \leq n$. Compute $\det(A)$.

3. **Putnam 2014. A3.** Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute
$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$
in closed form.

4. **Putnam 2009. A1.** Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

5. **Putnam 2009. A2.** Functions $f, g, h$ are differentiable on some open interval around 0 and satisfy the equations and initial conditions
$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$
Find an explicit formula for $f(x)$, valid in some open interval around 0.

6. **Putnam 2009. A3.** Let $d_n$ be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,
$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$
The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n \to \infty} d_n$.

7. **Putnam 2009. B1.** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

8. **Putnam 2009. B2.** A game involves jumping to the right on the real number line. If $a$ and $b$ are real numbers and $b > a$, the cost of jumping from $a$ to $b$ is $b^3 - ab^2$. For what real numbers $c$ can one travel from 0 to 1 in a finite number of jumps with total cost exactly $c$?

9. **Putnam 2009. B3.** Call a subset $S$ of $\{1, 2, \ldots, n\}$ *mediocre* if it has the following property: Whenever $a$ and $b$ are elements of $S$ whose average is an integer, that average is also an element of $S$. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots, n\}$. [For instance, every subset of $\{1, 2, 3\}$ except $\{1, 3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers $n$ such that $A(n + 2) - 2A(n + 1) + A(n) = 1$.