Problem Solving Seminar. Fall 2019.

Problem Set 4. Inequalities.

Classical results.

1. **AM-GM.** For any non-negative real numbers $x_1, x_2, \dots, x_n$,
$$\sqrt[n]{x_1 x_2 \dots x_n} \le \frac{x_1 + x_2 + \dots + x_n}{n}.$$
2. **Cauchy-Schwarz.** For any real $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$,
$$(x_1y_1 + x_2y_2 + \dots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \dots x_n^2)(y_1^2 + y_2^2 + \dots + y_n^2).$$
3. **Arithmetic-harmonic mean.** For any non-negative real numbers $x_1, x_2, \dots, x_n$,
$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} \le \frac{x_1 + x_2 + \dots + x_n}{n}.$$
4. **Jensen.** For any convex function f and any real $x_1, x_2, \dots, x_n$,
$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \le \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}.$$

Problems.

1. **Putnam 2003. A2.** Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be nonnegative real numbers. Show that
$$(a_1a_2 \cdots a_n)^{1/n} + (b_1b_2 \cdots b_n)^{1/n} \le [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$
2. **Putnam 2004. B2.** Let m and n be positive integers. Show that
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$
3. **Putnam 2014. B2.** Suppose that f is a function on the interval $[1, 3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?
4. **Putnam 2002. B3.** Show that, for all integers $n > 1$,
$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$
5. **USA 1997.** A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.
6. **Putnam 2003. A4.** Let a, b, c, A, B, C be real, a, A non-zero such that $|ax^2 + bx + c| \le |Ax^2 + Bx + C|$ for all real x. Show that $|b^2 - 4ac| \le |B^2 - 4AC|$.

7. **Putnam 2013. B4.** For any continuous real-valued function f defined on the interval $[0, 1]$, let

$$\mu(f) = \int_0^1 f(x)\,dx,\ \mathrm{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\,dx,$$
$$M(f) = \max_{0 \le x \le 1} |f(x)|\,.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0, 1]$, then

$$\mathrm{Var}(fg) \le 2\mathrm{Var}(f)M(g)^2 + 2\mathrm{Var}(g)M(f)^2.$$

8. **Put 2003. B6.** Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\,dx\,dy \ge \int_0^1 |f(x)|\,dx$$

for any continuous real-valued function f on $[0, 1]$.