Problem Seminar.

Geometry.

Classical results.

1. **Triangle area.** Let ABC be a triangle with side lengths $a = BC$, $b = CA$, and $c = AB$, and let r be its inradius and R be its circumradius. Let $s = (a+b+c)/2$ be its semiperimeter. Then its area is

$$sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = \frac{1}{2}ab\sin C.$$

2. Every polygon (not necessarily convex) has a triangulation.

3. **Art Gallery.** The floor plan of a single- floor art gallery can be considered as a (not necessarily convex) polygon with n vertices. Prove that it is always possible to position $\lfloor \frac{n}{3} \rfloor$ such that every point inside the gallery has a line-of-sight connection to some guard.

4. **Pick.** The area of any polygon with integer vertex coordinates is exactly $I + B/2 - 1$, where I is the number of lattice points in its interior, and B is the number of lattice points on its boundary.

Problems.

1. **Putnam 1946. B1.** Two circles C_1 and C_2 intersect at points A and B. The circle C_1 has radius 1. L denotes the arc AB of C_2 which lies inside C_1. L divides C_1 into two parts of equal area. Show L has length greater than 2.

2. **Putnam 1998. A1.** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

3. **Putnam 2008. B1.** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

4. **Putnam 1955. A2.** O is the center of a regular n-gon $P_1P_2 \ldots P_n$ and X is a point outside the n-gon on the line OP_1. Show that $|XP_1| \cdot |XP_2| \cdot \ldots \cdot |XP_n| + |OP_1|^n = |OX|^n$.

5. **Putnam 1957. A5.** Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

6. **Putnam 2017 . B5.** A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

7. **Putnam 2000. A5.** Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

8. **Putnam 1992. A6.** Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points?