Problem Solving Seminar. Fall 2019.

Problem Set 6. Combinatorics.

Classical results.

1. Show that the equation
$$x_1 + x_2 + \ldots + x_r = n$$
has exactly $\binom{n+r-1}{r-1}$ non-negative integer solutions.

2. Consider a convex polygon with n vertices so that no 3 diagonals go through the same point.

   (a) How many intersection points do the diagonals have?

   (b) Into how many regions do the diagonals divide the interior of the polygon?

3. **Catalan numbers.** Find a closed-form expression for the number of valid sequences containing n pairs of parentheses. For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

4. **Erdős-Ko-Rado.** Let $\mathcal{F}$ be a family of k element subsets of an n element set, with $n \geq 2k$, such that every two sets in $\mathcal{F}$ have a non-empty intersection. Then
$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Problems.

1. **Putnam 1964. B2.** Let S be a finite set, and suppose that a collection $\mathcal{F}$ of subsets of S has the property that any two members of $\mathcal{F}$ have at least one element in common, but $\mathcal{F}$ cannot be extended (while keeping this property). Prove that $\mathcal{F}$ contains exactly half of the subsets of S.

2. **Putnam 1992. B1.** Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

3. **Putnam 1996. B1.** Define a *selfish set* to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

4. **Putnam 2001. B1.** Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is
$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

5. **Putnam 1993. A3.** Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that
$$c(n, m) = \sum_{j=1}^{m} j^n$$

6. **Putnam 1996. A3.** Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

7. **Putnam 1997. A5.** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

8. **Putnam 1996. B5.** We call a finite string of the symbols X and O *balanced* if every substring of consecutive symbols has a difference of at most 2 between the number of X's and the number of O's. For example, $XOOXOOX$ is not balanced, because the substring $OOXOO$ has a difference of 3. Find the number of balanced strings of length n.

9. **Putnam 2018. B6.** Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most
$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$