Problem Seminar.
Combinatorics.

Classical results.

1. Consider a convex polygon with n vertices so that no 3 diagonals go through the same point.

   (a) How many intersection points do the diagonals have?

   (b) Into how many regions do the diagonals divide the interior of the polygon?

2. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

3. **Erdős-Ko-Rado.** Let $\mathcal{F}$ be a family of k element subsets of an n element set, with $n \geq 2k$, such that every two sets in $\mathcal{F}$ have a non-empty intersection. Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Problems.

1. **Putnam 1954. A2.** Given any five points in the interior of a square with side length 1, show that two of the points are a distance apart less than $k = 1/\sqrt{2}$. Is this result true for a smaller k?

2. **Putnam 2003. A1.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

3. **Putnam 1996. B1.** Define a *selfish set* to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

4. **Putnam 2001. B1.** Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

   Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

5. **Putnam 1993. A3.** Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n$$

6. **IMO 1978.** An international society has members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there exists at least one member whose number is the sum of the numbers of two members from his/her own country, or twice as large as the number of one member from his/her country.

7. **Putnam 1955. B5.** Let n be a positive integer. Suppose we have an infinite sequence of 0's and 1's is such that it only contains n different blocks of n consecutive terms. Show that it is eventually periodic.

8. **Putnam 2005. B4.** For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

9. **IMO 2014.** Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 squares.