Problem Seminar. Fall 2017.

Problem Set 6. Calculus.

Classical results.

1. Every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.
2. **Mean value theorem.** If $f : [a, b] \to \mathbb{R}$ is a differentiable function then there exists $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
3. **Leibniz formula.**
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$
4. **Gaussian integral.**
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Problems.

1. Compute
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \right).$$
2. **Putnam 1994. A1.** Suppose that a sequence $a_1, a_2, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. **Putnam 2015. B1.** Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.
4. **Putnam 2007. B2.** Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0, 1)$,
$$\left| \int_0^{\alpha} f(x)\, dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$
5. **Putnam 1955. B6.** Let $f : \mathbb{Z}_+ \to \mathbb{R}_+$ be a function which satisfies $\lim_{n\to\infty} f(n) = 0$. Show that there are only finitely many solutions to the equation $f(x) + f(y) + f(z) = 1$.
6. **Putnam 2013. A3.** Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy
$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$
Prove that there exists a real number y with $0 < y < 1$ such that
$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

7. **Putnam 2008. A4.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

8. **Putnam 2010. A6.** Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x \to \infty} f(x) = 0$. Prove that

$$\int_0^{\infty} \frac{f(x) - f(x+1)}{f(x)}\, dx$$

diverges.