MATH 340: Discrete Structures II. Winter 2017.

Assignment #5: Enumeration. Solutions.

1. *Combinatorial identities.*

a) Give an algebraic proof of the following identity:

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m}$$

b) Give a combinatorial (bijective) proof of the identity in a).

Solution:
a): We prove the identity for fixed m by induction on n.
Base case($n = m$): The left and right side both evaluate to 1.
Induction step: ($n - 1 \to n$). By the induction hypothesis we have

$$\binom{n}{m+1} = \sum_{k=m}^{n-1} \binom{k}{m}.$$

Adding $\binom{n}{m}$ to both sides we obtain

$$\binom{n}{m+1} + \binom{n}{m} = \sum_{k=m}^{n} \binom{k}{m}.$$

Finally,

$$\binom{n+1}{m+1} = \binom{n}{m+1} + \binom{n}{m},$$

as shown in class.

b): The binomial coefficient $\binom{n+1}{m+1}$ counts the number of subsets S of $[n+1] = \{1, 2, \ldots, n+1\}$ of size $m+1$. On the other hand, consider subsets S of $[n]$ with $|S| = m+1$, such that the largest element of S is $k+1$, for $m \leq k \leq n$. There are $\binom{k}{m}$ such subsets, as $S - \{k+1\}$ is an arbitrary subset of $\{1, 2, \ldots, k\}$ of size m. As every subset of $[n+1]$ of size

$m+1$ is counted among the subsets with the fixed largest element exactly once we have

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m},$$

as desired.

2. *Labelled trees.*
Let $f : [n] \to [n]$ be a function, and let T_f be a labelled tree on n vertices, constructed from f using the procedure demonstrated in class. Suppose that T_f contains a vertex of degree at least k. Show that f takes at most $n-k+2$ different values,

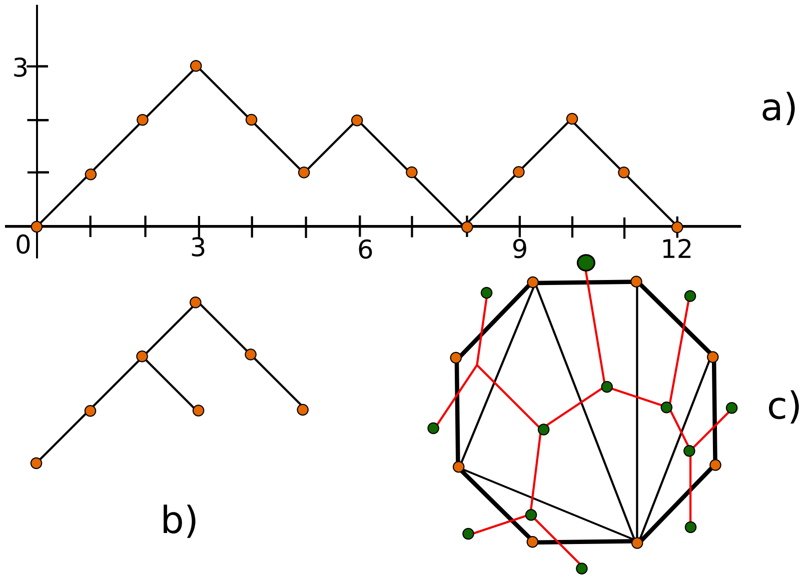
Solution: The map $f \to T_f$ associates to every function $f : [n] \to [n]$ a labeled tree T_f on n vertices with two vertices of T_f marked as red and blue. Let P be the path of T_f joining red and blue vertices. If $ij \in E(T_f)$ then either $ij \in E(P)$, or $i = f(j)$ and $j \notin V(P)$, or $j = f(i)$ and $i \notin V(P)$. Moreover, if $i \in V(P)$ then there exists $j \in V(P)$ such that $i = f(j)$. Let $i \in [n]$ be the vertex of degree at least k. If $i \in V(P)$, then $|f^{-1}(i)| \geq k-1$ by the above, and the same conclusion holds if $i \notin V(P)$. Thus f takes a value different from i for at most $n-k+1$ integers in $[n]$. The conclusion follows.

3. *Catalan numbers. I.*
Give a bijection to show that the following is counted by Catalan numbers. The number of orderings of numbers $\{1, 2, \ldots, 2n\}$, such that

- the numbers $\{1, 3, \ldots, 2n-1\}$ appear in order,
- the numbers $\{2, 4, \ldots, 2n\}$ appear in order,
- $2k-1$ precedes $2k$ for every $1 \leq k \leq n$.

Solution: We will map a permutation satisfying the properties above to a sequence of n pluses and n minuses, such that every initial segment of the sequence contains at least as many pluses as minuses, using the following simple rule:

- In a permutation $\pi_1\pi_2 \ldots \pi_{2n}$ replace π_i by a "+" if π_i is odd, and replace it by a "-" otherwise.



Note that the rule above provides a bijection between the permutations of the numbers $\{1, 2, \dots, 2n\}$, satisfying the first two properties above and *all* sequences of n pluses and n minuses, as the number corresponding to kth plus (or minus) in the sequence mus t be equal to $2k - 1$ (respectively, $2k$). Moreover, a permutation satisfies the third property if and only if every initial segment in the associated sequence of pluses and minuses contains at least as many pluses as minuses. (The third property holds if and only if the kth plus precedes the kth minus in the associated sequence for every k.)

It follows that the map above is a bijection as desired.

4. *Catalan numbers. II.*

Given a sequence $+ + + + \ + +$, construct

a) a Dyck path,

b) a rooted plane tree on 7 vertices,

c) a decomposition of a 8-gon into triangles,

corresponding to this sequence via the bijections shown in class.

Solution:

See the figure above.

5. *Generating functions.* For the following recurrences, find the ordinary generating function $F(x)$ and use it to obtain a closed formula for $f(n)$.

a) $f(n) = 6f(n-1) - 8f(n-2)$ for $n \geq 2$, $f(0) = 3, f(1) = 10$,

b) $f(n) = 4f(n-1) - 4f(n-2)$ for $n \geq 2$, $f(0) = 0, f(1) = 2$.

Solution:

a): $F(x) = \sum_{n=0}^{\infty} f(n)x^n$. We have

$$\begin{aligned}
f(n)x^n &= 6f(n-1)x^n - 8f(n-2)x^n, \\
\sum_{n=2}^{\infty} f(n)x^n &= 6x \sum_{n=2}^{\infty} f(n-1)x^{n-1} - 8x^2 \sum_{n=2}^{\infty} f(n-2)x^{n-2}, \\
F(x) - 3 - 10x &= 6x(F(x) - 3) - 8x^2 F(x), \\
F(x) &= \frac{3-8x}{1-6x+8x^2} = \frac{3-8x}{(1-2x)(1-4x)} = \frac{1}{1-2x} + \frac{2}{1-4x}, \\
F(x) &= \sum_{n=0}^{\infty} (2x)^n + 2\sum_{n=0}^{\infty} (4x)^n, \\
F(x) &= \sum_{n=0}^{\infty} (2^n + 2^{2n+1})x^n, \\
f(n) &= 2^n + 2^{2n+1}.
\end{aligned}$$

b): As in a) we have

$$\begin{aligned}
f(n)x^n &= 4f(n-1)x^n - 4f(n-2)x^n, \\
F(x) - 2x &= 4xF(x) - 4x^2 F(x), \\
F(x) &= \frac{2x}{1-4x+4x^2} = \frac{2x}{(1-2x)^2}, \\
F(x) &= 2x \sum_{n=0}^{\infty} (n+1)(2x)^n = \sum_{n=0}^{\infty} (n+1)2^{n+1}x^{n+1}, \\
f(n) &= n2^n.
\end{aligned}$$