Elementary amenable groups and the space of marked groups

Phillip Wesolek (joint with Jay Williams)

Binghamton University

McGill Seminar on Logic, Category Theory, and Computation

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Remark

If X is well-chosen, set-theoretic properties of X reflect mathematical properties of $\mathcal{X}$ and vice versa.

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Definition (Grigorchuk)

The **space of marked groups** is

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- $\mathscr{G}_\omega$ is a compact Polish space.
- $\mathscr{G}_\omega$ parametrizes the class of countable groups.

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Non-examples: Non-abelian free groups, non-elementary hyperbolic groups, $SL_n(\mathbb{Z})$,...

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Group-theoretic translation

Is there a “nice” characterization of elementary amenable groups?

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- There are sets that are coanalytic but non-Borel. (Souslin)
- Analytic or coanalytic sets require "uncountable information" to define, while Borel sets are definable with "countable information."

Theorem (W.–Williams, 15)

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Corollary (Grigorchuk, 83)

$\mathrm{EG} \subsetneq \mathrm{A}$.

Outline of proof

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1. Via a decomposition procedure, we build a Borel map $\Phi : \mathscr{G}_\omega \to \mathrm{Tr}$.
2. We prove that $G \in \mathrm{EG}$ if and only if $\Phi(G)$ is well-founded.
3. Applying classical results in descriptive set theory and group theory, we deduce that EG is coanalytic and non-Borel.

Decomposition trees

Question

How can we take apart an elementary amenable group “from above?”

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Observation (Chou, Osin)

A non-trivial finitely generated elementary amenable group has a non-trivial finite or abelian quotient.

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4. Repeat.

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Note

A marked group comes with a preferred enumeration.

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Definition

$T^k(G)$ is the **decomposition tree** of G with offset k.

The decomposition tree $T^k(G)$

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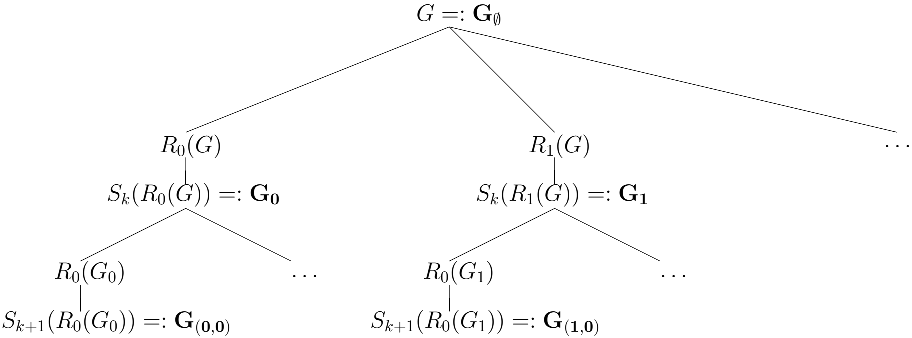

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(2) $\Rightarrow$ (1): Induction on the rank of a decomposition tree. □

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Corollary

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such that $K_n \neq \{e\}$ and $K_{n+1} \leq [K_n, K_n] \cap \bigcap \mathcal{N}_n(K_n)$ for all $n \geq 1$.

$\mathcal{EG}$ is coanalytic and non-Borel

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Proposition

EG is coanalytic in $\mathscr{G}_\omega$.

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Proposition

EG is coanalytic in $\mathscr{G}_\omega$.

Proof.

The set $WF \subset \mathrm{Tr}$ is a coanalytic set. □

Intuitive definition

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Intuitive boundedness theorem

If $A \subseteq X$ is Borel, then any coanalytic rank admits a countable bound.

- There is a coanalytic rank $\rho : \mathit{WF} \to \omega_1$.

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Conclusion

To show EG is non-Borel, it suffices to show the least upper bound of $\rho \circ \Phi_k$ is ω_1.

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Remark

Showing $\rho \circ \Phi_k$ is unbounded does not require finding groups in $\mathrm{A} \setminus \mathrm{EG}$.

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Corollary

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Questions

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Question (Cornulier)

What is the Cantor-Bendixson rank of $\mathscr{G}_\omega$ or $\mathscr{G}_2$?

Thank you