

① Category  $\boxed{\text{pFun}}$  ("pseudo-functor") defined:

→ object:  $(\Phi) = (X, A, \Phi)$  :

$X, A$  : small categories;  $\Phi: X \rightarrow A$  functor

→ arrow:  $(\tau, \Sigma, \theta): (X, A, \Phi) \rightarrow (X', A', \Phi')$  :

$$\begin{array}{c} \text{"} \\ \Sigma \end{array} : \begin{array}{c} \text{"} \\ \Phi \end{array} \rightarrow \begin{array}{c} \text{"} \\ \Phi' \end{array}$$

where:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & A \\ \tau \downarrow & \Downarrow \Sigma & \downarrow \theta \\ X' & \xrightarrow{\Phi'} & A' \end{array}$$

$\Sigma: \theta \Phi \xrightarrow{\cong} \Phi' \tau$  natural isomorphism

→ composition:  
(= pasting)

$$\begin{array}{ccccc} \begin{array}{c} \text{"} \\ \Phi \end{array} & \xrightarrow{\begin{array}{c} \text{"} \\ \Sigma \end{array}} & \begin{array}{c} \text{"} \\ \Phi' \end{array} & \xrightarrow{\begin{array}{c} \text{"} \\ \Sigma' \end{array}} & \begin{array}{c} \text{"} \\ \Phi'' \end{array} \\ & \searrow & \text{ } & \nearrow & \\ & \begin{array}{c} \text{"} \\ \Sigma' \end{array} \begin{array}{c} \text{"} \\ \Sigma \end{array} : ? & & & \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & A \\ \tau \downarrow & \Downarrow \Sigma & \downarrow \theta \\ X' & \xrightarrow{\Phi'} & A' \\ \tau' \downarrow & \Downarrow \Sigma' & \downarrow \theta' \\ X'' & \xrightarrow{\Phi''} & A'' \end{array}$$

pasting to:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & A \\ \tau' \tau \downarrow & \Downarrow \Sigma'' & \downarrow \theta' \theta \\ X'' & \xrightarrow{\Phi''} & A'' \end{array}$$

$$(\tau', \Sigma', \theta') \circ (\tau, \Sigma, \theta) \stackrel{\text{def}}{=} (\tau' \tau, \underbrace{(\Sigma' \tau)(\theta' \Sigma)}_{= \Sigma''}, \theta' \theta)$$

$$\begin{array}{ccc} X & \xrightarrow{\theta' \theta \Phi} & \\ \theta' \Phi' \tau \downarrow \theta' \Sigma & & \downarrow \Sigma'' \\ \downarrow \Sigma' \tau & & \downarrow \\ \Phi'' \tau' \tau & \xrightarrow{\quad} & A'' \end{array}$$

Theorem 1. (i)  $\underline{p}\text{Fun}$  is an accessible category.

(ii) More precisely:  $\underline{p}\text{Fun}$  is  $\kappa'_1$ -accessible and has  $(\kappa'_0-)$  filtered colimits.

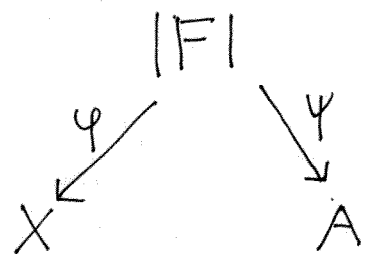
(iii) Even more precisely:  $\underline{p}\text{Fun}$  is dual-regular: for some (countable) regular category  $\mathbb{C}$ ,

$$\underline{p}\text{Fun} \cong \text{Reg}(\mathbb{C}, \text{Set}) \stackrel{\text{def}}{=} \overline{\text{def}}$$

$\overline{\text{def}}$  the category of all regular functors  $\mathbb{C} \rightarrow \text{Set}$ , a full subcategory of  $[\mathbb{C}, \text{Set}]$ .

2 Category Sana defined:

→ object:  $F = (|F|, X, A, \varphi, \psi)$ , a span,  
 where



of small categories and functors  
subject to 1), 2), 3) below:

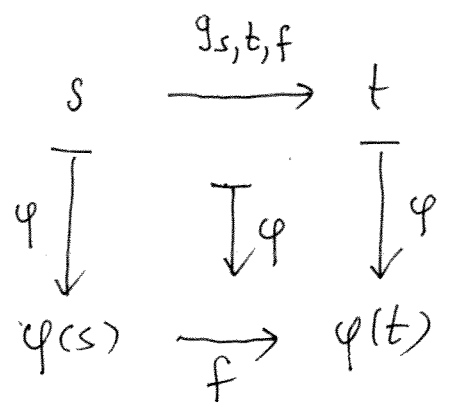
→ 1) Whenever  $s, t \in |F|$  (objects of  $|F|$ ), and  
 $f: \varphi(s) \rightarrow \varphi(t)$  (in  $X$ )

there is unique

$$g: s \rightarrow t \quad (\text{in } |F|)$$

such that  $\varphi(g) = f$

(notation:  $g = g_{s,t,f}$ )



Sana  
 $\rightarrow 2)$

For  $x \in X$  and  $b \in A$  (objects),

I define

$$|F|(x, b) \stackrel{\text{def}}{=} \varphi^{-1}(x) \cap \psi^{-1}(b)$$

$$= \{ t \in |F| : \varphi(t) = x \text{ \& } \psi(t) = b \}$$

and define, for  $s \in |F|$ ,  $b \in A$ , the map

$$F_{s,b} : |F|(\varphi(s), b) \longrightarrow \text{Iso}(\psi(s), b)$$

$$t \longmapsto \psi(g_{s,t}, 1_{\varphi(s)})$$

(see 1)  $\nearrow$

$\rightarrow$  we require (condition 2):

$$F_{s,b} \text{ is a bijection}$$

[  $\text{Iso}(a, b)$  is the set of  
 isomorphisms  $a \xrightarrow{\cong} b$  ]

Sana  
 $\rightarrow 3)$

$\varphi$  is surjective on objects:

for all  $x \in X$ , there is  $s \in |F|$

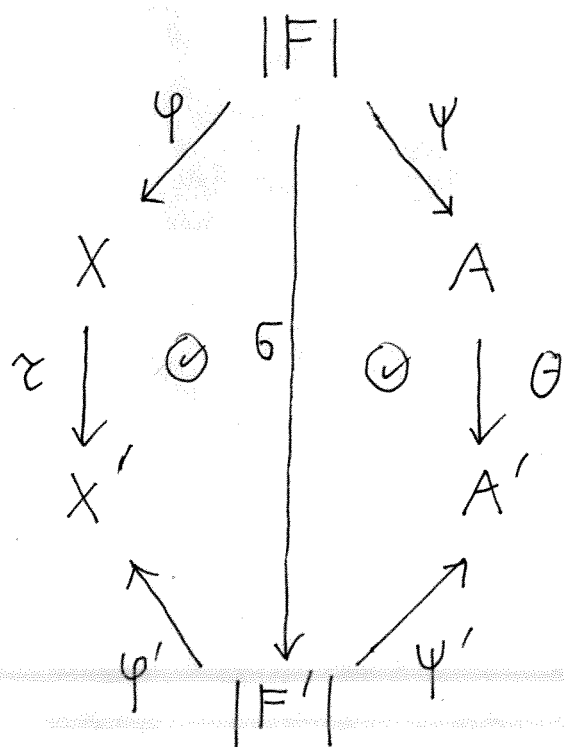
such that  $\varphi(s) = x$ .

("object" of  $|Sana|$  defined)

$\rightarrow$  arrow of  $\boxed{\text{Sana}}$  : defined:

$$\textcircled{\sigma} = (\tau, \sigma, \theta) : F \longrightarrow F'$$

same as a morphism of spans:



$$\tau \varphi = \varphi' \sigma$$

$$\theta \psi = \psi' \sigma$$

$\rightarrow$  composition : component-wise

□ category Sana defined.

Proposition: Sana is dual-regular

see p.  $\boxed{2}$

# Theorem 1. (continued)

(iv)  $\underline{pFun}$  and  $\underline{Sana}$

are equivalent categories.

③

→ On the proof of Theorem 1. (iv):

①⊙ For  $\Phi \in \underline{pFun}$  (object), the  
corresponding  $\Gamma(\Phi) \in \underline{Sana}$ :

$$F = \Gamma(\Phi) = (|\Gamma\Phi|, \varphi, \psi) :$$

$$\text{span: } \begin{array}{ccc} & |\Gamma\Phi| & \\ \varphi \swarrow & & \searrow \psi \\ X & & A \end{array} \quad \text{defined:}$$

→ object of  $|\Gamma\Phi|$ :

$$(S=) (x, a, v)$$

where  $x \in X, a \in A, v \in \text{Iso}(\Phi x, a)$

for object, the effects of  $\varphi$  and  $\psi$ :

$$\varphi((x, a, v)) = x$$

$$\psi((x, a, v)) = a$$

→ arrow of  $|R\Phi|$ :

$$(f, g) : (x, a, v) \longrightarrow (y, b, p)$$

where  $f : x \rightarrow y$  (in  $X$ )

$g : a \rightarrow b$  (in  $A$ )

and

$$a \xrightarrow{g} b$$

$$(*) \quad \begin{array}{ccc} v \uparrow \cong & \theta & \cong \uparrow p \\ \Phi(x) & \xrightarrow{\Phi(f)} & \Phi(y) \end{array} \quad \text{commutes}$$

$$\Phi(x) \xrightarrow{\Phi(f)} \Phi(y)$$

[note: given  $s = (x, a, v)$  and  $t = (y, b, p)$

and  $f : \varphi(s) = x \longrightarrow \varphi(t) = y$ ,

there is unique  $g : \psi(s) = a \longrightarrow \psi(t) = b$

such that  $(*)$  holds;

We define ,

$$\text{for } h = (f, g) : s = (x, a, v) \rightarrow t = (y, b, v)$$

$$\varphi(h) \stackrel{\text{def}}{=} f : \varphi(s) \rightarrow \varphi(t)$$

→ [Condition (1) holds; with

$$g_{s,t,f} = (f, g)$$

with  $g$  from (\*).

$$\begin{array}{ccc} (x, a, v) & \xrightarrow{(f, g)} & (y, b, v) \\ \varphi \downarrow & & \downarrow \varphi \quad \downarrow \varphi \\ x & \xrightarrow{f} & y \end{array}$$

[For condition 2) on objects of  $\mathcal{S}_{ana}$ :

For  $x \in X$ ,  $b \in A$ , we now have:

$$\rightarrow |F|(x, b) \stackrel{\text{now}}{=} \underbrace{\left\{ (x, b, v) : v \in \text{Iso}(\Phi x, b) \right\}}_t$$

$$\varphi(t) = x, \psi(t) = b$$

For  $s = (x, a, \mu)$ ,  $t = (x, b, v)$

$$\rightarrow \text{Iso}(\psi(s), b) \stackrel{\text{now}}{=} \text{Iso}(a, b)$$

and the arrow

$$g_{s,t,1_{\varphi(s)}} : s \longrightarrow t$$

is

$$\longrightarrow g_{s,t,1_{\varphi(s)}} = (1_x, g) :$$

$$\begin{array}{ccc} (x, a, \mu) & \xrightarrow{(1_x, g)} & (x, b, \nu) \\ \varphi \downarrow & & \downarrow \varphi \\ x & \xrightarrow{1_x} & x \end{array}$$

with  $g$  from:

$$\begin{array}{ccc} a & \xrightarrow{g} & b \\ \mu \uparrow \cong & \circlearrowleft & \cong \uparrow \nu \\ \Phi(x) & \xrightarrow{\cong} & \bar{\Phi}(x) \\ & \Phi(1_x) & \\ & = 1_{\bar{\Phi}(x)} & \end{array} \quad (\because g \text{ is } \cong)$$

$$\rightarrow \boxed{\text{and}} \quad \varphi(g_{s,t,1_{\varphi(s)}}) = g$$

Thus, indeed, for any given

$s = (x, a, \mu)$  and  $b \in A$  :

for any  $g \in \text{Iso}(\underbrace{\psi(s)}_a, b)$

there is unique  $t = (x, b, \nu)$  such that

$$t \xrightarrow[F_{s,b}]{} \psi(g_{s,t}, 1_{\psi(s)}) = g$$

→ Re: condition 3) (the "non-algebraic one")  
on objects in Sana:

given  $x \in X$ , we have

$$s = (x, \underline{\Phi}(x), 1_{\underline{\Phi}(x)}) \in |\Gamma \underline{\Phi}|$$

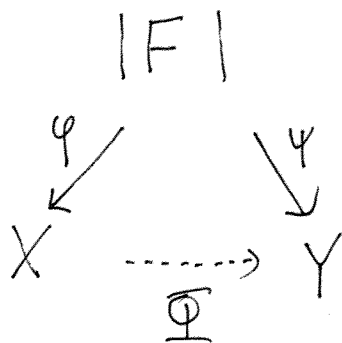
[end of verifying the object effect

of equivalence functor  $\Gamma: \text{pFun} \rightarrow \text{Sana}$ ]

end of ① started on p. 6

(2) For  $F \in \text{Sana}$ , the corresponding  $\Gamma^{\sim 1}(F) \in \text{pFun}$ :

$$F = (|F|, \varphi, \psi):$$



to define  $\Phi = \Gamma^{\sim 1}(F)$ :

Let, for any  $x \in X$ ,  $s_x \in |F|$  be chosen so that  $\varphi(s_x) = x$  (see condition 3)).

Define  $\boxed{\Phi(x) \stackrel{\text{def}}{=} \psi(s_x)}$ .

For  $f: x \rightarrow y$  in  $X$ , consider

$$g_f \stackrel{\text{def}}{=} g_{s_x, s_y, f} : s_x \rightarrow s_y;$$

$$\varphi(g_f) = f.$$

Define

$$\bar{\Phi}(f) \stackrel{\text{def}}{=} \psi(g_f): \bar{\Phi}(x) \longrightarrow \bar{\Phi}(y)$$

The uniqueness part in condition 1) ensures that

$$x \xrightarrow{f} y \xrightarrow{e} z \Rightarrow g_{ef} = g_e g_f$$

and  $g_{1_x} = 1_{s_x}$

$\therefore \bar{\Phi} : X \longrightarrow A$  is a functor.

For the rest of the proof of the equivalence

$$p\text{Fun} \simeq \text{Sana}$$

(the bigger part), we define a new category

④

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Category Sana pFun defined:

→ object:  $(F, \Phi, \mu)$

where  $F \in \text{Sana}$ ,  $\Phi \in \text{pFun}$

and

$$F = (|F|, X, A, \varphi, \psi)$$

$$\Phi ::= \begin{array}{ccc} & |F| & \\ \varphi \swarrow & & \searrow \psi \\ X & & A \\ \parallel & & \parallel \\ X & \xrightarrow[\Phi]{} & A \end{array}$$

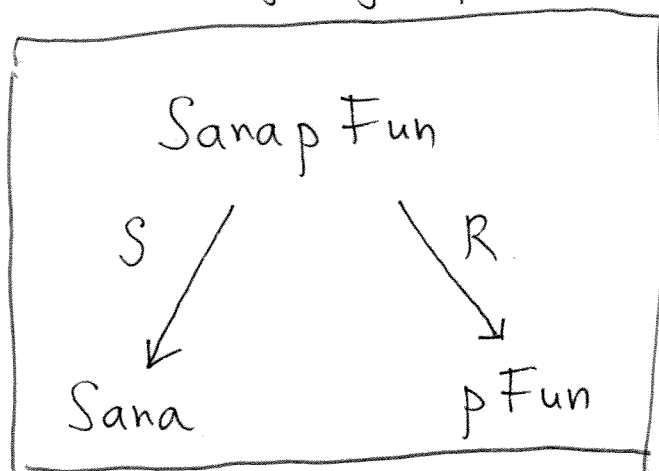
and

$$\boxed{\begin{array}{ccc} & |F| & \\ \varphi \swarrow & & \searrow \psi \\ X & \xrightarrow[\mu]{\Phi} & \Phi \end{array}}$$

$$\mu : \Phi \varphi \xrightarrow{\sim} \psi$$

natural isomorphism

We will have forgetful functors:



of course :  $S((F, \mathbb{Q}, \mu)) \stackrel{\text{def}}{=} F$

$R((F, \mathbb{Q}, \mu)) \stackrel{\text{def}}{=} \overline{\mathbb{Q}}$

→ arrow of SanapFun :

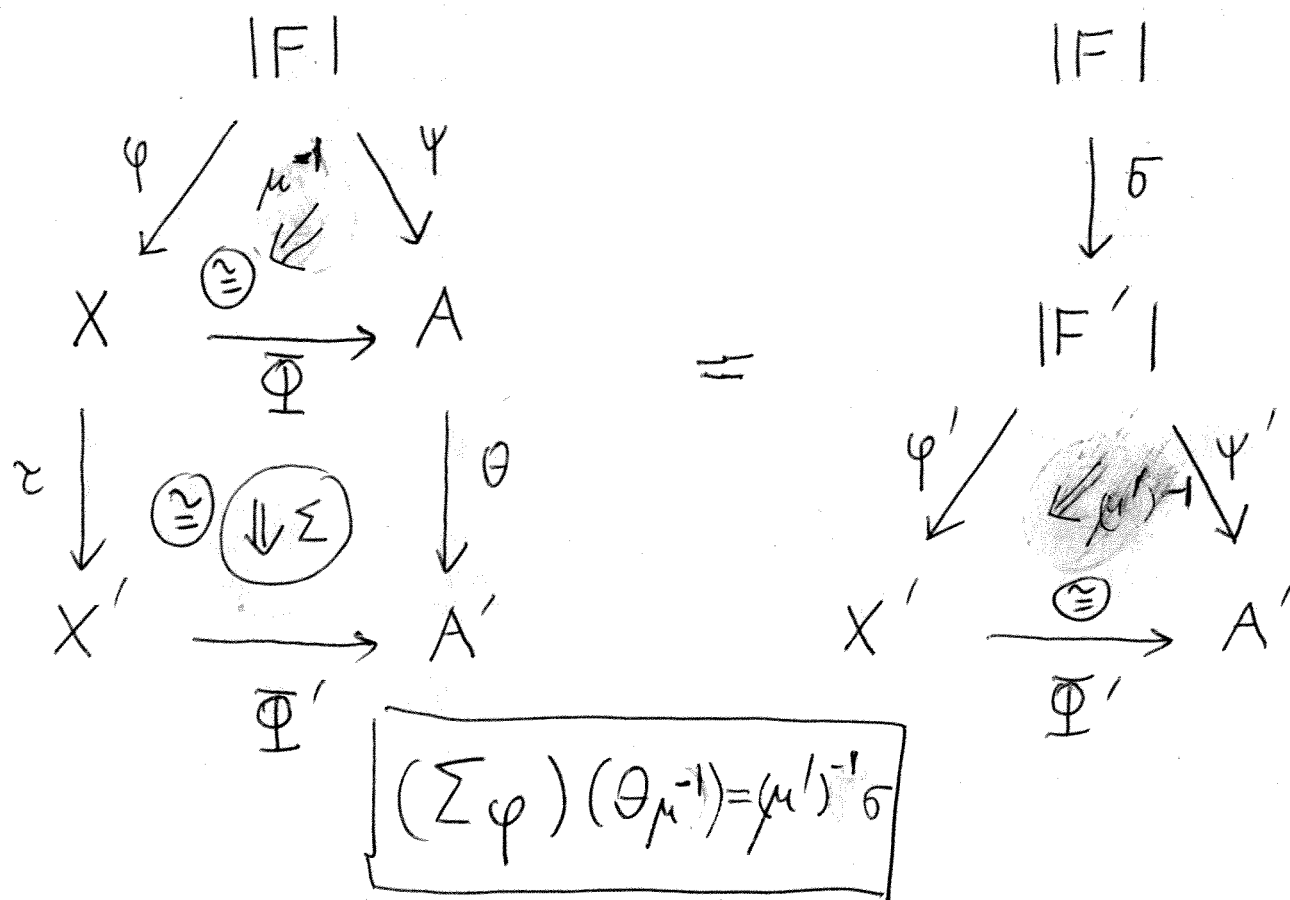
$$(\tau, \sigma, \Sigma, \theta) : (F, \mathbb{Q}, \mu) \longrightarrow (F', \mathbb{Q}', \mu')$$

where  $(\tau, \sigma, \theta) : F \longrightarrow F'$  in Sana

and

$$(\tau, \Sigma, \theta) : \mathbb{Q} \longrightarrow \mathbb{Q}' \text{ in } p\text{Fun}$$

and: compatibility of the natural transformations:



(note that by  $(z, \sigma, \theta): F \longrightarrow F'$  being an arrow in  $\text{Sana}$ , the equalities for the 1-cell composites are already known:

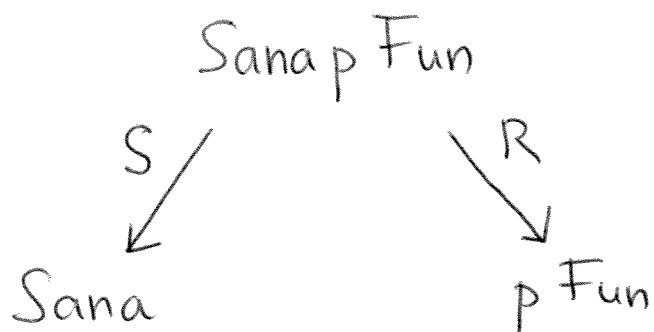
$$z\varphi = \varphi'\sigma$$

$$\theta\psi = \psi'\sigma)$$

end of def'n of  $\text{SanaFun}$

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Theorem 1 (v). In

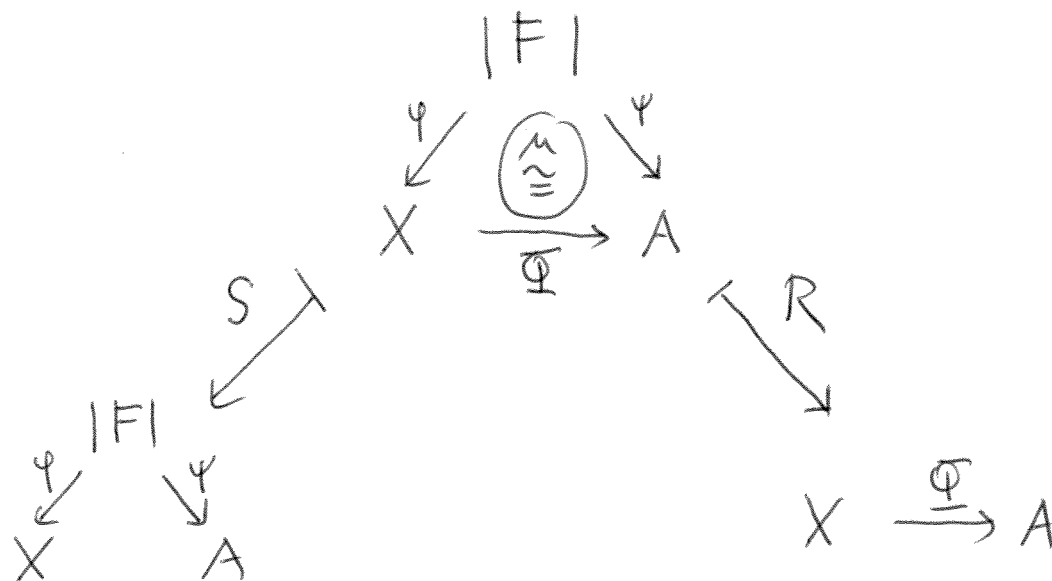


the forgetful functors  $R$  &  $S$  are full and faithful, and surjective (not just essentially surjective) on objects.

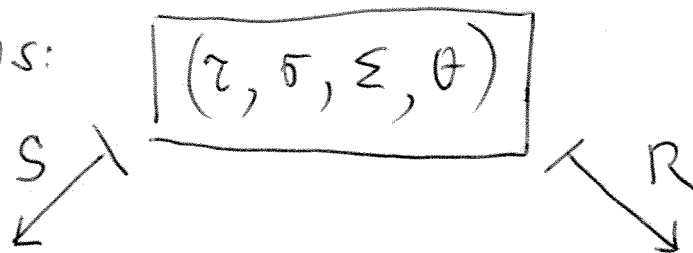
In particular,  $S$  &  $R$  are equivalences and

$$\text{Sana} \simeq \text{Sanap Fun} \simeq \text{p Fun}$$

Reminder: On objects:



On arrows:



$$(\tau, \sigma, \theta)$$

$$(\tau, \Sigma, \theta)$$

$|F|$

