(1) $\boxed{\text{Category } \boxed{\mathrm{pFun}}}$ ("pseudo-functor") defined:

→ object: $\Phi = (X, A, \Phi)$:

X, A : small categories; $\Phi : X \longrightarrow A$ functor

→ arrow: $(\tau, \Sigma, \theta) : (X, A, \Phi) \longrightarrow (X', A', \Phi')$:

$$\Sigma \;:\; \Phi \longrightarrow \Phi'$$

where:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\cong \Downarrow\Sigma"] & A \arrow[d, "\theta"] \\ X' \arrow[r, "\Phi'"'] & A' \end{tikzcd} \quad : \quad \Sigma : \theta\Phi \xrightarrow{\cong} \Phi'\tau \quad \text{natural isomorphism}$$

→ composition: (= pasting)

$$\begin{tikzcd} \Phi \arrow[r, "\Sigma"] \arrow[rr, bend right, "\Sigma'\Sigma\,:\,?"'] & \Phi' \arrow[r, "\Sigma'"] & \Phi'' \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\Downarrow\Sigma"] & A \arrow[d, "\theta"] \\ X' \arrow[r, "\Phi'"] \arrow[d, "\tau'"'] \arrow[dr, phantom, "\Downarrow\Sigma'"] & A' \arrow[d, "\theta'"] \\ X'' \arrow[r, "\Phi''"'] & A'' \end{tikzcd} \quad \text{pasting to :} \quad \begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\tau'\tau"'] \arrow[dr, phantom, "\Downarrow\Sigma''"] & A \arrow[d, "\theta'\theta"] \\ X'' \arrow[r, "\Phi''"'] & A'' \end{tikzcd}$$

$$(\tau', \Sigma', \theta') \circ (\tau, \Sigma, \theta) \overset{\text{def}}{=} (\tau'\tau, \underbrace{(\Sigma'\tau)(\theta'\Sigma)}_{=\Sigma''}, \theta'\theta)$$

$$\begin{tikzcd}[column sep=huge] X \arrow[r, bend left=60, "\theta'\theta\Phi"] \arrow[r, "\theta'\Phi'\tau" description] \arrow[r, bend right=60, "\Phi''\tau'\tau"'] & A'' \end{tikzcd} \qquad \Downarrow\theta'\Sigma,\ \Downarrow\Sigma'\tau,\ \Downarrow\Sigma''$$

$\boxed{\text{Theorem 1.}}$ (i) $\underline{\text{pFun}}$ is an accessible category.

(ii) More precisely: $\underline{\text{pFun}}$ is $\aleph_1$-accessible and has ($\aleph_0$-) filtered colimits.

(iii) Even more precisely: $\underline{\text{pFun}}$ is <u>d</u>ual-<u>r</u>egu<u>lar</u>: for some (countable) regular category $\mathbb{C}$,

$$\underline{\text{pFun}} \simeq \text{Reg}(\mathbb{C}, \text{Set}) \underset{\text{def}}{=}$$

$\underset{\text{def}}{=}$ the category of all regular functors $\mathbb{C} \to \text{Set}$, a full subcategory of $[\mathbb{C}, \text{Set}]$.

2 $\boxed{\text{Category } \boxed{\text{Sana}}}$ defined:

$\boxed{\to \text{object:}}$ $F = (|F|, X, A, \varphi, \psi)$, a span,

where

$$\begin{tikzcd}[column sep=small] & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

of small categories and functors

subject to $\boxed{1), 2), 3)}$ below:

$\boxed{\to 1)}$ Whenever $s, t \in |F|$ (objects of $|F|$), and

Sana

$$f : \varphi(s) \to \varphi(t) \quad (\text{in } X)$$

there is unique

$$g : s \to t \quad (\text{in } |F|)$$

such that $\varphi(g) = f$

(notation: $g = g_{s,t,f}$)

$$\begin{tikzcd} s \arrow[r, "g_{s,t,f}"] \arrow[d, mapsto, "\varphi"'] \arrow[dr, phantom, "\overline{\downarrow}\varphi"] & t \arrow[d, mapsto, "\varphi"] \\ \varphi(s) \arrow[r, "f"'] & \varphi(t) \end{tikzcd}$$

Sana

$\boxed{\rightarrow 2)}$ For $x \in X$ and $b \in A$ (objects),

I define

$$|F|(x,b) \underset{\text{def}}{=} \varphi^{-1}(x) \cap \psi^{-1}(b)$$

$$= \{ t \in |F| : \varphi(t) = x \ \& \ \psi(t) = b \}$$

and define, for $s \in |F|$, $b \in A$, the map

$$F_{s,b} : |F|(\varphi(s), b) \longrightarrow \mathrm{Iso}(\psi(s), b)$$

$$t \longmapsto \psi(g_{s,t}, 1_{\varphi(s)})$$

(see 1))

$\rightarrow$ we require (condition 2):

$$\boxed{F_{s,b} \text{ is a bijection}}$$

[$\mathrm{Iso}(a,b)$ is the set of isomorphisms $a \xrightarrow{\cong} b$]

Sana

$\boxed{\rightarrow 3)}$ φ is surjective on objects:

for all $x \in X$, there is $s \in |F|$ such that $\varphi(s) = x$.

("object" of $\boxed{\text{Sana}}$ defined)

[$\longrightarrow$ arrow of $\boxed{\text{Sana}}$: defined:

$$\sigma = (\tau, \sigma, \theta) : F \longrightarrow F'$$

same as a morphism of spans:

$$\begin{tikzcd}
 & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[ddd, "\sigma"] & \\
X \arrow[d, "\tau"'] \arrow[dr, phantom, "\checkmark"] & & A \arrow[d, "\theta"] \arrow[dl, phantom, "\checkmark"] \\
X' & & A' \\
 & {|F'|} \arrow[ul, "\varphi'"] \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

$$\tau\varphi = \varphi'\sigma$$
$$\theta\psi = \psi'\sigma$$

[$\longrightarrow$ composition : component-wise

$\square$ category Sana defined.

Proposition: Sana is dual-regular

see p. $\boxed{2}$

Theorem 1. (continued)

(iv) $\underline{\mathrm{pFun}}$ and $\underline{\mathrm{Sana}}$

are equivalent categories.

3

→ On the proof of Theorem 1. (iv):

1 · For $\overline{\Phi} \in \underline{\mathrm{pFun}}$ (object), the corresponding $\Gamma(\overline{\Phi}) \in \mathrm{Sana}$:

$$F = \Gamma(\overline{\Phi}) = (|\Gamma\overline{\Phi}|, \varphi, \psi) :$$

span:
$$\begin{tikzcd} & {|\Gamma\overline{\Phi}|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$
defined:

→ object of $|\Gamma\overline{\Phi}|$:

$$(s =)\ (x, a, \nu)$$

where $x \in X$, $a \in A$, $\nu \in \mathrm{Iso}(\overline{\Phi} x, a)$

for object, the effects of φ and ψ:

$$\varphi((x, a, \nu)) = x$$

$$\psi((x, a, \nu)) = a$$

→ arrow of $|\ulcorner \Phi|$:

$$(f, g) : (x, a, \nu) \longrightarrow (y, b, \rho)$$

where $f : x \to y$ (in X)

$g : a \to b$ (in A)

and

$$(*) \qquad \begin{array}{ccc} a & \xrightarrow{g} & b \\ {\scriptstyle \nu}\uparrow{\scriptstyle \cong} & \circlearrowleft & {\scriptstyle \cong}\uparrow{\scriptstyle \rho} \\ \Phi(x) & \xrightarrow[\Phi(f)]{} & \Phi(y) \end{array} \qquad \text{commutes}$$

note: given $s = (x, a, \nu)$ and $t = (y, b, \rho)$

and $f : \varphi(s) = x \longrightarrow \varphi(t) = y$,

there is unique $g : \psi(s) = a \longrightarrow \psi(t) = b$

such that $(*)$ holds;

We define ,

for $h = (f, g) : s = (x, a, \nu) \longrightarrow t = (y, b, \rho)$

$\varphi(h) \underset{\text{def}}{=} f : \varphi(s) \to \varphi(t)$

$$\begin{tikzcd}
(x,a,\nu) \arrow[r, "{(f,g)}"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\downarrow\varphi"] & (y,b,\rho) \arrow[d, "\varphi"] \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

→ Condition (1) holds; with

$$g_{s,t,f} = (f, g)$$

with g from $(*)$.

For condition 2) on objects of Sana:

For $x \in X$, $b \in A$, we now have:

→ $|F|(x, b) \underset{\text{now}}{=} \{ \underbrace{(x, b, \nu)}_{t} : \nu \in \operatorname{Iso}(\Phi x, b) \}$

$\varphi(t) = x$, $\psi(t) = b$

For $s = (x, a, \mu)$, $t = (x, b, \nu)$

→ $\operatorname{Iso}(\psi(s), b) \underset{\text{now}}{=} \operatorname{Iso}(a, b)$

and the arrow

$$g_{s,t,1_{\psi(s)}} : s \longrightarrow t$$

is

$\longrightarrow$ $g_{s,t,1_{\psi(s)}} = (1_x, g)$:

$$\begin{tikzcd}
(x, a, \mu) \arrow[rr, "(1_x, g)"] \arrow[d, mapsto, "\varphi"'] & {} \arrow[d, phantom, "\mapsto \varphi"] & (x, b, \nu) \arrow[d, mapsto, "\varphi"] \\
x \arrow[rr, "1_x"'] & {} & x
\end{tikzcd}$$

with g from:

$$\begin{tikzcd}
a \arrow[r, "g"] & b \\
\Phi(x) \arrow[u, "\mu", "\cong"'] \arrow[r, "\cong", "{\Phi(1_x) = 1_{\Phi(x)}}"'] \arrow[ur, phantom, "\checkmark"] & \Phi(x) \arrow[u, "\cong", "\nu"']
\end{tikzcd}$$

$(\therefore\ g \text{ is } \cong)$

$\longrightarrow$ $\boxed{\text{and}}$ $\psi(g_{s,t,1_{\varphi(s)}}) = g$

Thus, indeed, for any given

$s = (x, a, \mu)$ and $b \in A$:

for any $g \in \mathrm{Iso}(\underbrace{\varphi(s)}_{a}, b)$

there is unique $t = (x, b, \nu)$ such that

$$t \overset{}{\underset{F_{s,b}}{\longmapsto}} \varphi(g_{s,t,1_{\varphi(s)}}) = g$$

→ Re: condition 3) (the "non-algebraic one")
on objects in Sana:

given $x \in X$, we have

$$s = (x, \Phi(x), 1_{\Phi(x)}) \in |\Gamma\Phi|$$

[end of verifying the object effect
of equivalence functor $\Gamma : p\mathrm{Fun} \to \mathrm{Sana}$]

end of ① started on p. $\boxed{6}$

(2) ⊙ For $F \in \mathrm{Sana}$, the corresponding $\Gamma^{\sim 1}(F) \in p\,\mathrm{Fun}$:

$F = (|F|, \varphi, \psi)$:

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, dashed, "\Phi"'] & & Y \end{tikzcd}$$

to define $\Phi = \Gamma^{\sim 1}(F)$:

Let, for any $x \in X$, $s_x \in |F|$ be chosen so that $\varphi(s_x) = x$ (see condition 3)).

Define $\boxed{\Phi(x) \overset{\text{def}}{=} \psi(s_x).}$

For $f : x \longrightarrow y$ in X, consider

$$g_f \underset{\text{def}}{=} g_{s_x, s_y, f} : s_x \longrightarrow s_y ;$$

$$\varphi(g_f) = f.$$

Define
$$\overline{\Phi}(f) \overset{\text{def}}{=} \psi(g_f) : \overline{\Phi}(x) \longrightarrow \overline{\Phi}(y)$$

The uniqueness part in condition 1) ensures that

$$x \xrightarrow{f} y \xrightarrow{e} z$$

$$\Longrightarrow \quad g_{ef} = g_e g_f$$

and $g_{1_x} = 1_{S_x}$

$\therefore \quad \overline{\Phi} : X \longrightarrow A$ is a functor.

— · —

For the rest of the proof of the equivalence

$$p\text{Fun} \simeq \text{Sana}$$

(the bigger part), we define a new category

④

⊙ Category $\boxed{\text{Sana pFun}}$ defined:

$\boxed{\to \text{ object:}}$ $(F, \underline{\Phi}, \mu)$

where $F \in \text{Sana}$, $\underline{\Phi} \in \text{pFun}$

and

$$F = (|F|, X, A, \varphi, \psi)$$

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

$$\underline{\Phi} :=: \quad X \xrightarrow[\underline{\Phi}]{} A$$

and

$$\boxed{\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\overset{\mu}{\Rightarrow}\,\cong"] & \\ X \arrow[rr] & & \underline{\Phi} \end{tikzcd}}$$

$\mu : \underline{\Phi}\varphi \xrightarrow{\cong} \psi$

natural isomorphism

We will have forgetful functors:

$$\boxed{\begin{tikzcd} & \mathrm{SanapFun} \arrow[dl, "S"'] \arrow[dr, "R"] & \\ \mathrm{Sana} & & \mathrm{pFun} \end{tikzcd}}$$

<u>of course</u> : $S((F, \Phi, \mu)) \underset{\text{def}}{=} F$

$R((F, \Phi, \mu)) \underset{\text{def}}{=} \Phi$

$\longrightarrow$ arrow of SanapFun :

$$(\tau, \sigma, \Sigma, \theta) : (F, \Phi, \mu) \longrightarrow (F', \Phi', \mu')$$

where $(\tau, \sigma, \theta) : F \longrightarrow F'$ in Sana

and

$(\tau, \Sigma, \theta) : \Phi \longrightarrow \Phi'$ in pFun

and: compatibility of the natural transformations:

$$
\begin{tikzcd}
 & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\overset{\mu^{-1}}{\Swarrow}\cong"] & \\
X \arrow[rr, "\Phi"'] \arrow[d, "\tau"'] & \arrow[d, phantom, "\cong \; \Downarrow\Sigma"] & A \arrow[d, "\theta"] \\
X' \arrow[rr, "\Phi'"'] & & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & {|F|} \arrow[d, "\sigma"] & \\
 & {|F'|} \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] \arrow[d, phantom, "\overset{(\mu')^{-1}}{\Swarrow}\cong"] & \\
X' \arrow[rr, "\Phi'"'] & \; & A'
\end{tikzcd}
$$

$$\boxed{(\Sigma\varphi)(\theta\mu^{-1}) = (\mu')^{-1}\sigma}$$

(note that by $(\tau, \sigma, \theta) : F \longrightarrow F'$ being an arrow in Sana, the equalities for the 1-cell composites are already known:

$$\tau\varphi = \varphi'\sigma$$

$$\theta\psi = \psi'\sigma \;)$$

<u>end of def'n of SanapFun</u>

5 **Theorem 1 (v).** In

$$\begin{tikzcd} & \mathrm{SanapFun} \arrow[dl, "S"'] \arrow[dr, "R"] & \\ \mathrm{Sana} & & \mathrm{pFun} \end{tikzcd}$$

the forgetful functors R & S are full and faithful, and surjective (not just essentially surjective) on objects.
In particular, S & R are equivalences and

$$\mathrm{Sana} \simeq \mathrm{SanapFun} \simeq \mathrm{pFun}$$

Reminder: On objects:

$$\begin{tikzcd} & & & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\overset{\mu}{\cong}"] & & & \\ & & X \arrow[rr, "\Phi"'] \arrow[dl, mapsto, "S"'] & {} & A \arrow[dr, mapsto, "R"] & & \\ & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & & & & {} \arrow[d, phantom] & \\ X & & A & & & X \arrow[r, "\Phi"] & A \end{tikzcd}$$

On arrows:

$$
\begin{tikzcd}
 & \boxed{(\tau, \sigma, \Sigma, \theta)} \arrow[dl, mapsto, "S"'] \arrow[dr, mapsto, "R"] & \\
\boxed{(\tau, \sigma\, \theta)} & & \boxed{(\tau, \Sigma, \theta)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[ddd, "\sigma"] & \\
X \arrow[d, "\tau"'] \arrow[r, phantom, "\checkmark"] & & A \arrow[d, "\theta"] \arrow[l, phantom, "\checkmark"] \\
X' & & A' \\
 & {|F'|} \arrow[ul, "\varphi'"] \arrow[ur, "\psi'"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\Phi"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\cong\Sigma"] & A \arrow[d, "\theta"] \\
X' \arrow[r, "\Phi'"'] & A'
\end{tikzcd}
$$