

5. The universal property of the slice

As before, \mathcal{C} is a coherent category, X a fixed object of \mathcal{C} ; we consider $\Phi = \Phi_X : \mathcal{C} \rightarrow \mathcal{C}/X$ discussed above.

To abbreviate, let's write \mathcal{D} for \mathcal{C}/X , and use 'dot' for the effect of Φ : $\dot{Y} = \Phi(Y)$; for $Y \xrightarrow{f} Z$, $\dot{f} = \Phi(f)$.

The first - and main - observation is that the object \dot{X} has a global element $1_{\mathcal{D}} \xrightarrow{c_X} \dot{X}$ in \mathcal{D} .

As we know, $1_{\mathcal{D}}$ can be chosen as $(X, 1_X)$; we

have

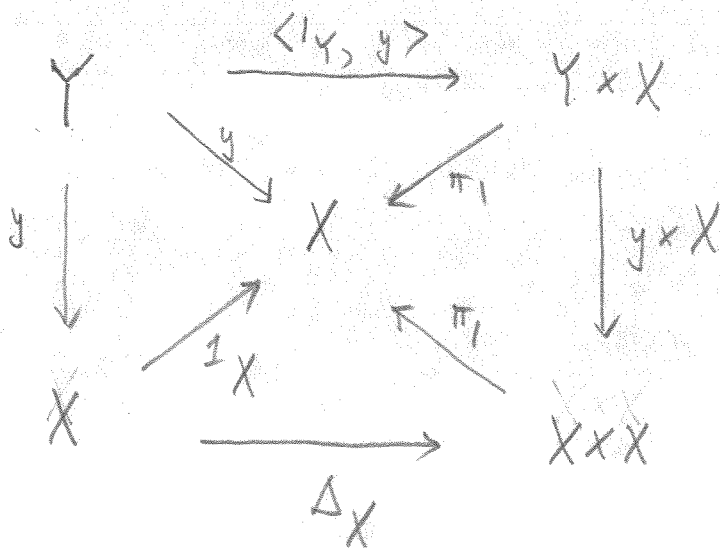
$$\begin{array}{ccc}
 X & \xrightarrow{\langle 1_X, 1_X \rangle} & X \times X \\
 \searrow 1_X & \ominus & \swarrow \pi_1 \\
 & X &
 \end{array}$$

thus $\boxed{c_X = \Delta_X} : (X, 1_X) \rightarrow (X \times X, \pi_1)$

usual notation
for $\langle 1_X, 1_X \rangle$

$c_X: 1_D \rightarrow X$ is a generic element of X in the sense that I will now explain — although we do not use this fact explicitly in our further work.

Using the generic element c_X , and the objects and arrows of \mathbb{C}/X that come from \mathbb{C} via \mathbb{I} , we can 'generate' all of the structure of \mathbb{C}/X . Let (Y, y) be an arbitrary object of \mathbb{C}/X , and consider the following ^{commutative} diagram in \mathbb{C} :



The outside square is a pullback — the checking of this is omitted (not really: see pages 43.1 and following!).

This means that we have the following pullback diagram in \mathcal{C}/X :

$$\begin{array}{ccc}
 (Y, y) & \xrightarrow{\langle 1_Y, y \rangle} & \dot{Y} \\
 y \downarrow & & \downarrow y \\
 1_D & \xrightarrow{c_X} & \dot{X}
 \end{array}$$

Now, assume that we have an arbitrary coherent functor $F: \mathcal{C} \rightarrow \mathcal{E}$, and let $1_{\mathcal{E}} \xrightarrow{e} F(X)$ be a global element of $F(X)$.

I claim that we can extend F to

$G: \mathcal{C}/X \rightarrow \mathcal{E}$ such that G is coherent,

$$G \circ \Phi \cong F$$

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\Phi} & \mathcal{C}/X \\
 F \searrow & \cong & \downarrow G \\
 & & \mathcal{E}
 \end{array}$$

by a natural isomorphism

$$\mu: G\Phi \xrightarrow{\cong} F$$

and also G maps $c_X: 1_D \rightarrow \Phi(X)$

to $e: 1_E \rightarrow F(X)$ modulo the isomorphism

μ , that is:

$$\begin{array}{ccc}
 G(1_D) & \xrightarrow{G(c_X)} & G\Phi(X) \\
 \downarrow \cong & \cong & \downarrow \mu_X \\
 1_E & \xrightarrow{e} & F(X)
 \end{array}$$

commutes.

The pullback in G/X on the previous page will have to be mapped to a pullback

$$\begin{array}{ccc}
 G(Y, y) & \xrightarrow{\sigma} & G\Phi(Y) \\
 \downarrow \lrcorner & & \downarrow G\Phi(y) \\
 G(1_D) & \xrightarrow{G(c_X)} & G\Phi(X)
 \end{array}$$

which, combined with isomorphisms:

$$\begin{array}{ccccc}
 G((Y, y)) & \longrightarrow & G\Phi(Y) & \xrightarrow[\cong]{\mu_Y} & F(Y) \\
 \downarrow \cong & & \downarrow G\Phi(y) & & \downarrow F(y) \\
 G(1_D) & \xrightarrow[G(c_X)]{} & G\Phi(X) & \xrightarrow[\cong]{\mu_X} & F(X) \\
 \downarrow \cong & \cong & \downarrow \mu_X & & \\
 1_E & \xrightarrow[e]{} & F(X) & &
 \end{array}$$

gives rise to pullback in \mathcal{E} :

$$\begin{array}{ccc}
 G((Y, y)) & \xrightarrow{\Gamma} & F(Y) \\
 \downarrow \cong & \lrcorner & \downarrow F(y) \\
 1_E & \xrightarrow[e]{} & F(X)
 \end{array}
 \quad (*)$$

This means that $G((Y, y))$ is determined, at least up to an isomorphism, by the requirements on G .

To actually define G , we pick, for any object

(Y, y) of \mathbb{C}/X , a pullback diagram $(*)$

(thus, not only the object $G((Y, y))$, but also

the morphism $G((Y, y)) \longrightarrow F(Y)$. It is seen

that the action of G on morphisms $(Y, y) \xrightarrow{f} (Z, z)$

will be uniquely determined so that the diagrams

$$\begin{array}{ccc}
 G((Y, y)) & \longrightarrow & F(Y) \\
 G(f) \downarrow & \text{\textcircled{=}} & \downarrow F(f) \\
 G((Z, z)) & \longrightarrow & F(Z)
 \end{array} \quad (*)$$

all commute.

Let me show why G preserves e.e. morphisms

Let $f: (Y, y) \longrightarrow (Z, z)$ be e.e. in \mathbb{C}/X .

The reason why $G(f)$ is e.e. is that $(*)$ is

actually a pullback, and, by assumption on F , $F(f)$

is e.e., and \mathbb{E} is coherent. The fact that

$(*)$ is a pullback comes from the fact that

the two squares on the left in what follows

compose to the quadrangle on the right:

$$\begin{array}{ccc}
 G((Y, y)) & \longrightarrow & F(Y) \\
 G(f) \downarrow & \textcircled{1} & \downarrow F(f) \\
 G((Z, z)) & \longrightarrow & F(Z) \\
 ! \downarrow & \textcircled{2} & \downarrow F(z) \\
 \mathbb{1}_E & \longrightarrow & F(X)
 \end{array}
 \qquad
 \begin{array}{ccc}
 G((Y, y)) & \longrightarrow & F(Y) \\
 ! \downarrow & \textcircled{3} & \downarrow F(y) \\
 \mathbb{1}_E & \longrightarrow & F(X)
 \end{array}$$

(since $Y \xrightarrow{f} Z$, and there is just one

$$\begin{array}{ccc}
 & \circlearrowleft & \\
 y \downarrow & & \downarrow z \\
 & X &
 \end{array}$$

arrow $G((Y, y)) \longrightarrow \mathbb{1}_E$)

and we have the general (and widely used rule) that if in a situation like the above, with knowing that $\textcircled{1}$ commutes, and $\textcircled{2}$ and $\textcircled{3}$ are pullbacks, then so is $\textcircled{1}$.

I will not pursue the genericity of $c_X: \mathbb{1}_{C/X} \rightarrow \mathbb{Q}(X)$ any more, since in the completeness proof this fact is not used.

6. Conservativeness

(6.1) Let \mathcal{C}, \mathcal{S} be categories, $M: \mathcal{C} \rightarrow \mathcal{S}$ a functor, \underline{A} a class of arrows in \mathcal{C} .

Let us say that M is conservative for \underline{A} if for all $f \in \underline{A}$, $M(f)$ is an isomorphism in \mathcal{S} (if and) only if f is an isomorphism in \mathcal{C} .

Proposition. Assume \mathcal{C} has finite limits, and M preserves finite limits. Assume M is conservative for (the class of) monomorphisms in \mathcal{C} . Then: M is conservative for (the class of) all arrows in \mathcal{C} .

Proof. Consider an arrow $A \xrightarrow{f} B$ in \mathcal{C} .

Consider:

$$\begin{array}{ccccc}
 & & \xrightarrow{1_A} & & \\
 & & \curvearrowright & & \\
 A & \xrightarrow{m} & K & \begin{array}{l} \xrightarrow{b_0} \\ \xrightarrow{b_1} \end{array} & A & \xrightarrow{f} & B \\
 & & & \triangleright & & & \\
 & & & \xrightarrow{b_1} & A & \xrightarrow{f} & B \\
 & & \curvearrowleft & & & & \\
 & & \xrightarrow{1_K} & & & &
 \end{array} \quad (*)$$

Here, (K, b_0, b_1) is the so-called kernel-pair of f :

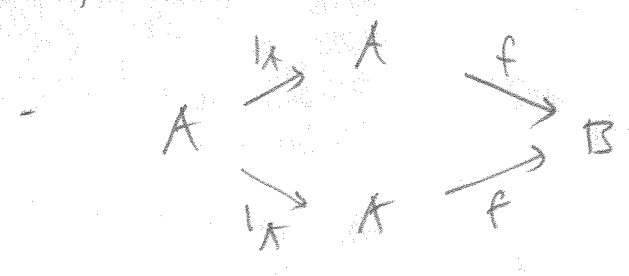
the pullback of f against itself. The morphism $m: A \rightarrow K$ is defined by using the universal property of K such that

$$k_0 m = k_1 m = 1_A. \quad m \text{ is a monomorphism.}$$

if $m: B \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} A$, and $mg = mh$, then $k_0 mg = k_0 mh$,

hence, $g = h$. Also, m is an isomorphism, iff

f is a monomorphism. Namely, if f is a monomorphism, then k_0, k_1 are isomorphisms (the pullback diagram is isomorphic to



in this case), and so m is an isomorphism. And if m is an iso, so are k_0 and k_1 , and f is a mono.

Now, assume $M(f)$ is an isomorphism. Consider the image of $(*)$ under M in \mathcal{B} . It is the same construct from $M(f)$ in \mathcal{B} as $(*)$ was

from f in \mathcal{C} . Therefore, $M(m)$ is an isomorphism, since $M(f)$ is, in particular, a monomorphism. But m is a mono, and M is conservative for mono's; therefore, m is an isomorphism. But then, by what we said above, f is a monomorphism. Now, since M is conservative for mono's, and $M(f)$ is an isomorphism, it follows that f is an isomorphism.

I now generalize the concept a bit. Suppose \underline{M} is a class of functors $M: \mathcal{C} \rightarrow \mathcal{D}_M$ from the fixed category \mathcal{C} to possibly-variable categories \mathcal{D}_M , $M \in \underline{M}$. I will say that

\underline{M} is conservative for \underline{A} (\underline{A} : as before)

if

for all $f \in \underline{A}$, $M(f)$ is an isomorphism

for all $M \in \underline{M}$ $\textcircled{\text{iff}}$ f is an isomorphism in \mathcal{C} .

The above proposition is generalised, with essentially unchanged proof, to the version obtained by replacing M by \underline{M} three times (first, it should be: "every $M \in \underline{M}$ preserves finite limits"). This generalization is not really a generalization, since, with given \underline{M} , I can consider the single

$$\underline{\tilde{M}} : \mathbb{C} \longrightarrow \prod_{M \in \underline{M}} \mathcal{S}_M$$

and the "generalization" is reduced to the original version.

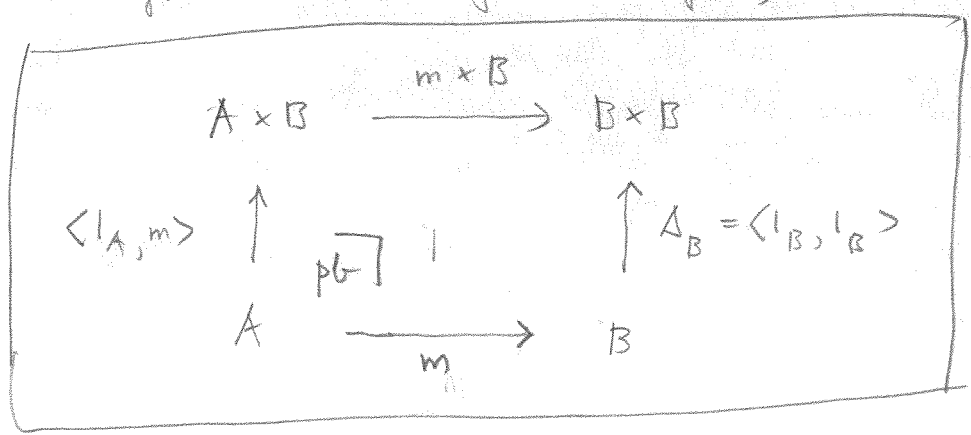
Looking at the "Gödel completeness theorem", p 10: by the present Proposition, it suffices to prove it for all monomorphisms $f: A \rightarrow B$.

(6.2) Lemma. Let $A \xrightarrow{m} B$ be a monomorphism in \mathbb{C} . Then $\Phi_B = \Phi : \mathbb{C} \rightarrow \mathbb{C}/B$ is conservative for m .

Proof. We use the 'generic element'

$$C_B: \mathbb{1}_{C/B} \longrightarrow \mathbb{Q}(B)$$

— and, actually, this proof will be used, rather than the lemma itself when it comes to the crunch, I start with the fact that in \mathcal{C} , for an arbitrary arrow $m: A \rightarrow B$ in \mathcal{C} , the following is a pullback diagram — and for a change, I will verify it:

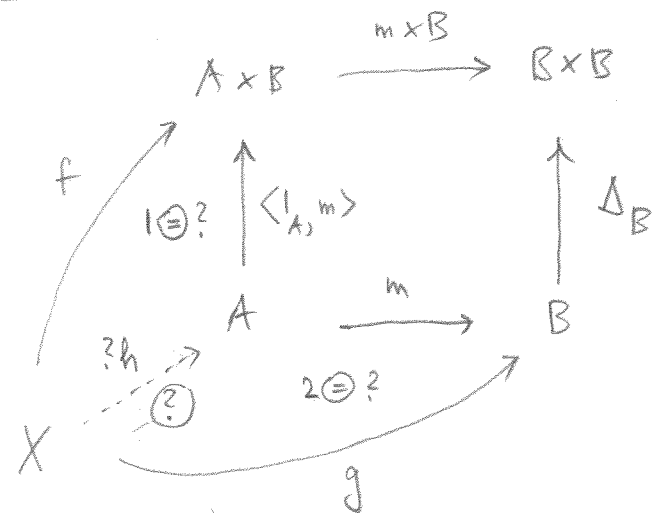


Same pullback as the one on p. 39.2!

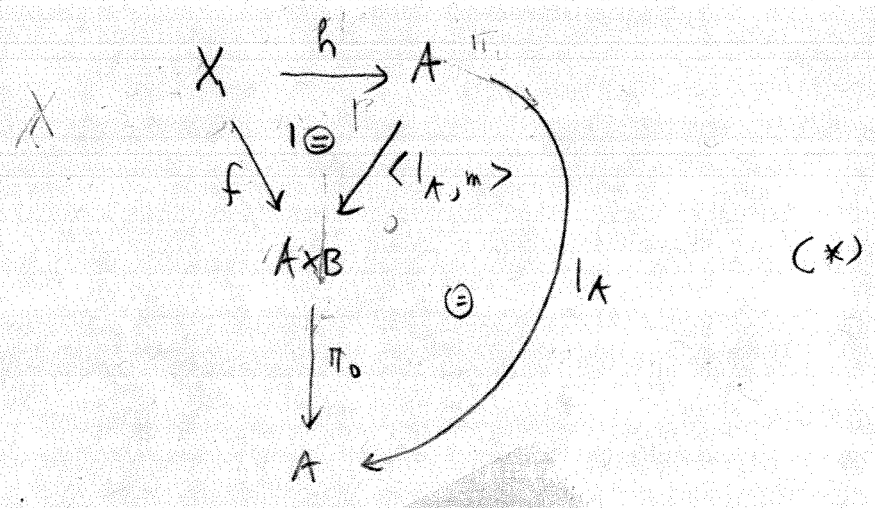
Suppose we have $f: X \rightarrow A \times B$, $g: X \rightarrow B$ such that

$$(m \times B) f = \Delta_B g$$

? is there h as required by ? ?

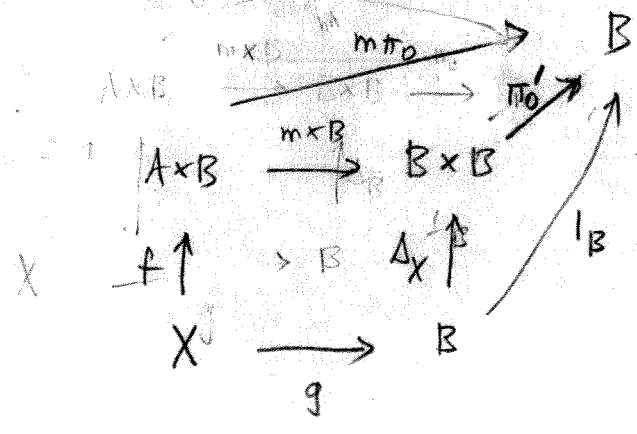


The required commutativity $\textcircled{1}$ when followed by the projection π_0 determines h :

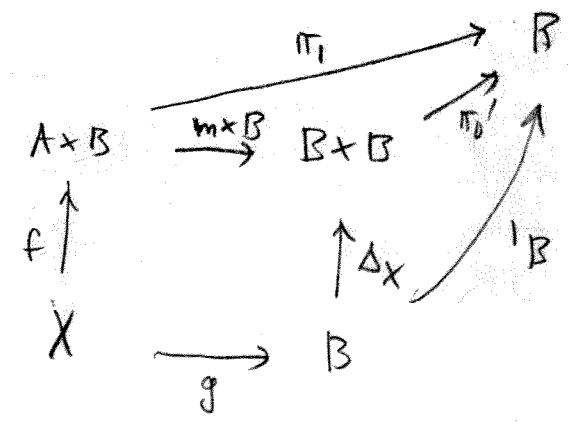


h must be $h = \pi_0 \circ f$.

The assumed equality (box, previous page), when followed by the two projections $B \times B \xrightarrow[\pi_1]{\pi_0} B$ gives us:



$m \pi_0 f = g$ (boxed)



$\pi_1 f = g$ (boxed)

We verify that $h = \pi_0 f$ indeed satisfies

1 \Leftrightarrow and 2 \Leftrightarrow :

$$\langle 1_A, m \rangle h \stackrel{?}{=} f \quad (1)$$

$$m h \stackrel{?}{=} g \quad (2)$$

(1) is the equality of two arrows into the product $A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$; it is equivalent to the conjunction of the following two

$$\pi_0 \langle 1_A, m \rangle h \stackrel{?}{=} \pi_0 f \quad (1.1)$$

$$\pi_1 \langle 1_A, m \rangle h \stackrel{?}{=} \pi_1 f \quad (1.2).$$

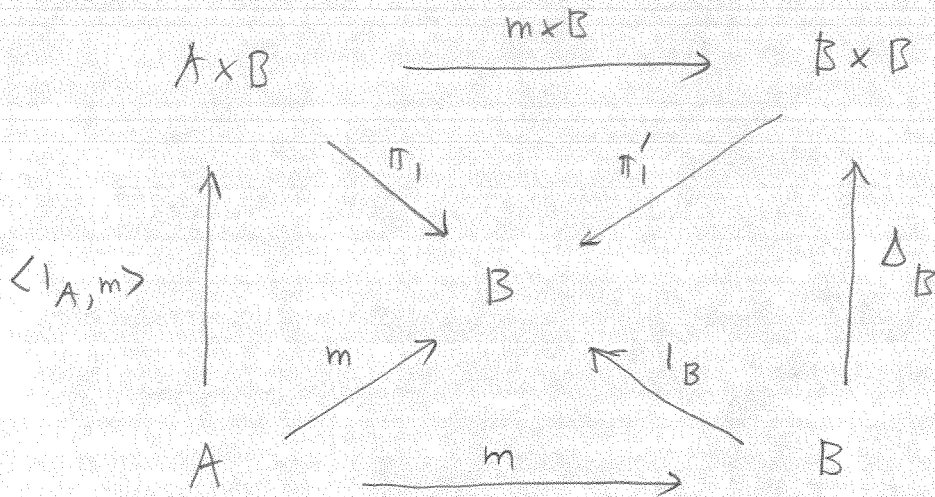
(1.1) was used, to get $h = \pi_0 f$. But $\pi_1 \langle 1_A, m \rangle = m$

thus, (1.2) is reduced to

$$m h \stackrel{?}{=} \pi_1 f \quad (1.2')$$

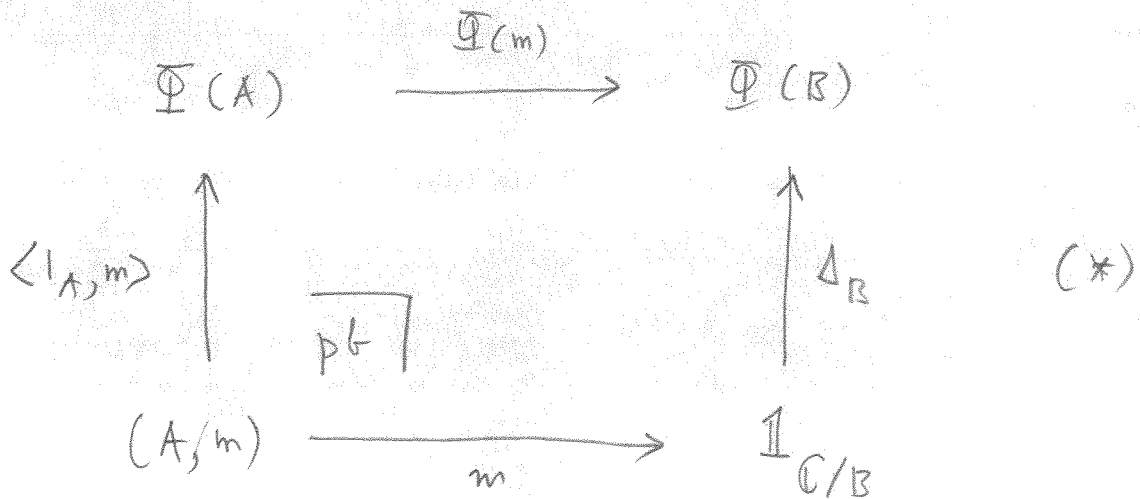
With $h = \pi_0 f$, (2) is $m \pi_0 f = g$ - which is true (see previous page), and (1.2') is $m \pi_0 f \stackrel{?}{=} \pi_1 f$ - and this holds, since both sides equal g (see previous page).

Next, we can complete the diagram to something taking place in \mathbb{C}/\mathbb{B} :



since all four triangles commute.

Therefore, we have the following pullback in \mathbb{C}/\mathbb{B} :



This is the pullback in \mathbb{C}/\mathbb{B} that we are going to use again. However, the assertion of 6.2 Lemma is clear:

if $\Phi(m)$ is an isomorphism, so is its pullback $m: (A,m) \rightarrow \mathbb{1}_{\mathbb{C}/\mathbb{B}}$ in \mathbb{C}/\mathbb{B} — and so is $m: A \rightarrow B$ in \mathbb{C} .

I must put here the conservativeness argument that uses the last-mentioned pullback,

6.2' Proposition Suppose: \mathcal{C} is coherent,

$m: A \twoheadrightarrow B$ is a mono in \mathcal{C} ;

and we have

$$\Sigma: \mathcal{C}/B \longrightarrow \mathcal{S},$$

a coherent functor that is conservative for the following subobject of the terminal object of \mathcal{C}/B :

$$\left\{ \begin{array}{l} (A, m) \xrightarrow{m} \mathbb{1}_{\mathcal{C}/B} = (B, l_B) \\ \\ \begin{array}{ccc} A & \xrightarrow{m} & B \\ & \searrow m & \swarrow l_B \\ & B & \end{array} \end{array} \right.$$

Then the composite $\Sigma \circ \Phi_B: \mathcal{C} \longrightarrow \mathcal{S}$ is conservative for $m: A \twoheadrightarrow B$ in \mathcal{C} :

$$(\mathcal{C} \xrightarrow{\Phi_B} \mathcal{C}/B \xrightarrow{\Sigma} \mathcal{S}).$$

Proof. Suppose $\Sigma \circ \Phi_B(m)$ is an isomorphism in \mathcal{S} , to derive that $m: A \twoheadrightarrow B$ is an iso in \mathcal{C} .

Consider the pullback (*) on p. 44, and \mathbb{C}/\mathbb{B} , and apply the coherent functor Σ to it, obtaining the pullback:

$$\begin{array}{ccc}
 \Sigma \Phi(A) & \xrightarrow[\cong]{\Sigma \Phi(m)} & \Sigma \Phi(B) \\
 \uparrow \Sigma(\langle 1_A, m \rangle) & & \uparrow \Sigma(\Delta_B) \\
 \Sigma(A, m) & \xrightarrow{\Sigma(m)} & \Sigma(1_{\mathbb{C}/\mathbb{B}})
 \end{array}$$

pr

in the category \mathcal{S} .

Since the pullback of an isomorphism is an isomorphism,

$\Sigma(m) : \Sigma(A, m) \rightarrow \Sigma(1_{\mathbb{C}/\mathbb{B}})$ is an isomorphism. in \mathbb{C}/\mathbb{B} .

We have assumed that Σ is conservative for $m : (A, m) \rightarrow 1_{\mathbb{C}/\mathbb{B}}$.

Therefore, $m : (A, m) \rightarrow 1_{\mathbb{C}/\mathbb{B}} = (B, 1_B)$ is an

isomorphism in \mathbb{C}/\mathbb{B} . Applying the forgetful functor

$$\Psi_B : \mathbb{C}/\mathbb{B} \rightarrow \mathbb{C}$$

to this, we get that $m : A \rightarrow B$ is an isomorphism in \mathbb{C} , as desired.

(6.3) Lemma. Suppose X has global support:

$$X \xrightarrow{!X} \mathbb{1} \text{ is e.e. (in } \mathbb{C} \text{). Then}$$

$$\Phi_X: \mathbb{C} \longrightarrow \mathbb{C}/X \text{ is conservative}$$

(for all arrows in \mathbb{C} ; enough: for mono's in \mathbb{C}).

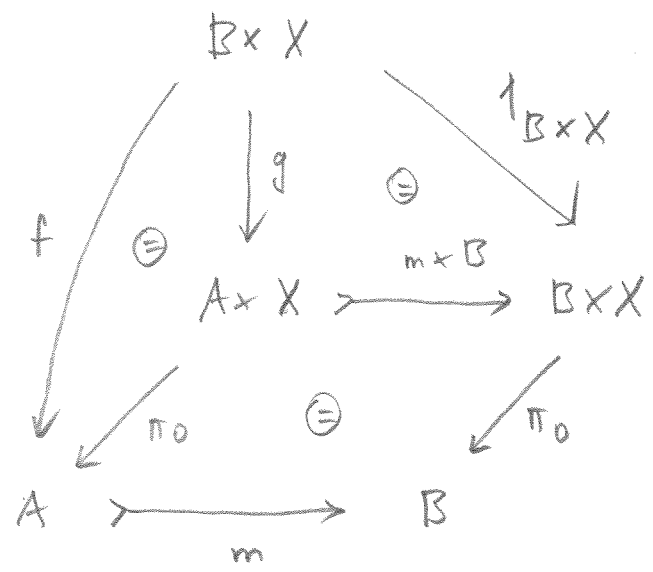
proof: Let $m: A \rightarrow B$ be any mono in \mathbb{C} . Having an inverse g of $\Phi(m)$

means

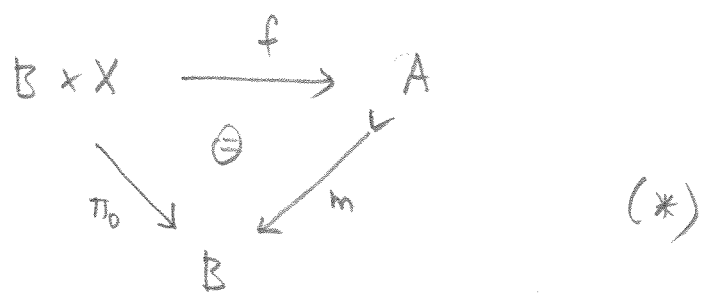
$$\begin{array}{ccc}
 A \times X & \begin{array}{c} \xleftarrow{g} \\ \xrightarrow{m \times X} \end{array} & B \times X \\
 \searrow \pi_1 & & \swarrow \pi_1' \\
 & B &
 \end{array}$$

$$(m \times X) \circ g = \mathbb{1}_{B \times X}, \quad \pi_1 \circ g = \pi_1'. \text{ Let } f = \pi_0 \circ g,$$

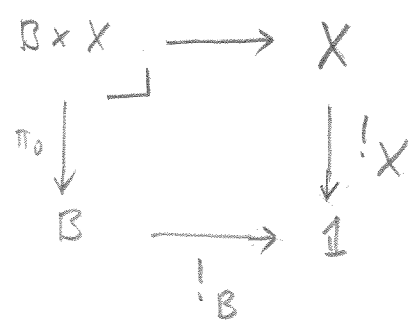
for $\pi_0: A \times X \rightarrow A$, We have



and



Now,

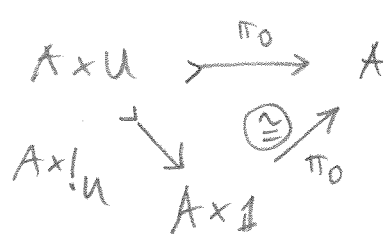


is a pullback. Therefore, since $!_X$ is assumed to be e.e., so is π_0 . From $(*)$, m must be an isomorphism.

(6.4) Lemma. Suppose U and V are subobjects of $\mathbb{1} = \mathbb{1}_C$ and $A \xrightarrow{m} B$ a monomorphism. Suppose $U \vee V = \mathbb{1}$. If $\Phi_U(m)$ and $\Phi_V(m)$ are both isomorphisms in C/U , resp., in C/V , then m is an isomorphism.

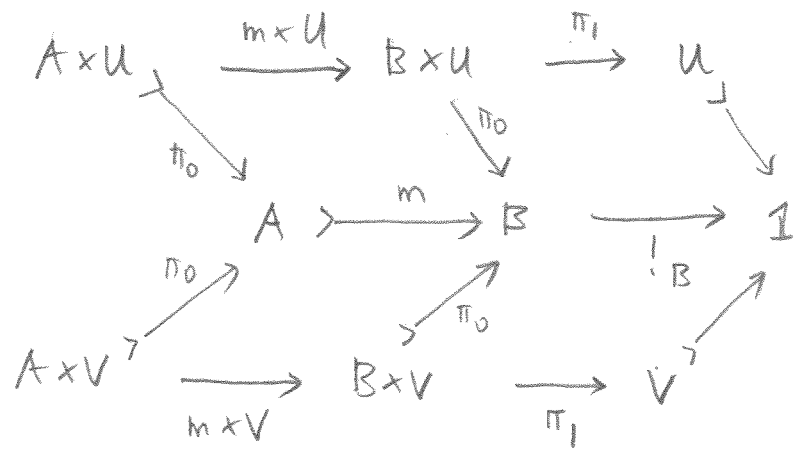
Proof. Of course, the monomorphism $U \rightarrow \mathbb{1}$ is the unique $!_U : U \rightarrow \mathbb{1}$; the assumption is that $!_U$ is a monomorphism; the same for $!_V : V \rightarrow \mathbb{1}$.

Then $A \times U \xrightarrow{\pi_0} A$ is a monomorphism, since



Similarly for other π_0 projections that appear below.

Consider:



All of the four quadrangles are pullbacks

$$\Phi_U(m) = m \times U; \quad \Phi_V(m) = m \times V; \text{ they are}$$

both isomorphisms. Hence the monomorphisms

$$A \times U \xrightarrow[\cong]{m \times U} B \times U \xrightarrow{\pi_0} B, \text{ and } B \times U \xrightarrow{\pi_0} B$$

determine the same subobject of B ; call it $\hat{U} \in \text{Sub}(B)$,

similarly for $\hat{V} = [A \times V, \pi_0(m \times V)] = [B \times V, \pi_0] \in \text{Sub}(B)$

Now, $\hat{U} = [B \times U, \pi_0] \in \text{Sub}(B)$ equals

$$\hat{U} = (!_B)^* U$$

for the map $!_B: B \rightarrow \mathbb{1}$; similarly

$$\hat{V} = (!_B)^* V.$$

$$\text{Then } \hat{U} \vee \hat{V} = (!_B)^* U \vee (!_B)^* V =$$

↑
sup in $\text{Sub}(B)$

$$= (!_B)^* (U \vee V) = (!_B)^* (\pi_{\mathbb{1}})$$

$$= \pi_B$$

But the subobject $\hat{A} = [A, m]$ ^{of B} has the property

that $\hat{U}, \hat{V} \leq \hat{A}$, because $\hat{U} = [A \times U, m \pi_0]$, $\hat{V} = [A \times V, m \pi_0]$.

Therefore, $\hat{U} \vee \hat{V} \leq \hat{A}$, i.e.,

$$\prod_B \leq \hat{A}, \quad \prod_B = \hat{A},$$

which means that m is an isomorphism.

(6.5) Lemma. $\Gamma = \mathbb{C}(\mathbb{1}, -): \mathbb{C} \rightarrow \text{Set}$

is conservative for all mono's $U \xrightarrow{m} \mathbb{1}$

into $\mathbb{1}$.

Proof: This is immediate: $\Gamma(m)$ being an isomorphism

means, at least, that there is an arrow $\Gamma(\mathbb{1}) \rightarrow \Gamma(U)$

in Set . Since $\Gamma(\mathbb{1}) \cong \mathbb{1}_{\text{Set}} \cong \{*\}$, this means that

there is an element of $\Gamma(U)$, i.e. an arrow $\mathbb{1}_{\mathbb{C}} \rightarrow U$

in \mathbb{C} . We know (see: middle of p 15, e.g.) that

this means that m is an isomorphism.

7. Directed colimits

The proof of Gödel completeness - our goal - requires the repeated application of the slice construction discussed above, and the assembling ^{of the resulting} infinitely many coherent categories and coherent functors into one final one, one which is complete and saturated in the sense given in section 3. The 'assembling' is done, formally, in the way of a directed colimit of coherent categories and functors. The directed colimit is constructed entirely in the category of sets; e.g., the set of objects of the colimit is the colimit of the ~~subject-sets~~ ^{objects} of the ingredient categories; the same for the set of arrows. The success of the operation (mainly, but not exclusively, the fact that the resulting category is in fact coherent) depends on a property of directed colimits in Set, the category of sets, which can be stated by saying that directed fact

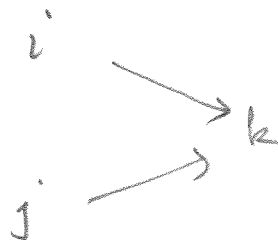
colimits commute with finite limits in Set.
 This is a very fundamental fact that has
 far-reaching application in category theory,
 namely, in category theory after Saunders
 Mac Lane's 'bible', the "Categories for the
 working mathematician" (1971). Said fundamental
 fact is duly proved towards the end of the book

(Chapter IX: Special Limits, section 2: Interchange of
 Limits, Theorem 1, p211), so that one can
 easily imagine a "second volume" of the
 book, starting with the Gabriel-Ulmer theory
 of locally presentable categories, in which said
 fundamental fact is at the heart of the matter.

- and going on to accessible categories if you like.
 If you visit the spot in Mac Lane's book, you'll
 find, I think, that the half-page proof is a
 bit breezy - understandable, since the theorem
 is not put to any serious use yet in the book!
 I will studiously avoid using the theorem; instead,
 I show the details of the construction of the
 directed colimit, and some details of the proof of

Mac Lane's theorem, details that I actually use. The Diligent Reader can then go to the book and acquire a better understanding of the Theorem 1 (p211), better than just by running through Mac Lane's words (unless the D.R. is a different intelligence from mine).

Let (I, \leq) be a ^{non-empty!} partially ordered set (\leq is reflexive, transitive and anti-symmetric) which is also directed: for all $i, j \in I$, there is $k \in I$ such that $i \leq k$ and $j \leq k$:

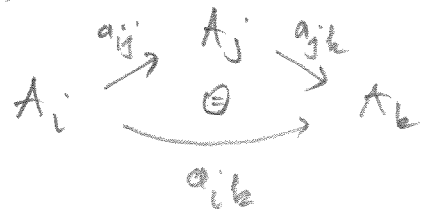


I use the arrow notation ($i \rightarrow k$ for $i \leq k$) since, of course, (I, \leq) is in fact a (rather special) category.

Let \mathcal{A} be a category: think, first, of $\mathcal{A} = \text{Set}$, but later others, e.g. $\mathcal{A} = \text{Cat}$ or $\mathcal{A} = \text{Coh}$.

Let $\underline{A} : \underline{I} \longrightarrow \mathcal{A}$ be a functor: or, as
 $\underline{I} = (\underline{I}, \leq)$

we say, a diagram; \underline{A} consists of objects $\underline{A}(i) = A_i$
of \mathcal{A} , and arrow $\underline{A}(i \leq j) = a_{ij} : A_i \rightarrow A_j$
such that $a_{ii} : A_i \rightarrow A_i$ is $a_{ii} = 1_{A_i}$, and
for $i \leq j \leq k$,



Commutates: $a_{ik} = a_{jk} \circ a_{ij}$.

Example: $\underline{I} = \mathbb{N}$ = the set of natural numbers;

\leq : usual ordering; $\mathcal{A} = \text{Set}$;

A_i : set for each $i \in \mathbb{N}$

and let

$$a_{ij} : A_i \rightarrow A_j$$

be an inclusion: $A_i \subseteq A_j$, and a_{ij} maps $x \in A_i$
to x itself in A_j .

Then, yes, we have a diagram $\underline{A} : \underline{I} \rightarrow \text{Set}$.

Now, it is very natural to consider

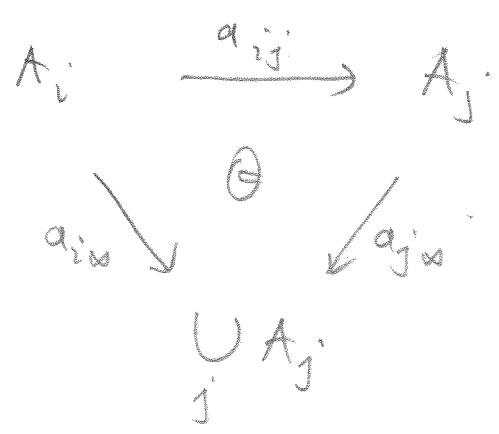
$$\bigcup_{i \in \mathbb{N}} A_i$$

the union of the sets, as a kind of "limit" (we will say: "colimit") of the diagram $\langle A_i \rangle_{i \in \mathbb{N}}$.

This is an example for the directed colimit we are advocating here. Note that in this example we have the further inclusions

$$a_{i\infty} : A_i \rightarrow \bigcup_{j \in \mathbb{N}} A_j$$

and, even,



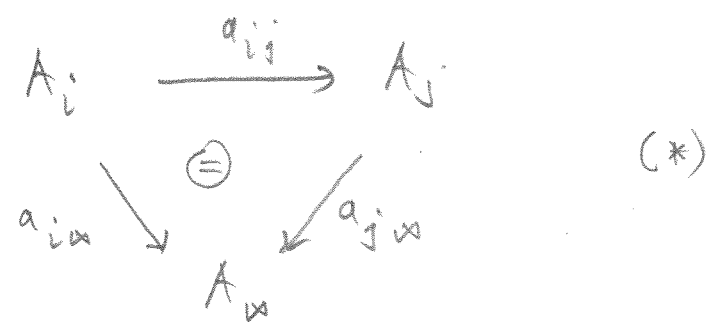
commuting - as of course all diagrams consisting of inclusions will commute - if they are of the right shape.

(always non-empty!)

Returning to a general directed poset $(I, \leq) = \underline{I}$ and a general \underline{I} -based diagram $\underline{A} : \underline{I} \rightarrow \text{Set}$ in the category of sets ($\mathcal{A} = \text{Set}$), now we construct a colimit

$A_\infty = \text{colim}_{i \in I} A_i$, a set, together with

maps $A_i \xrightarrow{a_{i\infty}} A_\infty$ such that



commutes. We could do this construction for an arbitrary poset (I, \leq) , without assuming directedness, or even for an arbitrary category (small) \underline{I} and diagram (functor) $\underline{A}: \underline{I} \rightarrow \text{Set}$ - but that would not be useful now, since the special features of this construction for the directed case are crucial for the applications. The colimit will have a universal property which determines it up to isomorphism - and which will also be very useful, although not enough: the universal property by itself will not suffice for our goals, the actual construction (or, if you wish, Mac Lane's Theorem 1 (p 210))

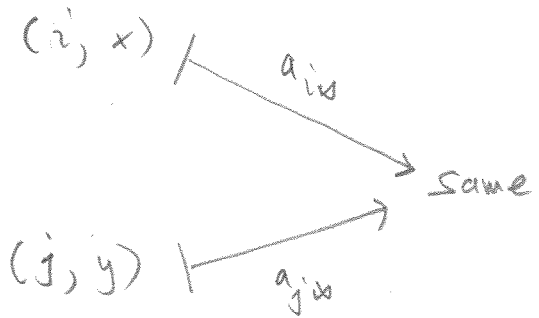
is necessary to keep in mind.

Here is the construction.

For each $i \in I$ and $x \in A_i$, we will have an element $a_{i\infty}(x)$ in A_∞ ; thus, we will have a map $(i, x) \mapsto a_{i\infty}(x)$; moreover, if

$$A_i \xrightarrow{a_{ij}} A_j$$

then because of the required commutativity $(*)$, p 54, I will have to have, for $y = a_{ij}(x)$

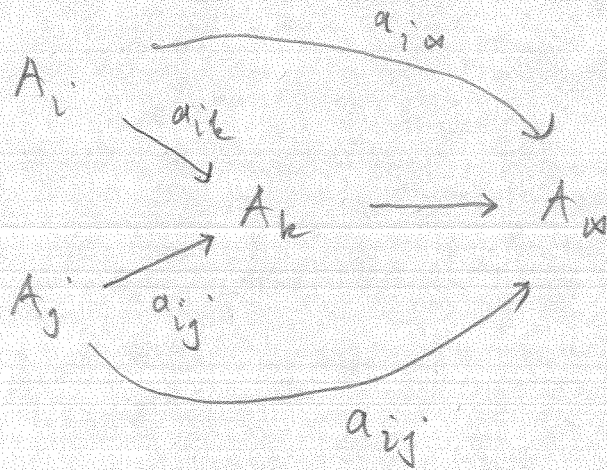


i.e., (i, x) and (j, y) will have to be identified in A_∞ .

However, we can "recognize" the necessity of this kind of identification more generally: for any i and j , there is at least one k with $i \leq k, j \leq k$, so that

(i, x) and (j, y) should be eventually identified if we see that $a_{ik}(x) = a_{jk}(y)$ $(*)$

in A_k :



thus, if $(*)$, then $a_{ix}(x) = a_{jx}(y)$.

Make this into a definition. Let B be the disjoint union of the sets A_i :

$$B = \bigsqcup_{i \in I} A_i = \{(i, x) : i \in I, x \in A_i\}$$

and define the binary relation R on B by:

$$b_1 R b_2 \stackrel{\text{def}}{\iff} \begin{aligned} & b_1 = (i, x) \text{ \& } b_2 = (j, y) \\ & \text{and there is } k, \\ & i \leq k \text{ \& } j \leq k \end{aligned}$$

such that

$$a_{ik}(x) = a_{jk}(y) \quad (x \in A_k)$$

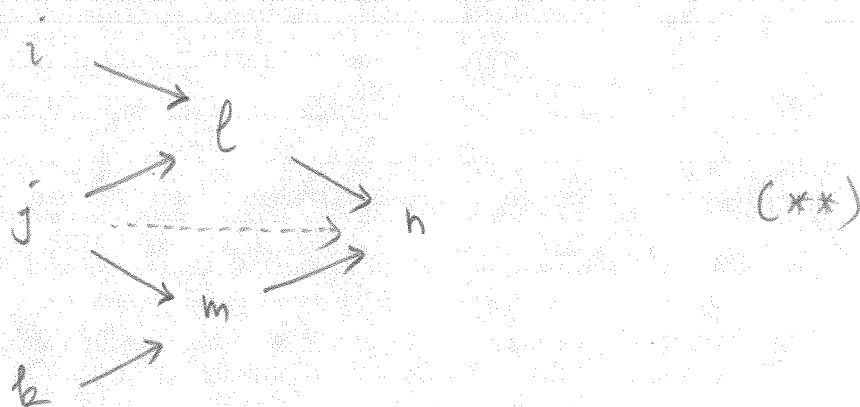
CLAIM: R is an equivalence relation on B .

Only transitivity is at issue.

Let:

$$(i, x) R (j, y) R (k, z) \quad (*)$$

We have: $l, m, n \in I$ such that



in I . (remember: $i \rightarrow l$ means $i \leq l$);

and, the choice of l and m : by the assumption $(*)$

so that

$$a_{il}(x) = a_{jl}(y) \stackrel{\text{def}}{=} u,$$

$$a_{jm}(y) = a_{km}(z) \stackrel{\text{def}}{=} v.$$

Let: $w \stackrel{\text{def}}{=} a_{jn}(y)$. The "diagram" $(**)$

when the functor \underline{A} is applied to it, becomes

commutative (since, of course $(**)$ commutes in I !)

and that means that

$$a_{in}(x) = a_{kn}(z) = w$$

- and this exhausts the requirement for the relationship $(i, x) R (k, z)$ - as desired.

I now define

$$A_\infty = \text{colim}_{i \in I} A_i = \text{colim } \underline{A}$$

abridged notation!

$$\stackrel{\text{def}}{=} B/R = \text{the set of equivalence}$$

classes of B/R .

The equivalence class containing (i, x) is written $[i, x]$. We put

$$\begin{matrix} A_i & \xrightarrow{a_{i\infty}} & A_\infty \\ x & \longmapsto & [i, x] \end{matrix}$$

Then

$$\begin{matrix} A_i & \xrightarrow{a_{ij}} & A_j \\ & \searrow a_{i\infty} & \swarrow a_{j\infty} \\ & & A_\infty \end{matrix} \quad (i \in j)$$

since, for $x \in A_i$ and $y \stackrel{\text{def}}{=} a_{ij}(x)$, we have

$(i, x) R (j, y)$ ("by the choice $k=j$ "), thus

$$[i, x] = [j, y].$$

Next, we formulate and verify the universal property of our construction — but for the formulation, we go to a general context, one that will have further special cases that are useful for us.

7.1

Definition

Let \mathcal{A} be a category, \mathcal{I} another category and $\underline{A} : \mathcal{I} \rightarrow \mathcal{A}$ a functor (although we will call it a 'diagram').

(i) A cocone on \underline{A} is given by an object B of \mathcal{A} , and a natural transformation

$$\underline{A} \xrightarrow{\varphi} \ulcorner B \urcorner$$

i.e.:

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\underline{A}} & \mathcal{A} \\ & \downarrow \varphi & \\ \mathcal{I} & \xrightarrow{\quad} & \ulcorner B \urcorner \end{array}$$

where $\ulcorner B \urcorner : \mathcal{I} \rightarrow \mathcal{A}$ is the constant functor:

with value B :

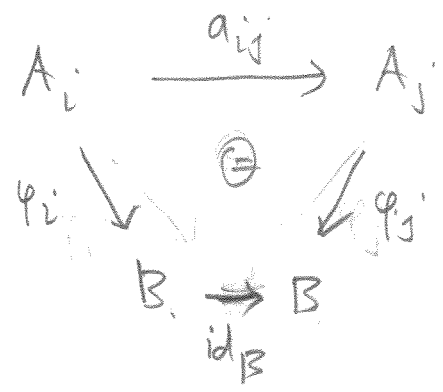
$$\begin{aligned} \overline{B}^1(i) &= B \quad \text{for all } i \\ (i \in \text{Ob}(\underline{I})) \end{aligned}$$

$$\overline{B}^1 \left(\begin{array}{c} i \\ \downarrow \alpha \\ j \end{array} \right) = \begin{array}{c} B \\ \downarrow \text{id}_B \\ B \end{array} \quad \text{for all } \alpha \in \text{Arr}(\underline{I}),$$

This means, in elementary terms, and using our previous notation, restricting ourselves to the special case of a poset $\underline{I} = (I, \leq)$, that:

$$\varphi = \langle \varphi_i : A_i \rightarrow B \rangle_{i \in I}$$

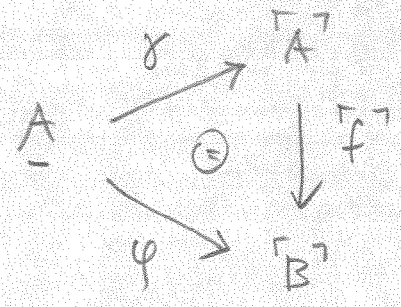
Such that (naturality!)



commutes.

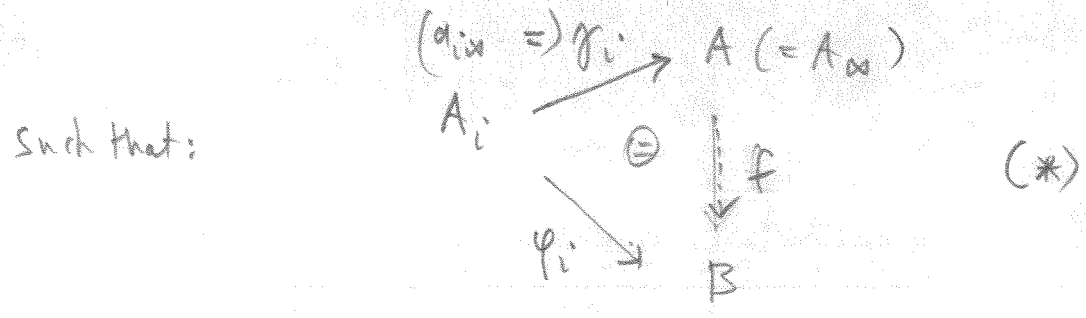
(ii) The cone $\underline{A} \xrightarrow{\gamma} \overline{A}$ is universal - and A is a colimit of the diagram \underline{A}

if the following holds: for an arbitrary cone $\underline{A} \xrightarrow{\varphi} \overline{B}$, there is a unique arrow $f: A \rightarrow B$ in \mathcal{A} such that



commutes ($\overline{f}: \overline{A} \rightarrow \overline{B}$ is the obvious natural transformation all whose components are equal to $f: A \rightarrow B$).

In practice, the above means that: \exists unique $f: A \rightarrow B$



commutes for all $i \in I$.

Let us check that in our construction in Set

above, we in fact have that $A_\infty = \text{colim}_{i \in I} A_i$

$= \text{colim } \underline{A}$, with the universal cocone

$\gamma = \langle a_{i\infty} : A_i \rightarrow A_\infty \rangle_{i \in I}$ (we also refer to

$\gamma_i : A_i \rightarrow A_\infty$ as (colimit) projections).

We need to define $f : A_\infty \rightarrow B$, i.e.,

$f([i, x])$ for all $i \in I, x \in A_i$. Obvious choice:

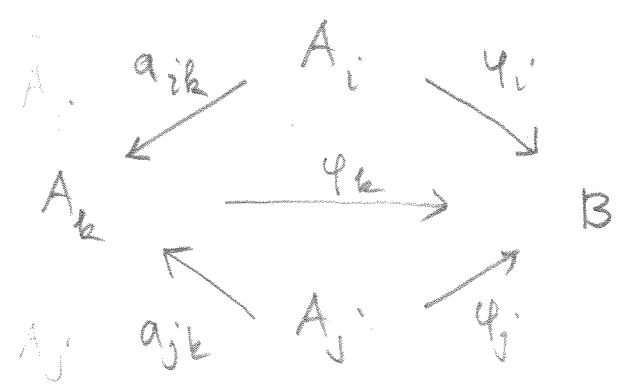
$$f([i, x]) \stackrel{\text{def}}{=} \varphi_i(x) \quad (*)$$

Is this well-defined? If $[i, x] = [j, y]$, we

have $i \leq k$ s.t. $i \leq k, j \leq k$ and $a_{ik}(x) = a_{jk}(y) = z$.

But, by the naturality of $\varphi : \underline{A} \rightarrow \underline{B}$, we

have



Commuting, showing

$$\varphi_i(x) = \varphi_k(a_{ik}(x)) = \varphi_k(z)$$

$$\varphi_j(y) = \varphi_k(a_{jk}(y)) = \varphi_k(z)$$

thus $\varphi_i(x) = \varphi_j(y)$ — as desired.

Also, clearly, f must satisfy $(*)$ — f is unique.

And, of course, $(*)$, p61 holds true.

There is an "internal" characterization of the colimit

cocone

$$\gamma = \langle a_{i\infty} \rangle_{i \in I} : A \longrightarrow \Gamma A_{\infty}$$

constructed on pages 55 and later:

1) "surjectivity": for any $x \in A_{\infty}$, there are $i \in I$ and $u \in A_i$ such that $x = a_{i\infty}(u)$;

2) "injectivity": whenever $i, j \in I$, $u \in A_i$

$v \in A_j$, and $a_{i\infty}(u) = a_{j\infty}(v)$, there is k such that

$i \leq k$ and $j \leq k$ and $a_{ik}(u) = a_{jk}(v)$.

One can see that, in the proof of the universal property of γ , all what we used was these facts about the construction.

We will prove - or, at least, sketch(?) a proof of - the fact that in the categories $\mathcal{A} = \text{Cat}$ (the category of small categories), and $\mathcal{A} = \text{Coh}$ directed colimits exist - moreover that they are calculated 'as in Set'. The latter means that, for instance, the object-set of the colimit category is the "corresponding" colimit of the object-sets of the ingredient categories (as it already said above).

I state the theorem that we need: - no more and no less!

7.2 Theorem Given a diagram

$$\mathbb{C} : \underline{I} \longrightarrow \text{Cat}$$

on a ^{non-empty} directed poset $\underline{I} = (I, \leq)$ (write \mathbb{C}_i for $\mathbb{C}(i)$, $i \in I$), and $F_{ij} : \mathbb{C}_i \rightarrow \mathbb{C}_j$ for $\mathbb{C}(i \leq j)$, there is $\mathbb{C}_\infty \in \text{Cat}$, and a cocone

$$\gamma = \langle F_{i\infty} : i \in I \rangle : \underline{I} \longrightarrow \mathbb{C}_\infty$$

such that :

1) ("surjectivity") : for every $X \in \text{Ob}(\mathbb{C}_\infty)$ there is i and $X' \in \text{Ob}(\mathbb{C}_i)$ such that $c_{i\infty}(X') = X$; and similarly for arrows;

2) ("injectivity") : whenever $i, j \in I$, and $X \in \mathbb{C}_i$ and $Y \in \mathbb{C}_j$, and $F_{i\infty}(X) = F_{j\infty}(Y)$, there is k such that $i \leq k$ and $j \leq k$ and $F_{ik}(X) = F_{jk}(Y)$ ($\in \mathbb{C}_k$).

Observe that the theorem does not mention the notion 'colimit'! However, in fact $\gamma : \underline{I} \longrightarrow \mathbb{C}_\infty$ is a colimit cocone - moreover, the properties described necessarily imply that this is so.

You will see(?) that the proof of the theorem will make good use of the universal property of the colimits in Set.

7.3 Corollary Suppose, in addition to the data in the theorem, that each $\mathcal{C}_i \in \text{Coh}$, and each functor $F_{ij} : \mathcal{C}_i \rightarrow \mathcal{C}_j$ is a coherent functor. Then \mathcal{C}_∞ is a coherent category, and each $F_{i\infty} : \mathcal{C}_i \rightarrow \mathcal{C}_\infty$ is a coherent functor.

The proof of the Corollary from the Theorem is completely elementary, but somewhat tedious.

I postpone the proof of both the Theorem and Corollary to the Appendix. The following three little lemmas also belong to the same circle of ideas; they are also postponed.

7.4 Lemma (i) Continuing the situation of the previous Corollary, let $X \xrightarrow{!_X} \mathbb{1}_{\mathcal{C}_\infty}$ be an e.e. morphism in \mathcal{C}_∞ , with codomain a terminal object of \mathcal{C}_∞ . Then: there are $i \in I$ and an e.e. morphism $!_{X'} : X' \longrightarrow \mathbb{1}_{\mathcal{C}_i}$ with terminal codomain in \mathcal{C}_i such that $F_{i\infty}(!_{X'}) = !_X$.

7.5 (ii) Suppose $U \vee V = \prod_{I} 1_{C_{\infty}}$ (U, V subobjects

of $1_{C_{\infty}}$). Then there is $i \in I$, $U', V' \in \text{Sub}(1_{C_i})$ such that $U' \xrightarrow{F_{iU}} U$, $V' \xrightarrow{F_{iV}} V$, and $U' \vee V' = \prod_{I} 1_{C_i}$.

(iii) Let $i_0 \in I$, and $A \xrightarrow{f} B$ an arrow in C_{i_0} .

Suppose that

for all $i \geq i_0$, the functor

$$F_{i_0 i} : C_{i_0} \longrightarrow C_i$$

is conservative for f ($F_{i_0 i}(f)$ iso $\Rightarrow f$ iso).

Then $C_{i_0, \infty} : C_{i_0} \longrightarrow C_{\infty}$ is also conservative for f .

8. Completion of the proof of the completeness theorem

By \mathbb{C} , I always mean a small coherent category.

8.1 Proposition. There exist: a small coherent category \mathbb{C}^* , a coherent functor $\Sigma (= \Sigma_{\mathbb{C}}^*) : \mathbb{C} \rightarrow \mathbb{C}^*$ with the following two properties:

- 1) Σ is conservative (for all arrows in \mathbb{C});
- 2) for every $X \in \text{Ob}(\mathbb{C})$ with full support ($X \xrightarrow{!x} \mathbb{1}_{\mathbb{C}}$ is e.e.), the object $\Sigma(X)$ in \mathbb{C}^* has a global element.

8.2 Proposition. Given a monomorphism $A \xrightarrow{m} B$ in \mathbb{C} , there exist: a small coherent category $\mathbb{C}^{\#} (= \mathbb{C}_{(m)}^{\#})$ and a coherent functor $\Sigma (= \Sigma_{\mathbb{C}, m}^{\#}) : \mathbb{C} \rightarrow \mathbb{C}^{\#}$ with the following two properties:

- 1) Σ is conservative for the arrow m in \mathbb{C} ;
- 2) for every pair (U, V) of subobjects of $\mathbb{1}_{\mathbb{C}}$ in \mathbb{C} such that $U \vee V = \prod_{\mathbb{1}_{\mathbb{C}}}$, at least one of the two subobjects $\Sigma(U), \Sigma(V)$ of $\mathbb{1}_{\mathbb{C}^{\#}}$ is equal to the top, $\prod_{\mathbb{1}_{\mathbb{C}^{\#}}}$.

Before saying anything about the proof of the propositions, I will use them to complete the proof of the completeness theorem (p. 10).

Let \mathcal{C} be a small coherent category, $m: A \twoheadrightarrow B$ a monomorphism in \mathcal{C} . We will prove:

there is $M: \mathcal{C} \rightarrow \text{Set}$
 M coherent, such that M is conservative for m :
 $M(m)$ isomorphism $\Rightarrow m$ is an isomorphism.

By (6.1) Proposition (p. 40), this will be sufficient.

Let $\mathcal{C}_0 \stackrel{\text{def}}{=} \mathcal{C}/B$, and $\Phi = \Phi_B: \mathcal{C} \rightarrow \mathcal{C}_0$.

Let $\hat{m} = \Phi(m) = m \times B: (A \times B, \pi_1) \rightarrow (B \times B, \pi_1)$.

Recursively, we define the coherent category \mathbb{C}_n for all $n \in \mathbb{N}$, together with the coherent functors

(69)

$$F_{kn} : \mathbb{C}_k \rightarrow \mathbb{C}_n \quad \text{for } k < n$$

($F_{nn} \stackrel{\text{def}}{=} \text{id}_{\mathbb{C}_n}$) satisfying the following: (i) & (ii):

(i) $\langle F_{kn} \rangle_{k < n}$ is compatible (a functor $\mathbb{N} \rightarrow \text{Coh}$):

$$\begin{array}{ccc}
 \mathbb{C}_k & \xrightarrow{F_{ke}} & \mathbb{C}_e \\
 & \searrow & \downarrow F_{en} \\
 & & \mathbb{C}_n \\
 & \searrow F_{kn} & \\
 & & \mathbb{C}_n
 \end{array}
 \quad \text{with } \textcircled{=} \text{ in the center}$$

$$F_{kn} = F_{en} \circ F_{ke} \quad (k < e < n)$$

(ii) $F_{0n} : \mathbb{C}_0 \rightarrow \mathbb{C}_n$ is conservative for \hat{m} .

\mathbb{C}_0 has been defined.

Suppose $n \in \mathbb{N}$, and the data for $k \leq n$ have all been defined. We need to define \mathbb{C}_{n+1} and the functors $F_{k, n+1} : \mathbb{C}_k \rightarrow \mathbb{C}_{n+1}$ ($k \leq n+1$).

Starting with \mathbb{C}_n , we apply the two "constructions" " $\mathbb{C} \mapsto \mathbb{C}^*$ ", and " $\mathbb{C} \mapsto \mathbb{C}_{(n)}^*$ " of the last two propositions, first the first, second the second, to arrive at \mathbb{C}_{n+1} .

Write $\hat{m}_n \stackrel{\text{def}}{=} F_{0n}(m)$, a mono in \mathbb{C}_n .

Applying 8.1, we have

$$\Sigma_{\mathbb{C}_n} : \mathbb{C}_n \longrightarrow \mathbb{C}_n^*$$

applying 8.2, we have

$$\Sigma_{\mathbb{C}_n^*, \hat{m}_n}^\# : \mathbb{C}_n^* \longrightarrow (\mathbb{C}_n^*)_{(\hat{m}_n)}^\#$$

Take the composite of these functors, to obtain

$$F_{n, n+1} : \mathbb{C}_n \longrightarrow \mathbb{C}_{n+1}$$

$$\text{(thus, } \mathbb{C}_{n+1} = (\mathbb{C}_n^*)_{(\hat{m}_n)}^\#)$$

As the composite of coherent functors, $F_{n, n+1}$ is coherent.

Define, for $k < n$, $F_{k, n+1} = F_{n, n+1} \circ F_{k, n}$; the $F_{k, n+1}$ are coherent and compatible (i)⊙ with $n+1$ for n .

Note that \forall ^{in general} if F is conservative for m , and G for $F(m)$, then $G \circ F$ is conservative for m . Since $F_{n, n+1}$ is the composite of functors with suitable conservativeness properties, by also using the induction hypothesis (ii)⊙ for n , we get (ii)⊙ for $n+1$.

This completes the construction of the diagram

(71)

$$\underline{C} : \mathbb{N} \longrightarrow \text{Coh}$$

where $\underline{C}(n) = C_n$, and $\underline{C}(k \leq n) = F_{kn}$.

\mathbb{N} is the classical directed order.

We define

$$C_\infty = \text{colim } \underline{C} = \text{colim}_{n \in \mathbb{N}} C_n$$

with

$$F_{n\infty} : C_n \longrightarrow C_\infty$$

the colimit coprojections.

7.2 Theorem with 7.3 Corollary (p.64, p.65) says that

C_∞ is a coherent category, $F_{n\infty}$ is a coherent functor ($n \in \mathbb{N}$).

7.4 (iii) (p.66) implies that $F_{0\infty} : C_0 \longrightarrow C_\infty$ is conservative for m .

CLAIM. C_∞ is complete and saturated (3.1; pp 12, 13)

To see that C_∞ is saturated, let $X \in C_\infty$ have

full support : $X \xrightarrow{!X} \mathbb{1}_{C_\infty}$ is e.e. By 7.4,

there are $n \in \mathbb{N}$, $X' \in C_n$, and $!X' : X' \rightarrow \mathbb{1}_{C_n}$

such that $F_{n\infty}(!X') = !X$. By the construction of \mathbb{C}_n^* (see 8.1), the object $\Sigma_{\mathbb{C}_n}^*(X')$ has a global element in \mathbb{C}_n^* . Therefore, the object

$$\Sigma_{\mathbb{C}_n^*, \hat{m}_n}^\# \left(\Sigma_{\mathbb{C}_n}^*(X') \right) = F_{n,n+1}(X') \in \mathbb{C}_{n+1}$$

also has a global element, and so does

$$X = F_{n\infty}(X') = F_{n+1,\infty}(F_{n,n+1}(X')) \text{ in } \mathbb{C}_\infty, \text{ which was to be proved.}$$

The proof that \mathbb{C}_∞ is complete is similar, using 7.4 (ii) (p.66).

CLAIM \square

The desired $M: \mathbb{C} \rightarrow \text{Set}$ is the composite

$$\mathbb{C} \xrightarrow{\Phi} \mathbb{C}/\mathcal{B} = \mathbb{C}_0 \xrightarrow{F_{0\infty}} \mathbb{C}_\infty \xrightarrow{\Gamma} \text{Set},$$

where Γ is the global-sections functor $\mathbb{C}_\infty(\mathbb{1}_{\mathbb{C}_\infty}, -)$.

Since the three factors are all coherent — Γ by "Summary", p 20; based on the CLAIM —, M is a model.

Write Σ for the composite

$$\mathcal{C}/B \xrightarrow{F_{0\omega}} \mathcal{C}_\omega \xrightarrow{\Gamma} \text{Set}.$$

and remember $\hat{m} := ((A, m) \xrightarrow{m} (B, 1_B))$ in \mathcal{C}/B .

$F_{0\omega}$ is conservative for \hat{m} , and Γ is conservative for

$F_{0\omega}(\hat{m})$, since $F_{0\omega}(\hat{m})$ is a monomorphism into a

terminal object in \mathcal{C}_ω , and we have 6.5 lemma (p 49).

Thus Σ is conservative for \hat{m} . We have the

situation of 6.2' Proposition (p 44.1), and we

can conclude that $\Sigma \overline{\Phi}_B = M: \mathcal{C} \rightarrow \text{Set}$

is conservative for m .

Q.E.D

The proofs of the 'Constructives' 8.1 and 8.2

are similar to the last proof - but they will

use ordinals, indexings by ordinals, and transfinite

recursion. For instance, for the proof of 8.1,

we start with an enumeration

$$\langle X_\beta \rangle_{\beta < \alpha}$$

of all objects of \mathcal{C} with full support, and

construct an ordinal-based diagram $\langle \mathbb{C}_\beta \rangle_{\beta \leq \alpha}$ (74)

$F_{\gamma\beta} : \mathbb{C}_\gamma \rightarrow \mathbb{C}_\beta$ ($\gamma < \beta < \alpha$) of coherent categories

and functors, where the main step of the construction,

$$F_{\beta, \beta+1} : \mathbb{C}_\beta \rightarrow \mathbb{C}_{\beta+1} \quad (\beta < \alpha)$$

will take $\mathbb{C}_{\beta+1} = \mathbb{C}_\beta / \hat{X}_\beta$ for $\hat{X}_\beta = F_{0\alpha}(X_\beta)$

$$\text{and } F_{\beta, \beta+1} = \Phi^{(\mathbb{C}_{\beta+1})} : \mathbb{C}_\beta \rightarrow \mathbb{C}_\beta / \hat{X}_\beta$$

thereby ensuring that $\hat{X}_{\beta+1} \stackrel{\text{def}}{=} F_{\beta, \beta+1}(\hat{X}_\beta) = F_{0, \beta+1}(X_\beta)$

has a global element (see p. 39.1).