5. The universal property of the slice

As before, $\mathbb{C}$ is a coherent category, X a fixed object of $\mathbb{C}$; we consider $\Phi = \Phi_X : \mathbb{C} \longrightarrow \mathbb{C}/X$ discussed above.

To abbreviate, let's write $\mathbb{D}$ for $\mathbb{C}/X$, and use 'dot' for the effect of Φ : $\dot{Y} = \Phi(Y)$; for $Y \xrightarrow{f} Z$, $\dot{f} = \Phi(f)$.

The first – and main – observation is that the object $\dot{X}$ has a <u>global</u> <u>element</u> $1_{\mathbb{D}} \xrightarrow{c_X} \dot{X}$ in $\mathbb{D}$.

As we know, $1_{\mathbb{D}}$ can be chosen as $(X, 1_X)$; we have

$$\begin{tikzcd} X \arrow[rr, "\langle 1_X, 1_X \rangle"] \arrow[dr, "1_X"'] & \arrow[d, phantom, "="] & X \times X \arrow[dl, "\pi_1"] \\ & X & \end{tikzcd}$$

thus $\boxed{c_X = \underbrace{\Delta_X}_{\text{usual notation for } \langle 1_X, 1_X \rangle}} : (X, 1_X) \longrightarrow (X \times X, \pi_1)$

$c_X : 1_{\mathbb{D}} \longrightarrow \dot{X}$ is a <u>generic</u> element of X in the sense that I will now explain — although we do not use this fact explicitly in our further work.

Using the generic element c_X, and the objects and arrows of $\mathbb{C}/X$ that come from $\mathbb{C}$ via Φ, we can 'generate' all of the structure of $\mathbb{C}/X$. Let (Y, y) be an arbitrary object of $\mathbb{C}/X$, and consider the following commutative diagram in $\mathbb{C}$:

$$\begin{tikzcd}
Y \arrow[rr, "\langle 1_Y, y\rangle"] \arrow[dr, "y"] \arrow[dd, "y"'] & & Y \times X \arrow[dl, "\pi_1"'] \arrow[dd, "y \times X"] \\
& X & \\
X \arrow[ur, "1_X"] \arrow[rr, "\Delta_X"'] & & X \times X \arrow[ul, "\pi_1"]
\end{tikzcd}$$

The outside square in a pullback — the checking of this ~~is~~ omitted (<u>not really</u>: see pages 43.1 and following!).

This means that we have the following pullback diagram in $\mathbb{C}/X$:

$$\begin{tikzcd}
(Y, y) \arrow[r, "\langle 1_Y, y\rangle"] \arrow[d, "y"'] & \dot{Y} \arrow[d, "\dot{y}"] \\
\mathbb{1}_{\mathbb{D}} \arrow[r, "c_X"'] & \dot{X}
\end{tikzcd}$$

Now, assume that we have a coherent functor $F : \mathbb{C} \longrightarrow \mathbb{E}$, and a global element $\mathbb{1}_{\mathbb{E}} \xrightarrow{e} F(X)$ of $F(X)$.

I claim that we can <u>extend</u> F to $G : \mathbb{C}/X \longrightarrow \mathbb{E}$ such that G is coherent, $G\Phi \cong F$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\Phi"] \arrow[dr, "F"'] & \mathbb{C}/X \arrow[d, "G"] \arrow[dl, phantom, "\cong"] \\
& \mathbb{E}
\end{tikzcd}$$

by a natural isomorphism

$$\mu : G\Phi \xrightarrow{\cong} F$$

and also G maps $c_X : 1_{\mathbb{D}} \longrightarrow \Phi(X)$ to $e : 1_{\mathbb{E}} \longrightarrow F(X)$ modulo the isomorphism μ, that is:

$$\begin{tikzcd}
G(1_{\mathbb{D}}) \arrow[r, "G(c_X)"] \arrow[d, "!"'] \arrow[d, phantom, shift left=3ex, "\cong"] \arrow[dr, phantom, "="] & G\Phi(X) \arrow[d, "\mu_X"] \arrow[d, phantom, shift right=3ex, "\cong"] \\
1_{\mathbb{E}} \arrow[r, "e"] & F(X)
\end{tikzcd}$$

commutes.

The pullback in $\mathbb{C}/X$ on the previous page will have to be mapped to a pullback

$$\begin{tikzcd}
G((Y,y)) \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, very near start, "\lrcorner"] & G\Phi(Y) \arrow[d, "G\Phi(y)"] \\
G(1_{\mathbb{D}}) \arrow[r, "G(c_X)"'] & G\Phi(X)
\end{tikzcd}$$

which, combined with isomorphisms:

$$\begin{tikzcd}
G((Y,y)) \arrow[r] \arrow[d, "!"'] & G\Phi(Y) \arrow[r, "\mu_Y", "\cong"'] \arrow[d, "G\Phi(y)"'] \arrow[dr, phantom, "="] & F(Y) \arrow[d, "F(y)"] \\
G(\mathbb{1}_{\mathbb{D}}) \arrow[r, "G(c_X)"'] \arrow[d, "!"', "\cong"] \arrow[dr, phantom, "="] & G\Phi(X) \arrow[r, "\mu_X", "\cong"'] \arrow[d, "\mu_X", "\cong"'] & F(X) \arrow[dl, equal] \\
\mathbb{1}_{\mathbb{E}} \arrow[r, "e"'] & F(X) &
\end{tikzcd}$$

gives rise to pullback in $\mathbb{E}$:

$$\begin{tikzcd}
G((Y,y)) \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(Y) \arrow[d, "F(y)"] \\
1_{\mathbb{E}} \arrow[r, "e"'] & F(X)
\end{tikzcd} \qquad (*)$$

This means that $G((Y,y))$ is determined, at least up an isomorphism, by the requirements on G.

To actually define G, we pick, for any object

(Y, y) of $\mathbb{C}/X$, a pullback diagram $(*)$

(thus, not only the object $G((Y,y))$, but also

the morphism $G((Y,y)) \longrightarrow F(Y)$). It is seen

that the action of G on morphisms $(Y,y) \xrightarrow{f} (Z,z)$

will be uniquely determined so that the

diagrams

$$\begin{tikzcd} G((Y,y)) \arrow[r] \arrow[d, "G(f)"'] \arrow[dr, phantom, "\ominus"] & F(Y) \arrow[d, "F(f)"] \\ G((Z,z)) \arrow[r] & F(Z) \end{tikzcd} \qquad (*)$$

all commute.

Let me show why G preserves e.e. morphisms

Let $f : (Y,y) \longrightarrow (Z,z)$ be e.e. in $\mathbb{C}/X$.

The reason why $G(f)$ is e.e. is that $(*)$ is

actually a pullback, and, by assumption on F, $F(f)$

is e.e., and $\mathbb{E}$ is coherent. The fact that

$(*)$ is a pullback comes from the fact that

the two squares on the left in what follows

compose to the quadrangle on the right:

$$\begin{tikzcd} G((Y,y)) \arrow[r] \arrow[d, "G(f)"'] \arrow[dr, phantom, "①"] & F(Y) \arrow[d, "F(f)"] \\ G((Z,z)) \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "②"] & F(Z) \arrow[d, "F(z)"] \\ \mathbb{1}_{\mathbb{E}} \arrow[r] & F(X) \end{tikzcd} \qquad \begin{tikzcd} G((Y,y)) \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "③"] & F(Y) \arrow[d, "F(y)"] \\ \mathbb{1}_{E} \arrow[r] & F(X) \end{tikzcd}$$

(since $\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "y"'] & \arrow[d, phantom, "="] & Z \arrow[dl, "z"] \\ & X & \end{tikzcd}$, and there is just one

arrow $G((Y,y)) \longrightarrow \mathbb{1}_{\mathbb{E}}$)

and we have the general (and widely used rule) that if in a situation like the above, with knowing that ① commutes, and ② and ③ are pullbacks, then so is ①.

I will not pursue the genericity of $c_X : \mathbb{1}_{\mathbb{C}/X} \to \Phi(X)$ any more, since in the completeness proof this fact is not used.

6. Conservativeness

(6.1) Let $\mathbb{C}$, $\mathbb{S}$ be categories, $M : \mathbb{C} \longrightarrow \mathbb{S}$ a functor, $\underset{\sim}{A}$ a class of arrows in $\mathbb{C}$.

Let us say that M is <u>conservative</u> for $\underset{\sim}{A}$ if for all $f \in \underset{\sim}{A}$, $M(f)$ is an isomorphism in $\mathbb{S}$ (if and) only if f is an isomorphism in $\mathbb{C}$.

<u>Proposition</u>. Assume $\mathbb{C}$ has finite limits, and M preserves finite limits. Assume M is conservative for (the class of) monomorphisms in $\mathbb{C}$. Then: M is conservative for (the class of) all arrows in $\mathbb{C}$.

<u>Proof</u>. Consider an arrow $A \xrightarrow{f} B$ in $\mathbb{C}$.

Consider:

$$\begin{tikzcd}
& & A \arrow[dr, "f"] & \\
A \arrow[r, dashed, "m"] \arrow[urr, bend left, "1_A"] \arrow[drr, bend right, "1_A"'] & K \arrow[ur, "k_0"] \arrow[dr, "k_1"'] & & B \\
& & A \arrow[ur, "f"'] &
\end{tikzcd} \qquad (*)$$

Here, (K, k_0, k_1) is the so-called kernel-pair of f:

the pullback of f against itself. The morphism $m : A \to K$ is defined by using the universal property of K such that $k_0 m = k_1 m = 1_A$. m is a monomorphism: if $B \overset{g}{\underset{h}{\rightrightarrows}} A$, and $mg = mh$, then $k_0 mg = k_0 mh$, hence, $g = h$. Also, m is an isomorphism iff f is a monomorphism. Namely, if f is a monomorphism, then k_0, k_1 are isomorphisms (the pullback diagram is isomorphic to

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "1_A"] \arrow[dr, "1_A"'] & & B \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

in this case), and so m is an isomorphism. And if m is an iso, so are k_0 and k_1, and f is a mono.

Now, assume $M(f)$ is an isomorphism. Consider the image of $(*)$ under M in $\mathbb{S}$. It is the same construct from $M(f)$ in $\mathbb{S}$ as $(*)$ was from f in $\mathbb{C}$. Therefore, $M(m)$ is an isomorphism, since $M(f)$ is, in particular, a monomorphism. But m is a mono, and M is conservative for mono's; therefore, m is an isomorphism. But then, by what we said above, f is a monomorphism. Now, since M is conservative for mono's, and $M(f)$ is an isomorphism, it follows that f is an isomorphism.

I now generalize the concept a bit. Suppose $\underline{M}$ is a class of functors $M : \mathbb{C} \longrightarrow \mathbb{S}_M$ from the fixed category $\mathbb{C}$ to – possibly – variable categories $\mathbb{S}_M$, $M \in \underline{M}$. I will say that $\underline{M}$ is conservative for $\underset{\sim}{A}$ ($\underset{\sim}{A}$: as before) if

for all $f \in \underset{\sim}{A}$, $M(f)$ is an isomorphism

for all $M \in \underline{M}$ iff f is an isomorphism in $\mathbb{C}$.

The above proposition is generalized, with essentially unchanged proof, to the version obtained by replacing $\mathbf{M}$ by $\underline{\mathbf{M}}$ three times (first, it should be: "every $M \in \underline{M}$ preserves finite limits"). This generalization is not really a generalization, since, with given $\underline{M}$, I can consider the single

$$\tilde{\underline{M}} : \mathbb{C} \longrightarrow \prod_{M \in \underline{M}} \mathbb{S}_M$$

and the "generalization" is reduced to the original version.

Looking at the "Gödel completeness theorem", p 10: by the present Proposition, it suffices to prove it for all monomorphisms $f : A \to B$.

(6.2) Lemma. Let $A \overset{m}{\rightarrowtail} B$ be a monomorphism in $\mathbb{C}$. Then $\Phi_B = \Phi : \mathbb{C} \longrightarrow \mathbb{C}/B$ is conservative for m.

<u>Proof.</u> We use the 'generic element'

$$c_B : \mathbb{1}_{\mathbb{C}/B} \longrightarrow \Phi(B)$$

— and, actually, this proof will be used, rather than the lemma itself when it comes to the crunch.

I start with the fact that in $\mathbb{C}$, for an arbitrary arrow $m : A \to B$ in $\mathbb{C}$, the following in a pullback diagram — and for a change, I will verify it:

$$\boxed{\begin{tikzcd}
A \times B \arrow[r, "m \times B"] & B \times B \\
A \arrow[u, "\langle 1_A, m\rangle"] \arrow[r, "m"'] \arrow[ur, phantom, "pb"] & B \arrow[u, "\Delta_B = \langle 1_B, 1_B\rangle"']
\end{tikzcd}}$$

'Same pullback as the one on p. 39.2!'

Suppose we have $f : X \to A \times B$, $g : X \longrightarrow B$ such that

$\boxed{(m \times B) f = \Delta_B g :}$? is there h as required by ? ?

$$\begin{tikzcd}
 & A \times B \arrow[r, "m \times B"] & B \times B \\
 & A \arrow[u, "\langle 1_A, m\rangle"'] \arrow[u, phantom, "1 (=) ?" left, shift right=4ex] \arrow[r, "m"] & B \arrow[u, "\Delta_B"'] \\
X \arrow[uur, bend left, "f"] \arrow[ur, dashed, "?h"] \arrow[urr, bend right, "g"'] \arrow[ur, phantom, "?" near end, shift right=2ex] \arrow[urr, phantom, "2 (=) ?"] & &
\end{tikzcd}$$

The required commutativity 1 $=$ when followed by the projection π_0 determines h:

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "1 ="] & A \arrow[dl, "\langle 1_A, m\rangle"] \arrow[ddl, bend left=60, "1_A"] \arrow[ddl, phantom, "="] \\
A\times B \arrow[d, "\pi_0"] & \\
A &
\end{tikzcd} \qquad (*)$$

h <u>must</u> be $h = \pi_0 f$.

The assumed equality (box, previous page), when followed by the two projections $B\times B \overset{\pi_0'}{\underset{\pi_1'}{\rightrightarrows}} B$ gives us:

$$\begin{tikzcd}
 & & B \\
A\times B \arrow[urr, "m\pi_0"] \arrow[r, "m\times B"] & B\times B \arrow[ur, "\pi_0'"] & \\
X \arrow[u, "f"] \arrow[r, "g"'] & B \arrow[u, "\Delta_X"] \arrow[uur, "1_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & B \\
A\times B \arrow[urr, "\pi_1"] \arrow[r, "m\times B"] & B\times B \arrow[ur, "\pi_0'"] & \\
X \arrow[u, "f"] \arrow[r, "g"'] & B \arrow[u, "\Delta_X"] \arrow[uur, "1_B"'] &
\end{tikzcd}$$

$$\boxed{m\pi_0 f = g} \qquad\qquad \boxed{\pi_1 f = g}$$

We verify that $h = \pi_0 f$ indeed satisfies 1 $\equiv$ and 2 $\equiv$:

$$\langle 1_A, m\rangle h \overset{?}{=} f \qquad (1)$$

$$m\, h \overset{?}{=} g \qquad (2)$$

(1) is the equality of two arrows into the product $A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$; it is equivalent to the conjunction of the following two

$$\pi_0 \langle 1_A, m\rangle h \overset{?}{=} \pi_0 f \qquad (1.1)$$

$$\pi_1 \langle 1_A, m\rangle h \overset{?}{=} \pi_1 f \qquad (1.2).$$

(1.1) was used, to get $h = \pi_0 f$. But $\pi_1 \langle 1_A, m\rangle = m$ thus, (1.2) is reduced to

$$m\, h \overset{?}{=} \pi_1 f \qquad (1.2')$$

With $h = \pi_0 f$, (2) is $m\pi_0 f = g$ – which is true (see previous page), and (1.2') is $m\pi_0 f \overset{?}{=} \pi_1 f$ – and this holds, since both sides equal g (see previous page).

Next, we can complete the diagram to something taking place in $\mathbb{C}/B$:

$$\begin{tikzcd}
A \times B \arrow[rr, "m \times B"] \arrow[dr, "\pi_1"] & & B \times B \arrow[dl, "\pi_1'"'] \\
& B & \\
A \arrow[uu, "\langle 1_A, m \rangle"] \arrow[ur, "m"] \arrow[rr, "m"] & & B \arrow[uu, "\Delta_B"'] \arrow[ul, "1_B"']
\end{tikzcd}$$

since all four triangles commute.

Therefore, we have the following pullback in $\mathbb{C}/B$:

$$\begin{tikzcd}
\Phi(A) \arrow[r, "\Phi(m)"] & \Phi(B) \\
(A, m) \arrow[u, "\langle 1_A, m \rangle"] \arrow[r, "m"'] \arrow[ur, phantom, "pb"] & \mathbb{1}_{\mathbb{C}/B} \arrow[u, "\Delta_B"']
\end{tikzcd} \qquad (*)$$

This is the pullback in $\mathbb{C}/B$ that we are going to use again. However, the assertion of 6.2 Lemma is clear: if $\Phi(m)$ is an isomorphism, so is its pullback $m : (A, m) \to \mathbb{1}_{\mathbb{C}/B}$ in $\mathbb{C}/B$ — and so is $m : A \to B$ in $\mathbb{C}$.

I must put here the conservativeness argument that uses the last-mentioned pullback.

6.2′ Proposition Suppose: $\mathbb{C}$ is coherent,

$$m : A \rightarrowtail B \text{ is a mono in } \mathbb{C};$$

and we have

$$\Sigma : \mathbb{C}/B \longrightarrow \mathbb{S},$$

a coherent functor that is conservative for the following subobject of the terminal object of $\mathbb{C}/B$:

$$(A, m) \xrightarrow{m} \mathbb{1}_{\mathbb{C}/B} = (B, 1_B)$$

$$\begin{tikzcd} A \arrow[rr, "m"] \arrow[dr, "m"'] & \arrow[d, phantom, "\ominus"] & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

Then the composite $\Sigma \Phi_B : \mathbb{C} \longrightarrow \mathbb{S}$ is conservative for $m : A \to B$ in $\mathbb{C}$

$$(\mathbb{C} \xrightarrow{\Phi_B} \mathbb{C}/B \xrightarrow{\Sigma} \mathbb{S}).$$

Proof. Suppose $\Sigma \Phi_B(m)$ is an isomorphism in $\mathbb{S}$, to derive that $m : A \rightarrowtail B$ is an iso in $\mathbb{C}$.

Consider the pullback (*) on p. 44, in $\mathbb{C}/B$, and apply the coherent functor Σ to it, obtaining the pullback:

$$\begin{tikzcd}
\Sigma\,\Phi(A) \arrow[rr, "\Sigma\,\Phi(m)", "\cong"'] & & \Sigma\,\Phi(B) \\
\Sigma(A, m) \arrow[u, "\Sigma(\langle 1_A, m\rangle)"] \arrow[rr, "\Sigma(m)"'] \arrow[urr, phantom, "\mathrm{pb}"] & & \Sigma(\mathbb{1}_{\mathbb{C}/B}) \arrow[u, "\Sigma(\Delta_B)"']
\end{tikzcd}$$

in the category $\mathcal{S}$.
Since the pullback of an isomorphism is an isomorphism,

$\Sigma(m) : \Sigma(A, m) \longrightarrow \Sigma(\mathbb{1}_{\mathbb{C}/B})$ is an isomorphism.

We have assumed that Σ is conservative for $m : (A, m) \to \mathbb{1}_{\mathbb{C}/B}$ in $\mathbb{C}/B$.

Therefore, $m : (A, m) \longrightarrow \mathbb{1}_{\mathbb{C}/B} = (B, 1_B)$ is an isomorphism in $\mathbb{C}/B$. Applying the forgetful functor

$$\Psi_B : \mathbb{C}/B \longrightarrow \mathbb{C}$$

to this, we get that $m : A \to B$ is an isomorphism in $\mathbb{C}$, as desired.

6.3 Lemma. Suppose X has global support: $X \xrightarrow{!_X} \mathbb{1}$ is e.e. (in $\mathbb{C}$). Then

$$\Phi_X : \mathbb{C} \longrightarrow \mathbb{C}/X \quad \text{is conservative}$$

(for all arrows in $\mathbb{C}$; enough: for mono's in $\mathbb{C}$).

proof: almost the same as before. Let $m : A \rightarrowtail B$ be any mono in $\mathbb{C}$. Having an inverse g of $\Phi(m)$ means

$$\begin{tikzcd}
A \times X \arrow[dr, "\pi_1"'] \arrow[rr, shift right, "m \times X"'] & & B \times X \arrow[ll, shift right, "g"'] \arrow[dl, "\pi_1'"] \\
& B &
\end{tikzcd}$$

$(m \times X) \circ g = 1_{B \times X}$, $\pi_1 g = \pi_1'$. Let $f = \pi_0 g$, for $\pi_0 : A \times X \longrightarrow A$. We have

$$\begin{tikzcd}
 & B \times X \arrow[ddl, bend right, "f"'] \arrow[d, "g"] \arrow[dr, "1_{B\times X}"] \arrow[d, phantom, shift right=6ex, "="] \arrow[dr, phantom, shift right=1ex, "="'] & \\
 & A \times X \arrow[r, tail, "m \times B"] \arrow[dl, "\pi_0"] \arrow[dr, phantom, "="] & B \times X \arrow[dl, "\pi_0"] \\
A \arrow[r, tail, "m"'] & B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \times X \arrow[rr, "f"] \arrow[dr, "\pi_0"'] \arrow[rr, phantom, shift right=4ex, "="] & & A \arrow[dl, tail, "m"] \\
 & B &
\end{tikzcd} \qquad (*)$$

Now,

$$\begin{tikzcd}
B \times X \arrow[r] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "!_X"] \\
B \arrow[r, "!_B"'] & 1
\end{tikzcd}$$

is a pullback. Therefore, since $!_X$ is assumed to be e.e., so is π_0. From $(*)$, m must be an isomorphism.

6.4 Lemma. Suppose U and V are subobjects of $\mathbb{1} = \mathbb{1}_{\mathbb{C}}$ and $A \overset{m}{\rightarrowtail} B$ a monomorphism. Suppose $U \vee V = \top$. If $\Phi_U(m)$ and $\Phi_V(m)$ are both isomorphisms in $\mathbb{C}/U$, resp., in $\mathbb{C}/V$, then m is an isomorphism.

Proof. Of course, the monomorphism $U \rightarrowtail \mathbb{1}$ is the unique $!_U : U \to \mathbb{1}$; the assumption is that $!_U$ is a monomorphism; the same for $!_V : V \to \mathbb{1}$. Then $A \times U \xrightarrow{\pi_0} A$ is a monomorphism, since

$$\begin{tikzcd}
A \times U \arrow[rr, tail, "\pi_0"] \arrow[dr, tail, "A \times !_U"'] & & A \\
& A \times \mathbb{1} \arrow[ur, "\pi_0"', "\cong"] &
\end{tikzcd}$$

Similarly for other π_0 projections that appear below.

Consider:

$$\begin{tikzcd}
A \times U \arrow[r, "m \times U"] \arrow[d, tail, "\pi_0"'] & B \times U \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & U \arrow[d, tail] \\
A \arrow[r, tail, "m"] & B \arrow[r, "!_B"'] & \mathbb{1} \\
A \times V \arrow[r, "m \times V"'] \arrow[u, tail, "\pi_0"] & B \times V \arrow[r, "\pi_1"'] \arrow[u, tail, "\pi_0"'] & V \arrow[u, tail]
\end{tikzcd}$$

All of the four quadrangles are pull backs

$\Phi_U(m) = m \times U$; $\Phi_V(m) = m \times V$; they are

both isomorphisms. Hence the monomorphisms

$A \times U \xrightarrow[\cong]{m \times U} B \times U \xrightarrow{\pi_0} B$, and $B \times U \xrightarrow{\pi_0} B$

determine the same subobject of B; call it $\hat{U} \in \mathrm{Sub}(B)$,

similarly for $\hat{V} = [A \times V, \pi_0(m \times V)] = [B \times V, \pi_0] \in \mathrm{Sub}(B)$

Now, $\hat{U} = [B \times U, \pi_0] \in \mathrm{Sub}(B)$ equals

$$\hat{U} = (!_B)^* U$$

for the map $!_B : B \to \mathbb{1}$; similarly

$$\hat{V} = (!_B)^* V .$$

Then $\underset{\substack{\uparrow \\ \text{sup in } \mathrm{Sub}(B)}}{\hat{U} \vee \hat{V}} = (!_B)^* U \vee (!_B)^* V =$

$$= (!_B)^* (U \vee V) = (!_B)^* (\top_{\mathbb{1}})$$

$$= \top_B$$

But the subobject $\hat{A} = [A, m]$ of B has the property

that $\hat{U}, \hat{V} \leq \hat{A}$, because $\hat{U} = [A \times U, m\pi_0]$, $\hat{V} = [A \times V, m\pi_0]$.

Therefore, $\hat{U} \vee \hat{V} \leq \hat{A}$, i.e.,

$$\top_B \leq \hat{A}, \quad \top_B = \hat{A},$$

which means that m is an isomorphism.

(6.5) Lemma. $\Gamma = \mathbb{C}(\mathbb{1}, -) : \mathbb{C} \longrightarrow \mathbf{Set}$

is conservative for all mono's $U \overset{m}{\rightarrowtail} \mathbb{1}$

into $\mathbb{1}$.

<u>proof</u>: This is immediate: $\Gamma(m)$ being an isomorphism

means, at least, that there is an arrow $\Gamma(\mathbb{1}) \to \Gamma(U)$

in Set. Since $\Gamma(\mathbb{1}) \cong \mathbb{1}_{Set} \cong \{*\}$, this means that

there is an element of $\Gamma(U)$, i.e. an arrow $\mathbb{1}_{\mathbb{C}} \to U$

in $\mathbb{C}$. We know (see: middle of p.15, e.g.) that

this means that m is an isomorphism.

7. Directed colimits

The proof of Gödel completeness – our goal – requires the repeated application of the slice construction discussed above, and the assembling of the resulting infinitely many coherent categories and coherent functors into one final one, one which is complete and saturated in the sense given in section 3. The 'assembling' is done, formally, in the way of a <u>directed</u> <u>colimit</u> of coherent categories and functors. The directed colimit is constructed entirely in the category of sets; e.g., the set of objects of the colimit is the colimit of the ~~object-sets~~ of their ingredient categories; the same for the set of arrows. The success of the operation (mainly, but not exclusively, the fact that the resulting category is in fact coherent) depends on a property of directed colimits in <u>Set</u>, the category of sets, which fact can be stated by saying that directed colimits commute with finite limits in <u>Set</u>. This is a very fundamental fact that has far-reaching application in category theory, namely, in category theory <u>after</u> Saunders MacLane's 'bible', the "Categories for the working mathematician" (1971). Said fundamental fact is duly proved towards the end of the book (Chapter IX: Special Limits, section 2.: Interchange of Limits, Theorem 1, p211), so that one can easily imagine a "second volume" of the book, starting with the Gabriel-Ulmer theory of locally presentable categories, in which said fundamental fact is at the heart of the matter. — and going on to accessible categories if you like. If you visit the spot in MacLane's book, you'll find, I think, that the half-page proof is a bit breezy — understandable, since the theorem is not put to any serious use yet in the book! I will studiously avoid <u>using</u> the theorem; instead, I show the details of the construction of the directed colimit, and some details of the proof of

Mac Lane's theorem, details that I actually use. The Diligent Reader can then go to the book and acquire a better understanding of the Theorem 1 (p211), better than just by running through Mac Lane's words (unless the D.R. is a different intelligence from mine).

Let $(I, \leq)$ be a (non-empty!) partially ordered set ($\leq$ is reflexive, transitive and anti-symmetric) which is also directed: for all $i, j \in I$, there is $k \in I$ such that $i \leq k$ and $i \leq k$:

$$\begin{tikzcd} i \arrow[dr] & \\ & k \\ j \arrow[ur] & \end{tikzcd}$$

I use the arrow notation ($i \to k$ for $i \leq k$) since, of course, $(I, \leq)$ is in fact a (rather special) category.

Let $\mathbb{A}$ be a category: think, first, of $\mathbb{A}$ = Set, but later others, e.g. $\mathbb{A}$ = Cat or $\mathbb{A}$ = Coh.

Let $\underline{A} : \underset{\sim}{\underline{I}} \longrightarrow \mathbb{A}$ be a functor: or, as

$\underline{I} = (I, \leq)$

we say, a diagram; $\underline{A}$ consists of objects $\underline{A}(i) = A_i$ of $\mathbb{A}$, and arrow $\underline{A}(i \leq j) = a_{ij} : A_i \to A_j$ such that $a_{ii} : A_i \to A_i$ is $a_{ii} = 1_{A_i}$, and for $i \leq j \leq k$,

$$\begin{tikzcd} & A_j \arrow[dr, "a_{jk}"] & \\ A_i \arrow[ur, "a_{ij}"] \arrow[rr, bend right, "a_{ik}"'] & \circleddash & A_k \end{tikzcd}$$

commutes: $a_{ik} = a_{jk} \circ a_{ij}$.

$\boxed{\text{Example}}$: $I = \mathbb{N}$ = the set of natural numbers;

$\leq$: usual ordering; $\mathbb{A} = Set$;

A_i : set for each $i \in \mathbb{N}$

and let

$a_{ij} : A_i \longrightarrow A_j$

be an inclusion: $A_i \subseteq A_j$, and a_{ij} maps $x \in A_i$ to x itself in A_j.

Then, yes, we have a diagram $\underline{A} : \underline{I} \to Set$.

Now, it is very natural to consider

$$\bigcup_{i \in \mathbb{N}} A_i$$

the union of the sets, as a kind of "limit" (we will say: "colimit") of the diagram $\langle A_i \rangle_{i \in \mathbb{N}}$. This is an example for the 'directed colimit' we are advocating here. Note that in this example we have the further inclusions

$$a_{i\infty} : A_i \longrightarrow \bigcup_{j \in \mathbb{N}} A_j$$

and, even,

$$\begin{tikzcd} A_i \arrow[rr, "a_{ij}"] \arrow[dr, "a_{i\infty}"'] & \arrow[d, phantom, "\circleddash"] & A_j \arrow[dl, "a_{j\infty}"] \\ & \bigcup_j A_j & \end{tikzcd}$$

commuting — as of course all diagrams consisting of inclusions will commute – if they are of the right shape.

Returning to a general (always non-empty!) directed poset $(I, \leq) = \underline{I}$ and a general $\underline{I}$-based diagram $\underline{A} : \underline{I} \to \mathrm{Set}$ in the category of sets ($\mathbb{A} = \mathrm{Set}$), now we construct a colimit

$A_\infty = \operatorname{colim}_{i \in I} A_i$, a set, together with

maps $A_i \xrightarrow{a_{i\infty}} A_\infty$ such that

$$\begin{tikzcd}
A_i \arrow[rr, "a_{ij}"] \arrow[dr, "a_{i\infty}"'] & \arrow[d, phantom, "="] & A_j \arrow[dl, "a_{j\infty}"] \\
& A_\infty &
\end{tikzcd} \qquad (*)$$

commutes. We *will* do this construction for an arbitrary poset $(I, \leq)$, without assuming directedness, or even for an arbitrary *category* (small) $\underline{I}$ and diagram (functor) $\underline{A} : \underline{I} \to \mathrm{Set}$ — but that would not be useful now, since the special features of the construction for the directed case are crucial for the applications. The colimit will have a *universal* property which determines it up to isomorphism — and which will also be very useful, although *not enough*: the universal property by itself will not suffice for our goals, the actual construction (or, if you wish, Mac Lane's Theorem 1 (p 211)) is necessary to keep in mind.

Here is the construction.

For each $i \in I$ and $x \in A_i$, we will have an element $a_{i\infty}(x)$ in A_∞; thus, we will have a map $(i,x) \longmapsto a_{i\infty}(x)$; moreover, if

$$A_i \xrightarrow{a_{ij}} A_j$$

then because of the required commutativity (*), p 54, I will have to have, for $y = a_{ij}(x)$

$$\begin{tikzcd}
(i,x) \arrow[dr, mapsto, "a_{i\infty}"] & \\
& \text{same} \\
(j,y) \arrow[ur, mapsto, "a_{j\infty}"'] &
\end{tikzcd}$$

i.e., (i,x) and (j,y) will have to be <u>identified</u> in A_∞. However, we "recognize" the necessity of this kind of identification more generally: for any i and j, there is at least one k with $i \le k$, $j \le k$, so that (i,x) and (i,j) should be eventually identified if we see that $a_{ik}(x) = a_{jk}(x)$ (*)

in A_k:

$$\begin{tikzcd}
A_i \arrow[dr, "a_{ik}"] \arrow[drr, bend left, "a_{i\infty}"] & & \\
& A_k \arrow[r] & A_\infty \\
A_j \arrow[ur, "a_{ij}"'] \arrow[urr, bend right, "a_{ij}"'] & &
\end{tikzcd}$$

thus, if $(*)$, then $a_{i\infty}(x) = a_{j\infty}(y)$.

We make this into a definition. Let B be the disjoint union of the sets A_i:

$$B = \coprod_{i\in I} A_i = \{(i, x) : i \in I,\ x \in A_i\}$$

and define the binary relation R on B by:

$$b_1 R b_2 \underset{\text{def}}{\iff} b_1 = (i,x) \;\&\; b_2 = (j,y)$$

and <u>there is</u> k,

$i \leq k \;\&\; j \leq k$

such that

$$a_{ik}(x) = a_{jk}(y) \quad (\in A_k)$$

<u>CLAIM</u>: R is an equivalence relation on B.

Only transitivity is at issue.

Let:

$$(i, x)\ R\ (j, y)\ R\ (k, z) \qquad (*)$$

We have: $\ell, m, n \in \underline{I}$ such that

$$\begin{tikzcd}
i \arrow[dr] & & \\
& \ell \arrow[dr] & \\
j \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & n \\
& m \arrow[ur] & \\
k \arrow[ur] & &
\end{tikzcd} \qquad (**)$$

in $\underline{I}$. (remember: $i \to \ell$ means $i \leq \ell$);

and, the choice of ℓ and m: by the assumption $(*)$

so that

$$a_{i\ell}(x) = a_{j\ell}(y) \underset{\text{def}}{=} u,$$

$$a_{jm}(y) = a_{km}(z) \underset{\text{def}}{=} v.$$

Let: $w \underset{\text{def}}{=} a_{jn}(y)$. The "diagram" $(**)$ when the functor $\underline{A}$ is applied to it, becomes commutative (since, of course $(**)$ commutes in $\underline{I}$!)

and that means that

$$a_{in}(x) = a_{kn}(z) = w$$

— and this exhausts the requirement for the relationship $(i, x)\, R\, (k, z)$ — as desired.

I now define

$$A_\infty = \underbrace{\operatorname*{colim}_{i \in I} A_i}_{} = \operatorname{colim} \underline{A}$$

abbreviated notation!

$$\underset{\text{def}}{=} \quad B / R = \text{the set of equivalence}$$

classes of R.

The equivalence class containing (i, x) is written $[i, x]$. We put

$$\begin{array}{ccc} A_i & \xrightarrow{a_{i\infty}} & A_\infty \\ x & \longmapsto & [i, x] \end{array}$$

Then

$$\begin{tikzcd} A_i \arrow[rr, "a_{ij}"] \arrow[dr, "a_{i\infty}"'] & \circleddash & A_j \arrow[dl, "a_{j\infty}"] \\ & A_\infty & \end{tikzcd} \qquad (i \leq j)$$

since, for $x \in A_i$ and $y \underset{\text{def}}{=} a_{ij}(x)$, we have

$(i, x) \, R \, (j, y)$ ("by the choice $k = j$"), thus

$[i, x] = [j, y]$.

Next, we formulate and verify the universal property of our construction — but for the formulation, we go to a general context, one that will have further special cases that are useful for us.

(7.1) $\boxed{\underline{\text{Definition}}}$ Let $\mathbb{A}$ be a category, $\underline{\underline{I}}$ another category and $\underline{A} : \underline{\underline{I}} \longrightarrow \mathbb{A}$ a functor (although we will call it a 'diagram').

(i) A <u>cocone</u> on $\underline{A}$ is given by an object B of $\mathbb{A}$, and a natural transformation

$$\underline{A} \xrightarrow{\ \varphi\ } \ulcorner B \urcorner$$

i.e.

$$\begin{tikzcd} \underline{\underline{I}} \arrow[r, bend left, "\underline{A}"] \arrow[r, bend right, "\ulcorner B \urcorner"'] \arrow[r, phantom, "\Downarrow \varphi"] & \mathbb{A} \end{tikzcd}$$

where $\ulcorner B \urcorner : \underline{\underline{I}} \longrightarrow \mathbb{A}$ is the constant functor:

with value B:

$$\ulcorner B \urcorner (i) = B \quad \text{for all } i$$

$(i \in Ob(\underline{I}))$

$$\ulcorner B \urcorner \left(\begin{array}{c} i \\ \downarrow \alpha \\ j \end{array} \right) = \begin{array}{c} B \\ \downarrow id_B \\ B \end{array} \quad \text{for all } \alpha \in Arr(\underline{I}).$$

This means, in elementary terms, and using our previous notation, restricting ourselves to the special case of a poset $\underline{I} = (I, \leq)$, that:

$$\varphi = \langle \varphi_i : A_i \longrightarrow B \rangle_{i \in I}$$

such that (naturality!)

$$\begin{tikzcd} A_i \arrow[rr, "a_{ij}"] \arrow[d, "\varphi_i"'] & = & A_j \arrow[d, "\varphi_j"] \\ B \arrow[rr, "id_B"'] & & B \end{tikzcd}$$

commutes.

(ii) The cocone $\underline{A} \xrightarrow{\gamma} \ulcorner A \urcorner$ is <u>universal</u> — and A is a <u>colimit</u> of the diagram $\underline{A}$ if the following holds: for an arbitrary cocone $\underline{A} \xrightarrow{\varphi} \ulcorner B \urcorner$, there is a unique arrow $f: A \to B$ in $\mathbb{A}$ such that

$$\begin{tikzcd} & \ulcorner A \urcorner \arrow[d, "\ulcorner f \urcorner"] \\ \underline{A} \arrow[ur, "\gamma"] \arrow[dr, "\varphi"'] \arrow[r, phantom, "="] & \ulcorner B \urcorner \end{tikzcd}$$

commutes ($\ulcorner f \urcorner : \ulcorner A \urcorner \to \ulcorner B \urcorner$ is the obvious natural transformation all whose components are equal to $f: A \to B$).

In practice, the above means that: $\exists$ unique $f: A \to B$

Such that:

$$\begin{tikzcd} & A \;(= A_\infty) \arrow[d, dashed, "f"] \\ A_i \arrow[ur, "(a_{i\infty} =)\, \gamma_i"] \arrow[dr, "\varphi_i"'] \arrow[r, phantom, "="] & B \end{tikzcd} \qquad (*)$$

commutes for all $i \in I$.

Let us check that in our construction in $\underline{Set}$ above, we in fact have that $A_\infty = \operatorname{colim}_{i \in I} A_i$

$= \operatorname{colim} \underline{A}$, with the universal cocone

$\gamma = \langle a_{i\infty} : A_i \to A_\infty \rangle_{i \in I}$ (we also refer to

$\gamma_i : A_i \to A_\infty$ as (colimit) coprojections).

We need to define $f : A_\infty \to B$, i.e.,

$f([i, x])$ for all $i \in I$, $x \in A_i$. Obvious choice:

$$f([i, x]) \underset{\text{def}}{=} \varphi_i(x) \qquad (*)$$

<u>Is this well-defined?</u> If $[i, x] = [j, y]$, we have ~~k~~ k s.t. $i \le k$, $j \le k$ and $a_{ik}(x) = a_{jk}(y) = z$.

But, by the naturality of $\varphi : \underline{A} \to \ulcorner B \urcorner$, we have

$$\begin{tikzcd}
& A_i \arrow[dl, "a_{ik}"'] \arrow[dr, "\varphi_i"] & \\
A_k \arrow[rr, "\varphi_k"] & & B \\
& A_j \arrow[ul, "a_{jk}"] \arrow[ur, "\varphi_j"'] &
\end{tikzcd}$$

commuting, showing

$$\varphi_i(x) = \varphi_k(a_{ik}(x)) = \varphi_k(z)$$

$$\varphi_j(y) = \varphi_k(a_{jk}(y)) = \varphi_k(z)$$

thus $\varphi_i(x) = \varphi_j(y)$ — as desired.

Also, clearly, f must satisfy $(*)$ — f is unique.

And, of course, $(*)$, p61 holds true.

There is an "internal" characterization of the colimit cocone

$$\gamma = \langle a_{i\infty} \rangle_{i \in I} : \underline{A} \longrightarrow \ulcorner A_\infty \urcorner$$

constructed on pages 55 and later:

1) "surjectivity": for any $x \in A_\infty$, there are $i \in I$ and $u \in A_i$ such that $x = a_{i\infty}(u)$;

2) "injectivity": whenever $i, j \in I$, $u \in A_i$, $v \in A_j$, and $a_{i\infty}(u) = a_{j\infty}(v)$, there is k such that $i \le k$ and $j \le k$ and $a_{ik}(u) = a_{jk}(v)$.

One can see that, in the proof of the universal property of γ, all what we used was these facts about the construction.

We will prove — or, at least, sketch(?) a proof of — the fact that in the categories $\mathbb{A} = \mathrm{Cat}$ (the category of small categories), and $\mathbb{A} = \mathrm{Coh}$ directed colimits exist — moreover that they are calculated "as in Set". The latter means that, for instance, the object-set of the colimit category is the "corresponding" colimit of the object-sets of the ingredient categories (as it already said above).

I state the theorem that we need: — no more and no less!

(7.2) <u>Theorem</u> Given a diagram

$$\mathbb{C} : \underline{I} \longrightarrow \mathit{Cat}$$

on a non-empty directed poset $\underline{I} = (I, \leq)$ (write $\mathbb{C}_i$ for $\mathbb{C}(i)$, $i \in \underline{I}$, and $F_{ij} : \mathbb{C}_i \to \mathbb{C}_j$ for $\mathbb{C}(i \leq j)$),

there is $\mathbb{C}_\infty \in \mathit{Cat}$, and a cocone

$$\gamma = \langle F_{i\infty} : i \in I \rangle : \quad \underline{I} \longrightarrow \ulcorner \mathbb{C}_\infty \urcorner$$

such that:

1) ("surjectivity") : for every $X \in Ob(\mathbb{C}_\infty)$ there is i and $X' \in Ob(\mathbb{C}_i)$ such that $c_{i\infty}(X') = X$; and similarly for arrows;

2) ("injectivity") : <u>whenever</u> $i, j \in I$, and $X \in \mathbb{C}_i$ and $Y \in \mathbb{C}_j$, and $F_{i\infty}(X) = F_{j\infty}(Y)$, <u>there</u> is k such that $i \leq k$ and $j \leq k$ and

$$F_{ik}(X) = F_{jk}(Y) \quad (\in \mathbb{C}_k).$$

Observe that the theorem does not mention the notion 'colimit'! However, in fact $\gamma : \underline{I} \longrightarrow \ulcorner \mathbb{C}_\infty \urcorner$ <u>is</u> a colimit cocone — moreover, the properties described necessarily imply that this is so.

You will see(?) that the proof of the theorem will make good use of the universal property of the colimits in $\underline{\text{Set}}$.

(7.3) __Corollary__ Suppose, in addition to the data in the theorem, that each $\mathbb{C}_i \in \mathrm{Coh}$, and each functor $F_{ij} : \mathbb{C}_i \to \mathbb{C}_j$ is a coherent functor. Then $\mathbb{C}_\infty$ is a coherent category, and each $F_{i\infty} : \mathbb{C}_i \to \mathbb{C}_\infty$ is a coherent functor.

The proof of the Corollary from the Theorem is completely elementary, but somewhat tedious.

I postpone the proof ~~of~~ both the Theorem and Corollary to the Appendix. The following three little lemmas also belong to the same circle of ideas; they are also postponed.

(7.4) __Lemma__ (i) Continuing the situation of the previous corollary, let $X \xrightarrow{!_X} 1_{\mathbb{C}_\infty}$ be an e.e. morphism in $\mathbb{C}_\infty$, with codomain a terminal object of $\mathbb{C}_\infty$. Then: there are $i \in I$ and an e.e. morphism $!_{X'} : X' \longrightarrow 1_{\mathbb{C}_i}$ with terminal codomain in $\mathbb{C}_i$ such that

$$F_{i\infty}(!_{X'}) = !_X .$$

7.5 (ii') Suppose $\mathcal{U} \vee \mathcal{V} = \top_{\mathbb{1}_{\mathbb{C}_\infty}}$ ($\mathcal{U}, \mathcal{V}$ subobjects of $\mathbb{1}_{\mathbb{C}_\infty}$). Then there is $i \in I$, $\mathcal{U}', \mathcal{V}' \in \mathrm{Sub}(\mathbb{1}_{\mathbb{C}_i})$ such that $\mathcal{U}' \xmapsto{F_{i\infty}} \mathcal{U}$, $\mathcal{V}' \xmapsto{F_{i\infty}} \mathcal{V}$, and $\mathcal{U}' \vee \mathcal{V}' = \top_{\mathbb{1}_{\mathbb{C}_i}}$.

(iii) Let $i_0 \in I$, and $A \xrightarrow{f} B$ an arrow in $\mathbb{C}_{i_0}$. Suppose that

for all $i \geq i_0$, the functor

$$F_{i_0 i} : \mathbb{C}_{i_0} \longrightarrow \mathbb{C}_i$$

is conservative for f ($F_{i_0 i}(f)$ iso $\Rightarrow$ f iso).

Then $c_{i_0\infty} : \mathbb{C}_{i_0} \longrightarrow \mathbb{C}_\infty$ is also conservative for f.

8. Completion of the proof of the completeness theorem

By $\mathbb{C}$, I always mean a small coherent category.

(8.1) Proposition. There exist: a small coherent category $\mathbb{C}^*$, a coherent functor $\Sigma \, (= \Sigma^*_{\mathbb{C}}) : \mathbb{C} \to \mathbb{C}^*$ with the following two properties:

1) Σ is conservative (for all arrows in $\mathbb{C}$);

2) for every $X \in Ob(\mathbb{C})$ with full support ($X \xrightarrow{!_X} \mathbb{1}_{\mathbb{C}}$ is e.e.), the object $\Sigma(X)$ in $\mathbb{C}^*$ has a global element.

(8.2) Proposition. Given a monomorphism $A \overset{m}{\rightarrowtail} B$ in $\mathbb{C}$, there exist: a small coherent category $\mathbb{C}^\# \, (= \mathbb{C}^\#_{(m)})$ and a coherent functor $\Sigma \, (= \Sigma^\#_{\mathbb{C}, m}) : \mathbb{C} \to \mathbb{C}^\#$ with the following two properties:

1) Σ is conservative for the arrow m in $\mathbb{C}$;

2) for every pair (U, V) of subobjects of $\mathbb{1}_{\mathbb{C}}$ in $\mathbb{C}$ such that $U \vee V = \top_{\mathbb{1}_{\mathbb{C}}}$, at least one of the two subobjects $\Sigma(U)$, $\Sigma(V)$ of $\mathbb{1}_{\mathbb{C}^\#}$ is equal to the top, $\top_{\mathbb{1}_{\mathbb{C}^\#}}$.

Before saying anything about the proof of the propositions, I will use them to complete the proof of the completeness theorem (p. 10).

Let $\mathbb{C}$ be a small coherent category, $m: A \rightarrowtail B$ a monomorphism in $\mathbb{C}$. We will prove:

$$\left[\begin{array}{l} \quad\quad\text{there is } M : \mathbb{C} \longrightarrow \mathbf{Set} \\ M \text{ coherent, such that } M \text{ is conservative for } m: \\ \quad\quad M(m) \text{ isomorphism} \Rightarrow m \text{ is an isomorphism.} \end{array}\right.$$

By (6.1) Proposition (p 40), this will be sufficient.

Let $\mathbb{C}_0 \underset{\text{def}}{=} \mathbb{C}/B$, and $\Phi = \Phi_B : \mathbb{C} \longrightarrow \mathbb{C}_0$.

Let $\hat{m} = \Phi(m) = m \times B : (A \times B, \pi_1) \longrightarrow (B \times B, \pi_1)$.

Recursively, we define the coherent category $\mathbb{C}_n$ for all $n \in \mathbb{N}$, together with the coherent functors

$$F_{kn} : \mathbb{C}_k \longrightarrow \mathbb{C}_n \quad \text{for } k<n$$

$(F_{nn} \underset{\text{def}}{=} id_{\mathbb{C}_n})$ satisfying the following: (i) & (ii):

(i) ⊙ $\langle F_{kn} \rangle_{k<n}$ is compatible (a functor $\mathbb{N} \to \mathrm{Coh}$):

$$\begin{tikzcd} \mathbb{C}_k \arrow[r, "F_{k\ell}"] \arrow[dr, bend right, "F_{kn}"'] \arrow[dr, phantom, "\circledcirc"] & \mathbb{C}_\ell \arrow[d, "F_{\ell n}"] \\ & \mathbb{C}_n \end{tikzcd} \qquad F_{kn} = F_{\ell n} \circ F_{k\ell}$$

$$(k<\ell<n)$$

(ii) ⊙ $F_{0n} : \mathbb{C}_0 \longrightarrow \mathbb{C}_n$ is conservative for $\hat{m}$.

$\mathbb{C}_0$ has been defined.

Suppose $n \in \mathbb{N}$, and the data for $k \leq n$ have all been defined. We need to define $\mathbb{C}_{n+1}$ and the functors $F_{k,n+1} : \mathbb{C}_k \longrightarrow \mathbb{C}_{n+1}$ $(k<n+1)$.

Starting with $\mathbb{C}_n$, we apply the two "constructions" "$\mathbb{C} \mapsto \mathbb{C}^*$", and "$\mathbb{C} \longmapsto \mathbb{C}^*_{(m)}$" of the last two propositions, first the first, second the second, to arrive at $\mathbb{C}_{n+1}$.

Write $\hat{m}_n \overset{\text{def}}{=} F_{0n}(m)$, a mono in $\mathbb{C}_n$.

Applying 8.1, we have

$$\Sigma_{\mathbb{C}_n} : \mathbb{C}_n \longrightarrow \mathbb{C}_n^* ;$$

applying 8.2, we have

$$\Sigma^{\#}_{\mathbb{C}_n^*, \hat{m}_n} : \mathbb{C}_n^* \longrightarrow (\mathbb{C}_n^*)^{\#}_{(\hat{m}_n)} .$$

Take the composite of these functors, to obtain

$$F_{n,n+1} : \mathbb{C}_n \longrightarrow \mathbb{C}_{n+1} .$$

(Thus, $\mathbb{C}_{n+1} = (\mathbb{C}_n^*)^{\#}_{(\hat{m}_n)}$).

As the composite of coherent functors, $F_{n,n+1}$ is coherent.

Define, for $k<n$, $F_{k,n+1} = F_{n,n+1} \circ F_{k,n}$; the $F_{k,n+1}$ are coherent and compatible ((i)⊙ with $n+1$ for n).

Note that in general if F is conservative for m, and G for $F(m)$, then $G\circ F$ is conservative for m. Since $F_{n,n+1}$ is the composite of functors with suitable conservativeness properties, by also using the induction hypothesis ((ii)⊙ for n), we get (ii)⊙ for $n+1$.

This completes the construction of the diagram

$$\underline{\mathbb{C}} : \mathbb{N} \longrightarrow \mathrm{Coh}$$

where $\underline{\mathbb{C}}(n) = \mathbb{C}_n$, and $\underline{\mathbb{C}}(k \le n) = F_{kn}$.

$\mathbb{N}$ is the classical directed order.

We define

$$\mathbb{C}_\infty = \operatorname{colim} \underline{\mathbb{C}} = \operatorname*{colim}_{n \in \mathbb{N}} \mathbb{C}_n$$

with

$$F_{n\infty} : \mathbb{C}_n \longrightarrow \mathbb{C}_\infty$$

the colimit coprojections.

7.2 Theorem with 7.3 Corollary (p 64, p 65) says that $\mathbb{C}_\infty$ is a coherent category, $F_{n\infty}$ is a coherent functor ($n \in \mathbb{N}$).

7.4 (iii) (p. 66) implies that $F_{0\infty} : \mathbb{C}_0 \longrightarrow \mathbb{C}_\infty$ is conservative for m.

$\boxed{\text{CLAIM.}}$ $\mathbb{C}_\infty$ is complete and saturated (3.1; pp 12, 13)

To see that $\mathbb{C}_\infty$ is saturated, let $X \in \mathbb{C}_\infty$ have full support : $X \xrightarrow{!_X} \mathbb{1}_{\mathbb{C}_\infty}$ is e.e. By 7.4, there are $n \in \mathbb{N}$, $X' \in \mathbb{C}_n$, and $!_{X'} : X' \to \mathbb{1}_{\mathbb{C}_n}$

such that $F_{n\infty}(!_{X'}) = !_X$. By the construction of $\mathbb{C}_n^*$ (see 8.1), the object $\Sigma^*_{\mathbb{C}_n}(X')$ has a global element in $\mathbb{C}_n^*$. Therefore, the object

$$\Sigma^{\#}_{\mathbb{C}_n^*,\,\hat{m}_n}\left(\Sigma^*_{\mathbb{C}_n}(X')\right)$$

$$=$$

$$F_{n,n+1}(X') \quad \in \mathbb{C}_{n+1}$$

also has a global element, and so does

$X = F_{n\infty}(X') = F_{n+1,\infty}(F_{n,n+1}(X'))$ in $\mathbb{C}_\infty$, which was to be proved.

The proof that $\mathbb{C}_\infty$ is complete is similar, using 7.4 (ii) (p. 66).

$\boxed{\text{CLAIM } \square}$

The desired $M : \mathbb{C} \longrightarrow \mathbf{Set}$ is the composite

$$\mathbb{C} \xrightarrow{\Phi} \mathbb{C}/B = \mathbb{C}_0 \xrightarrow{F_{0\infty}} \mathbb{C}_\infty \xrightarrow{\Gamma} \mathbf{Set},$$

where Γ is the global-sections functor $\mathbb{C}_\infty(1_{\mathbb{C}_\infty}, -)$.

Since the three factors are all coherent — Γ by "Summary", p 20, based on the CLAIM —, M is a model.

Write Σ for the composite

$$\mathbb{C}/B \xrightarrow{F_{0\alpha}} \mathbb{C}_\alpha \xrightarrow{\Gamma} \text{Set}.$$

and remember $\hat{m} :=: \left((A, m) \xrightarrow{m} (B, 1_B)\right)$ in $\mathbb{C}/B$.

$F_{0\alpha}$ is conservative for $\hat{m}$, and Γ is conservative for $F_{0\alpha}(\hat{m})$, since $F_{0\alpha}(\hat{m})$ is a monomorphism into a terminal object in C_α, and we have 6.5 lemma (p 49).

Thus Σ is conservative for $\hat{m}$. We have the situation of 6.2′ Proposition (p 44.1), and we can conclude that $\Sigma\,\Phi_B = M : \mathbb{C} \longrightarrow \text{Set}$ is conservative for m.

$\boxed{\text{QED}}$

The proofs of the 'constructions' 8.1 and 8.2 are similar to the last proof — but they will use ordinals, indexings by ordinals, and transfinite recursion. For instance, for the proof of 8.1, we start with an enumeration

$$\langle X_\beta \rangle_{\beta<\alpha}$$

of all objects of $\mathbb{C}$ with full support, and construct an ordinal-based diagram $\langle \mathbb{C}_\beta \rangle_{\beta \leq \alpha}$,

$F_{\gamma\beta} : \mathbb{C}_\gamma \to \mathbb{C}_\beta \quad (\gamma < \beta < \alpha)$ of coherent categories

and functors, where the main step of the construction,

$$F_{\beta,\beta+1} : \mathbb{C}_\beta \longrightarrow \mathbb{C}_{\beta+1} \quad (\beta < \alpha)$$

will take $\mathbb{C}_{\beta+1} = \mathbb{C}_\beta / \hat{X}_\beta$ for $\hat{X}_\beta = F_{0\alpha}(X_\beta)$

and $F_{\beta,\beta+1} = \Phi^{(\mathbb{C}_\alpha)} : \mathbb{C}_\beta \longrightarrow \mathbb{C}_\beta / \hat{X}_\beta$

thereby ensuring that $\hat{X}_{\beta+1} \underset{\text{def}}{=} F_{\beta,\beta+1}(\hat{X}_\beta) = F_{0,\beta+1}(X_\beta)$

has a global element (see p. 39.1).