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Traveling wavefronts for time-delayed reaction–diffusion equation: (II) Nonlocal nonlinearity

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ABSTRACT

This is the second part of a series of study on the stability of traveling wavefronts of reaction–diffusion equations with time delays. In this paper we will consider a nonlocal time-delayed reaction–diffusion equation. When the initial perturbation around the traveling wave decays exponentially as $x \to -\infty$ (but the initial perturbation can be arbitrarily large in other locations), we prove the asymptotic stability of all traveling waves for the reaction–diffusion equation, including even the slower waves whose speed are close to the critical speed. This essentially improves the previous stability results by Mei and So [M. Mei, J.W.-H. So, Stability of strong traveling waves for a nonlocal time-delayed reaction–diffusion equation, Proc. Roy. Soc. Edinburgh Sect. A 138 (2008) 551–568] for the speed $c > 2\sqrt{D_m(3\varepsilon p - 2d_m)}$ with a small initial perturbation. The approach we use here is the weighted energy method, but the weight function is more tricky to construct due to the property of the critical wavefront, and the difficulty arising from the nonlocal nonlinearity is also overcome. Finally, by using the Crank–Nicholson scheme, we present some numerical results which confirm our theoretical study.

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1. Introduction

Subsequently to [16], as the second part of this series of study, we further investigate the nonlocal time-delayed reaction–diffusion equation

$$\frac{\partial v}{\partial t} - D_m \frac{\partial^2 v}{\partial x^2} + d_m v = \varepsilon \int_{-\infty}^{\infty} b\big(v(t-r, x-y)\big) f_\alpha(y)\, dy, \quad (t,x) \in (0,\infty) \times R, \tag{1.1}$$

with the initial condition

$$v(s,x) = v_0(s,x), \quad (s,x) \in [-r, 0] \times R. \tag{1.2}$$

The model (1.1) describes a single-species population with age-structure and diffusion (cf. [24]). Here, as explained in [16], $v(t,x)$ denotes the total mature population of the species (after the maturation age $r > 0$) at time t and position x. $D_m > 0$ is the diffusion rate of the mature species, $d_m > 0$ is the death rate of the matured, $\alpha > 0$ is the total amount of diffusion for the immature species, and it is assumed to satisfy

$$\alpha \leqslant r D_m, \tag{1.3}$$

namely, the immatured is less mobile than the matured, and $\varepsilon > 0$ represents the impact of the death rate of the immature. The birth function $b(v)$ is one of the following two important types (see [12, 16])

$$b_1(v) = pve^{-av^q}, \quad \text{or} \quad b_2(v) = \frac{pv}{1+av^q} \tag{1.4}$$

with $a > 0$, $p > 0$ and $q > 0$. When $b(v) = b_1(v)$ with $q = 1$, Eq. (1.1) is called the nonlocal Nicholson's blowflies equation. Lastly, the function $f_\alpha(y)$ is the heat kernel

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-y^2/4\alpha} \quad \text{with} \int_{-\infty}^{\infty} f_\alpha(y)\, dy = 1. \tag{1.5}$$

By solving the equation

$$d_m v = \varepsilon p \int_{-\infty}^{\infty} b\big(v(t-r, x-y)\big) f_\alpha(y)\, dy,$$

we obtain immediately the two constant equilibria $v_\pm$ of (1.1)

$$\text{for } b_1(v): \quad v_- = 0 \quad \text{and} \quad v_+ = \left(\frac{1}{a} \ln \frac{\varepsilon p}{d_m}\right)^{1/q}, \tag{1.6}$$

$$\text{for } b_2(v): \quad v_- = 0 \quad \text{and} \quad v_+ = \left(\frac{\varepsilon p - d_m}{a d_m}\right)^{1/q}. \tag{1.7}$$

When $\varepsilon p > d_m$, we have $v_+ > v_- = 0$.

A traveling wavefront of Eq. (1.1) connecting the constant states $v_\pm$ is an increasing solution of the form of $\phi(x+ct)$, where $c>0$ is the wave speed. Thus, it satisfies

$$\begin{cases} -c\phi'(\xi) - D_m\phi''(\xi) + d_m\phi(\xi) = \varepsilon p \displaystyle\int_{-\infty}^{\infty} b\big(\phi(\xi - cr - y)\big) f_\alpha(y)\,dy, \\ \phi(\pm\infty) = v_\pm, \end{cases} \tag{1.8}$$

where $\xi = x+ct$ and $' = \frac{d}{d\xi}$. The existence of such traveling wavefronts was first proved by So, Wu and Zou [23] for $b(v) = b_1(v)$ with $q=1$ by means of the method of the upper–lower solutions, and later extended by Liang and Wu [12] to the general case $b(v) = b_1(v)$, or $b(v) = b_2(v)$. It was proved that there exists a number $c_* > 0$, called the critical wave speed (i.e., the minimum speed), such that for $c > c_*$, the wavefront $\phi(x+ct)$ exists. The boundedness of the critical wave speed c_* and the asymptotic behavior of c_* with respect to the delay time r and the diffusion rate D_m were further analyzed by Wu, Wei and Mei [31]. They proved that $c_* \to 2\sqrt{D_m(\varepsilon p - d_m)}$ as $r \to 0$, and $c_* = O(r^{-1/2}) \to 0$ as $r \to \infty$, while $c_* = O(\sqrt{D_m}) \to \infty$ as $D_m \to \infty$. Then, Mei and So [17] showed that, when the wavefront is as fast as $c > 2\sqrt{D_m(3\varepsilon p - 2d_m)}$ and the initial perturbation around the wavefront in a weighted Sobolev space is small enough, then the wavefront is asymptotically stable. However, for the gap $c_* < c \leqslant 2\sqrt{D_m(3\varepsilon p - 2d_m)}$, the stability of these slower wavefronts remains open. Obviously, as we know, such a stability result for the slower wavefronts is much more significant and challenging from both the mathematical and physical points of view. The purpose of the present paper is to resolve this case.

By using the weighted energy method, as in [16–18], the crucial step is to establish the basic L^2-energy estimate for $v - \phi$. However, different from the case of local nonlinearity in [16], the technique with a piecewise weight function cannot be applied to get the stability for all $c > c_*$ in the nonlocal case. This is because the nonlocal integral term will produce a large upper bound which forces us to look for a wave with a big enough wave-speed in order to eliminate it (see, for example, the stability result given in [17] with $c > 2\sqrt{D_m(3\varepsilon p - 2d_m)}$). Here, we develop a new technique to overcome this difficulty caused by the nonlocality. First of all, we introduce a non-piecewise weight function $w(x) = e^{-kx}$ for some carefully selected positive number k, which is related to the critical wave speed c_*. Next we prove that, for any given wave with wave speed $c > c_*$, the solution $v(t,x)$ converges to the corresponding traveling wavefront $\phi(x+ct)$ in the weighted Sobolev space $H^1_w(R)$. Hence, we have the convergence result: $\sup_{x\in I} |v(t,x) - \phi(x+ct)| = O(e^{-\mu t})$ for some positive constant μ, where $I = (-\infty, \bar{x})$ is any interval with $\bar{x} \gg 1$, due to the shortage of $w(x) \to 0$ as $x \to +\infty$. After that, we further prove $\lim_{x\to+\infty} |v(t,x) - \phi(x+ct)| = O(e^{-\mu t})$. By combining these two convergence results, we obtain the stability of the wavefront in the whole space $(-\infty, \infty)$.

The rest of the paper is organized as follows. In Section 2, we state the existence result for traveling wavefronts. Based on the property of the critical wavefront, we then introduce a weight function which is ideal for our purpose. In Section 3, we combine the weighted energy method with the comparison principle together and prove the stability for all traveling wavefronts ($c > c_*$). There will be no restriction on the delay time r, the wave speed c and the initial perturbation. This essentially improves and develops the previous stability results in [17]. In Section 4, we present some numerical results based on the Crank–Nicholson scheme. These numerical results confirm our theoretical results. Finally, in Section 5, we give a remark on the first part of this series of study [16] that the $b'(v_+) \ll 1$ can be removed by our new technique showed in this paper, even so $b'(v_+) \ll 1$ is reasonable as explained in [16].

For the notations adopted in the present paper, in particular the weighted Sobolev space $H^l_w(R)$ as well as the space $C([0,T]; H^l_w(R))$, we refer the reader to the first part of this series of study [16].

For other interesting research works related to this topic, please refer to [1–34] and the references therein.

2. Nonlinear stability

In this section, we first state the existence of traveling wavefronts and then introduce the nonlinear stability result for traveling wavefronts.

By using the method of the upper–lower solutions, Liang and Wu [12] (see also the early work by So, Wu and Zou [23]) proved the existence of the traveling wavefronts.

Proposition 2.1 *(Existence of traveling wavefronts). (See [12,23].) Assume that*

$$\text{for } b(v)=b_1(v):\quad 1<\frac{\varepsilon p}{d_m}\leqslant e^{1/q}, \tag{2.1}$$

$$\begin{aligned}\text{for } b(v)=b_2(v):\quad &\text{either } 1<\frac{\varepsilon p}{d_m}\leqslant\frac{q}{q-1}\quad \text{if } q>1,\\ &\text{or } 1<\frac{\varepsilon p}{d_m}<\infty\quad \text{if } 0<q\leqslant 1.\end{aligned} \tag{2.2}$$

Then there exist a minimum speed $c_*=c_*(r,\alpha,\varepsilon,D_m,d_m,p)\in(0,2\sqrt{D_m(\varepsilon p-d_m)})$ and a corresponding number $\lambda_*=\lambda(c_*)>0$ satisfying

$$\Delta(\lambda_*,c_*)=0,\qquad \frac{\partial}{\partial\lambda}\Delta(\lambda_*,c_*)=0, \tag{2.3}$$

where

$$\Delta(\lambda,c)=\varepsilon p e^{\alpha\lambda^2-\lambda cr}-\left[c\lambda-D_m\lambda^2+d_m\right], \tag{2.4}$$

such that for all $c>c_*$, the traveling wavefront $\phi(x+ct)$ of Eq. (1.1) connecting $v_\pm$ exists uniquely (up to shift).

Furthermore, for $c=c_*$, it holds that $\Delta(\lambda_*,c)=0$, i.e.,

$$\varepsilon p e^{\alpha\lambda_*^2-\lambda_* c_* r}=c_*\lambda_*-D_m\lambda_*^2+d_m, \tag{2.5}$$

and for $c>c_*$, it holds that $\Delta(\lambda_*,c)<0$, i.e.,

$$\varepsilon p e^{\alpha\lambda_*^2-\lambda_* cr}<c\lambda_*-D_m\lambda_*^2+d_m. \tag{2.6}$$

We now define a weight function as

$$w(x)=e^{-2\lambda_* x}, \tag{2.7}$$

where $\lambda_*=\lambda_*(c_*)$ is the positive constant determined in Proposition 2.1. As showed in [31], we know that $\frac{c_*}{2D_m}<\lambda_*<\frac{c_*}{D_m}$. Obviously, it satisfies $w(x)\to+\infty$ as $x\to-\infty$ and $w(x)\to 0$ as $x\to+\infty$.

Next, we state our stability result.

Theorem 2.2 *(Nonlinear stability). Let $\frac{\varepsilon p}{d_m}$ satisfy (2.1) for $b(v)=b_1(v)$, or (2.2) for $b(v)=b_2(v)$. For any given wavefront $\phi(x+ct)$ with a speed $c>c_*$, if c satisfies*

$$e^{\lambda_*^2\alpha}<\frac{c\lambda_*-D_m\lambda_*^2+d_m}{c_*\lambda_*-D_m\lambda_*^2+d_m}, \tag{2.8}$$

the initial data holds

$$v_- \leqslant v_0(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R, \tag{2.9}$$

and the initial perturbation is $v_0(s,x) - \phi(x+cs) \in C([-r,0]; H^1_w(R))$*, then the solution of* (1.1) *and* (1.2) *satisfies* $v(t,x) - \phi(x+ct) \in C([0,\infty); H^1_w(R))$*, and*

$$v_- \leqslant v(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R, \tag{2.10}$$

and

$$\big\|(v-\phi)(t)\big\|_{H^1_w(R)} \leqslant Ce^{-\mu t}, \quad t \geqslant 0, \tag{2.11}$$

for some positive constant μ.

In particular, the solution $v(t,x)$ *also converges asymptotically to the wavefront* $\phi(x+ct)$ *in the* L^∞*-norm*:

$$\sup_{x\in R}\big|v(t,x) - \phi(x+ct)\big| \leqslant Ce^{-\mu t}, \quad t \geqslant 0. \tag{2.12}$$

Remark 1.

1. The condition (2.8) is equivalent to

$$\alpha < \frac{1}{\lambda_*^2} \ln \frac{c\lambda_* - D_m\lambda_*^2 + d_m}{c_*\lambda_* - D_m\lambda_*^2 + d_m}. \tag{2.13}$$

   Here α is independent of c, D_m and d_m, but both λ_* and c_* are related to α. For given α, we need the wave speed c to be large such as in (2.13). Conversely, when c is sufficiently large, one can easily verify that $\frac{1}{\lambda_*^2} \ln \frac{c\lambda_* - D_m\lambda_*^2 + d_m}{c_*\lambda_* - D_m\lambda_*^2 + d_m}$ is sufficiently large. This ensures α is sufficiently large as well. When c is sufficiently close to the critical wave speed c_*, then one can recognize that $\frac{1}{\lambda_*^2} \ln \frac{c\lambda_* - D_m\lambda_*^2 + d_m}{c_*\lambda_* - D_m\lambda_*^2 + d_m} \ll 1$, which means α needed to be sufficiently small. Thus, when α is small enough, we may obtain the stability for those slower waves with the speed $c \in (c_*, 2\sqrt{D_m(3\varepsilon p - 2d_m)}]$, which is the unsolved gap in [17]. So, here we improve the previous stability results in [17].
2. The smallness for the initial perturbation and the condition $b'(v_+) = 0$ both required in [17] are removed for our new stability result.
3. By a deep analysis on the wavefront $\phi(x+ct)$, it can be verified that

$$\big|\phi(\xi) - v_-\big| = O(1)e^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty, \tag{2.14}$$

   where $\lambda_1 = \lambda_1(c) > 0$ is determined by $\Delta(\lambda_1, c) = 0$ and satisfies $0 < \lambda_1 < \lambda_*$. However, different from the local case studied in [16], the weight function $w(x)$ cannot be taken as $w(x) = e^{2\lambda|x|}$ for $\lambda \in (\lambda_1, \lambda_*)$ as $x \to -\infty$, namely, the decay rate of the initial perturbation around the wavefront cannot be released to

$$\big|v_0(s,x) - \phi(x+cs)\big| = O(1)e^{-\lambda|x|}, \quad \text{for } \lambda \in (\lambda_1, \lambda_*), \text{ as } x \to -\infty.$$

   The reason is that the nonlocal term (integration) produces a big upper bound which cannot be eliminated by the weight $w(x) = e^{2\lambda|x|}$ with $\lambda \in (\lambda_1, \lambda_*]$ (see (3.29) in the proof of the key Lemma 3.3). That is why here we need to select a stronger weight function as $w(x) = e^{2\lambda_*|x|}$ for $x \to -\infty$.

3. Proof of nonlinear stability

As shown in [16] (see also [9,15,27]), one can similarly prove the following boundedness and the comparison principle.

Lemma 3.1 *(Boundedness). Let the initial data satisfy*

$$v_- = 0 \leqslant v_0(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R. \tag{3.1}$$

Then the solution $v(t,x)$ of the Cauchy problem (1.1) and (1.2) satisfies

$$v_- \leqslant v(t,x) \leqslant v_+, \quad \text{for } (t,x) \in [0,\infty) \times R. \tag{3.2}$$

Lemma 3.2 *(Comparison principle). Let $\bar{v}(t,x)$ and $\underline{v}(t,x)$ be the solutions of (1.1) and (1.2) with the initial data $\bar{v}_0(s,x)$ and $\underline{v}_0(s,x)$, respectively. If*

$$v_- \leqslant \underline{v}_0(s,x) \leqslant \bar{v}_0(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R, \tag{3.3}$$

then

$$v_- \leqslant \underline{v}(t,x) \leqslant \bar{v}(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R. \tag{3.4}$$

Let the initial data $v_0(s,x)$ be such that $v_- \leqslant v_0(s,x) \leqslant v_+$ for $(s,x) \in [-r,0] \times R$, and let

$$\begin{cases} V_0^+(s,x) = \max\{v_0(s,x), \phi(x+cs)\}, \\ V_0^-(s,x) = \min\{v_0(s,x), \phi(x+cs)\}, \end{cases} \quad \text{for } (s,x) \in [-r,0] \times R. \tag{3.5}$$

This implies

$$v_- \leqslant V_0^-(s,x) \leqslant v_0(s,x) \leqslant V_0^+(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R, \tag{3.6}$$

$$v_- \leqslant V_0^-(s,x) \leqslant \phi(x+cs) \leqslant V_0^+(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R. \tag{3.7}$$

Define $V^+(t,x)$ and $V^-(t,x)$ as the corresponding solutions of (1.1) and (1.2) with respect to initial data $V_0^+(s,x)$ and $V_0^-(s,x)$, respectively, i.e.,

$$\begin{cases} \dfrac{\partial V^\pm}{\partial t} - D_m \dfrac{\partial^2 V^\pm}{\partial x^2} + d_m V^\pm = \varepsilon \displaystyle\int_{-\infty}^{\infty} b\big(V^\pm(t-r,x-y)\big) f_\alpha(y)\,dy, & (t,x) \in R_+ \times R, \\ V^\pm(s,x) = V_0^\pm(s,x), & x \in R,\ s \in [-r,0]. \end{cases} \tag{3.8}$$

By Lemma 3.2, it follows that

$$v_- \leqslant V^-(t,x) \leqslant v(t,x) \leqslant V^+(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R, \tag{3.9}$$

$$v_- \leqslant V^-(t,x) \leqslant \phi(x+ct) \leqslant V^+(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R. \tag{3.10}$$

As in [16], we also need the following three steps to prove stability.

Step 1: The convergence of $V^+(t,x)$ to $\phi(x+ct)$.

Let $\xi := x + ct$ and

$$u_0(s,\xi) := V_0^+(s,x) - \phi(x+cs), \qquad u(t,\xi) := V^+(t,x) - \phi(x+ct). \tag{3.11}$$

Then by (3.6) and (3.9), we have

$$u(t,\xi) \geqslant 0 \quad \text{and} \quad u_0(s,\xi) \geqslant 0. \tag{3.12}$$

It can be verified that $u(t,\xi)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial \xi} - D_m \dfrac{\partial^2 u}{\partial \xi^2} + d_m u - \varepsilon \displaystyle\int_{-\infty}^{\infty} b'\big(\phi(\xi - y - cr)\big) u(t-r, \xi - y - cr) f_\alpha(y)\, dy \\ \quad = \varepsilon \displaystyle\int_{-\infty}^{\infty} Q(t-r, \xi - y - cr) f_\alpha(y)\, dy, \quad (t,x) \in R_+ \times R, \\ u(s,\xi) = u_0(s,\xi), \quad (s,x) \in [-r,0] \times R, \end{cases} \tag{3.13}$$

where

$$Q(t-r, \xi - y - cr) = b(\phi + u) - b(\phi) - b'(\phi)u \tag{3.14}$$

with $\phi = \phi(\xi - y - cr)$ and $u = u(t-r, \xi - y - cr)$.

Multiplying (3.13) by $e^{2\mu t} w(\xi) u(t,\xi)$, one obtains

$$\begin{aligned} &\left\{\frac{1}{2} e^{2\mu t} w u^2\right\}_t + e^{2\mu t}\left\{\frac{1}{2} c w u^2 - D_m w u u_\xi\right\}_\xi + D_m e^{2\mu t} w u_\xi^2 + D_m e^{2\mu t} w' u_\xi u \\ &\quad + \left\{-\frac{c}{2}\frac{w'}{w} + d_m - \mu\right\} e^{2\mu t} w u^2 \\ &\quad - \varepsilon e^{2\mu t} w(\xi) u(\xi,t) \int_{-\infty}^{\infty} b'\big(\phi(\xi - y - cr)\big) u(t-r, \xi - y - cr) f_\alpha(y)\, dy \\ &= \varepsilon e^{2\mu t} w(\xi) u(\xi,t) \int_{-\infty}^{\infty} Q(t-r, \xi - y - cr) f_\alpha(y)\, dy. \end{aligned} \tag{3.15}$$

Notice that, by the Cauchy–Schwarz inequality,

$$\left|D_m e^{2\mu t} w' u u_\xi\right| = D_m e^{2\mu t} w \left| u_\xi \cdot \frac{w'}{w} v\right| \leqslant D_m e^{2\mu t} w u_\xi^2 + \frac{D_m}{4} e^{2\mu t} \left(\frac{w'}{w}\right)^2 w u^2,$$

then (3.15) is reduced to

$$\begin{aligned} &\left\{\frac{1}{2} e^{2\mu t} w u^2\right\}_t + \left\{\frac{1}{2} e^{2\mu t} c w u^2 - D_m e^{2\mu t} w u u_\xi\right\}_\xi \\ &\quad + \left\{-\frac{c}{2}\frac{w'}{w} + d_m - \mu - \frac{D_m}{4}\left(\frac{w'}{w}\right)^2\right\} e^{2\mu t} w u^2 \end{aligned}$$

$$
\begin{aligned}
&-\varepsilon e^{2\mu t}w(\xi)u(\xi,t)\int_{-\infty}^{\infty} b'\big(\phi(\xi-y-cr)\big)u(t-r,\xi-y-cr)f_\alpha(y)\,dy \\
&\leqslant \varepsilon e^{2\mu t}w(\xi)u(\xi,t)\int_{-\infty}^{\infty} Q(t-r,\xi-y-cr)f_\alpha(y)\,dy. \qquad (3.16)
\end{aligned}
$$

Integrating the above inequality over $R\times[0,t]$ with respect to ξ and t, one further has

$$
\begin{aligned}
&e^{2\mu t}\|u(t)\|_{L_w^2}^2+\int_0^t\int_R e^{2\mu\tau}\left\{-c\frac{w'(\xi)}{w(\xi)}+2d_m-2\mu-\frac{D_m}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2\right\}w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad-2\varepsilon\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-y-cr)\big)u(\tau,\xi)u(\tau-r,\xi-y-cr)f_\alpha(y)\,dy\,d\xi\,d\tau \\
&\leqslant\|u_0(0)\|_{L_w^2}^2+2\varepsilon\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)u(\tau,\xi)Q(\tau-y-r,\xi-cr)f_\alpha(y)\,dy\,d\xi\,d\tau. \qquad (3.17)
\end{aligned}
$$

We now turn to estimate the third term in (3.17). First of all, by using the change of variables $y\mapsto y$, $\xi-y-cr\mapsto\xi$, $\tau-r\mapsto\tau$, one can get

$$
\begin{aligned}
&\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-y-cr)\big)u^2(\tau-r,\xi-y-cr)f_\alpha(y)\,dy\,d\xi\,d\tau \\
&=\int_{-r}^{t-r}\int_R\int_R e^{2\mu(\tau+r)}w(\xi+y+cr)b'\big(\phi(\xi)\big)u^2(\tau,\xi)f_\alpha(y)\,dy\,d\xi\,d\tau \\
&=e^{2\mu r}\int_0^{t-r}\int_R e^{2\mu\tau}\left[\frac{b'(\phi(\xi))}{w(\xi)}\int_R w(\xi+y+cr)f_\alpha(y)\,dy\right]w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad+e^{2\mu r}\int_{-r}^0\int_R e^{2\mu\tau}\left[\frac{b'(\phi(\xi))}{w(\xi)}\int_R w(\xi+y+cr)f_\alpha(y)\,dy\right]w(\xi)u_0^2(\tau,\xi)\,d\xi\,d\tau \\
&\leqslant e^{2\mu r}\int_0^t\int_R e^{2\mu\tau}\left[\frac{b'(\phi(\xi))}{w(\xi)}\int_R w(\xi+y+cr)f_\alpha(y)\,dy\right]w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad+e^{2\mu r}\int_{-r}^0\int_R e^{2\mu\tau}\left[\frac{b'(\phi(\xi))}{w(\xi)}\int_R w(\xi+y+cr)f_\alpha(y)\,dy\right]w(\xi)u_0^2(\tau,\xi)\,d\xi\,d\tau. \qquad (3.18)
\end{aligned}
$$

Using the Cauchy–Schwarz inequality again,

$$
\big|u(\tau,\xi)u(\tau-r,\xi-y-cr)\big|\leqslant\frac{\eta}{2}u^2(\tau,\xi)+\frac{1}{2\eta}u^2(\tau-r,\xi-y-cr) \qquad (3.19)
$$

for a positive constant η which will be selected later (see Lemma 3.3 below), and using (3.18) and the fact $b'(\phi) > 0$ (which is proved in [16]), one can estimate the third term in (3.17) as follows:

$$\begin{aligned}
&2\varepsilon\left|\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-y-cr)\big)u(\tau,\xi)u(\tau-r,\xi-y-cr)f_\alpha(y)\,dy\,d\xi\,d\tau\right| \\
&\leqslant \varepsilon\left|\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-y-cr)\big)\left[\eta u^2(\tau,\xi)+\frac{1}{\eta}u^2(\tau-r,\xi-y-cr)\right]f_\alpha(y)\,dy\,d\xi\,d\tau\right| \\
&= \varepsilon\eta\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-y-cr)\big)u^2(\tau,\xi)f_\alpha(y)\,dy\,d\xi\,d\tau \\
&\quad+\frac{\varepsilon}{\eta}\int_R\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-y-cr)\big)u^2(\tau-r,\xi-y-cr)f_\alpha(y)\,dy\,d\xi\,d\tau \\
&\leqslant \varepsilon\eta\int_0^t\int_R e^{2\mu\tau}w(\xi)\left[\int_R b'\big(\phi(\xi-y-cr)\big)f_\alpha(y)\,dy\right]u^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad+\frac{\varepsilon e^{2\mu r}}{\eta}\int_0^t\int_R e^{2\mu\tau}\left[b'\big(\phi(\xi)\big)\int_R\frac{w(\xi+y+cr)}{w(\xi)}f_\alpha(y)\,dy\right]w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad+\frac{\varepsilon e^{2\mu r}}{\eta}\int_{-r}^0\int_R e^{2\mu\tau}\left[b'\big(\phi(\xi)\big)\int_R\frac{w(\xi+y+cr)}{w(\xi)}f_\alpha(y)\,dy\right]w(\xi)u_0^2(\tau,\xi)\,d\xi\,d\tau. \qquad (3.20)
\end{aligned}$$

Substituting (3.20) into (3.17) yields

$$\begin{aligned}
&e^{2\mu t}\|u(t)\|_{L_w^2}^2+\int_0^t\int_R e^{2\mu\tau}B_{\eta,\mu,w}(\xi)w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau \\
&\leqslant \|u_0(0)\|_{L_w^2}^2+\frac{\varepsilon e^{2\mu r}}{\eta}\int_{-r}^0\int_R e^{2\mu\tau}\left[b'\big(\phi(\xi)\big)\int_R\frac{w(\xi+y+cr)}{w(\xi)}f_\alpha(y)\,dy\right]w(\xi)u_0^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad+2\varepsilon\int_0^t\int_R\int_R e^{2\mu\tau}w(\xi)u(\tau,\xi)Q(\tau-y-r,\xi-cr)f_\alpha(y)\,dy\,d\xi\,d\tau, \qquad (3.21)
\end{aligned}$$

where

$$B_{\eta,\mu,w}(\xi):=A_{\eta,w}(\xi)-2\mu-\frac{\varepsilon}{\eta}\big(e^{2\mu r}-1\big)b'\big(\phi(\xi)\big)\int_R\frac{w(\xi+y+cr)}{w(\xi)}f_\alpha(y)\,dy \qquad (3.22)$$

and

$$A_{\eta,w}(\xi):=-c\frac{w'(\xi)}{w(\xi)}+2d_m-\frac{D_m}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2$$

$$
\begin{aligned}
&- \varepsilon\eta \int_R b'\big(\phi(\xi - y - cr)\big) f_\alpha(y)\,dy \\
&- \frac{\varepsilon b'(\phi(\xi))}{\eta} \int_R \frac{w(\xi + y + cr)}{w(\xi)} f_\alpha(y)\,dy.
\end{aligned}
\tag{3.23}
$$

As shown in [16], by the Taylor's formula and the fact $b''(v) \leqslant 0$ for $v_- \leqslant v \leqslant v_+$, one can similarly obtain

$$
Q(t - r, x - y) = b(\phi + u) - b(\phi) - b'(\phi)u \leqslant 0,
\tag{3.24}
$$

which leads, with the fact $u > 0$ (see (3.12)), that

$$
2\varepsilon \int_0^t \int_R \int_R e^{2\mu\tau} w(\xi) u(\tau,\xi) Q(\tau - y - r, \xi - cr) f_\alpha(y)\,dy\,d\xi\,d\tau \leqslant 0.
\tag{3.25}
$$

On the other hand, since $0 \leqslant b'(\phi) \leqslant p$ (see (3.18) in [16]), and $\frac{w(\xi+y+cr)}{w(\xi)} = e^{-2\lambda_*(y+cr)}$, it follows that

$$
\begin{aligned}
&\frac{\varepsilon e^{2\mu r}}{\eta} \int_{-r}^0 \int_R e^{2\mu\tau} b'\big(\phi(\xi)\big)\bigg[\int_R \frac{w(\xi + y + cr)}{w(\xi)} f_\alpha(y)\,dy\bigg] w(\xi) u_0^2(\tau,\xi)\,d\xi\,d\tau \\
&\leqslant \frac{\varepsilon p e^{2\mu r}}{\eta} \int_{-r}^0 e^{2\mu\tau} \int_R \bigg[\int_R e^{-2\lambda_*(y+cr)} f_\alpha(y)\,dy\bigg] w(\xi) u_0^2(\tau,\xi)\,d\xi\,d\tau \\
&= \frac{\varepsilon p e^{2\mu r}}{\eta} \int_{-r}^0 e^{2\mu\tau} \int_R \bigg[e^{4\alpha\lambda_*^2 - 2\lambda_* cr} \int_R \frac{1}{\sqrt{4\pi\alpha}} e^{-(\frac{y}{\sqrt{4\alpha}} + \sqrt{4\alpha}\lambda_*)^2}\,dy\bigg] w(\xi) u_0^2(\tau,\xi)\,d\xi\,d\tau \\
&= \frac{\varepsilon p}{\eta} e^{2\mu r + 4\alpha\lambda_*^2 - 2\lambda_* cr} \int_{-r}^0 e^{2\mu\tau} w(\xi) u_0^2(\tau,\xi)\,d\xi\,d\tau \\
&\leqslant C \int_{-r}^0 \big\|u_0(\tau)\big\|_{L_w^2}^2\,d\tau.
\end{aligned}
\tag{3.26}
$$

Applying (3.25) and (3.26) in (3.21), one then obtains

$$
\begin{aligned}
&e^{2\mu t}\big\|u(t)\big\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu\tau} B_{\eta,\mu,w}(\xi) w(\xi) u^2(\tau,\xi)\,d\xi\,d\tau \\
&\quad \leqslant \big\|u_0(0)\big\|_{L_w^2}^2 + C \int_{-r}^0 \big\|u_0(\tau)\big\|_{L_w^2}^2\,d\tau.
\end{aligned}
\tag{3.27}
$$

Next, we will carefully select the numbers η, μ and the weight function $w(\xi)$ to prove $B_{\eta,\mu,w}(\xi) > 0$. This is the crucial step in the present paper. For that purpose, we need the following key lemma.

Lemma 3.3. *Let $\eta = e^{2\lambda_*^2\alpha - \lambda_* cr}$. Then $A_{\eta,w}(\xi)$ defined in (3.23) satisfies*

$$A_{\eta,w}(\xi) \geqslant C_1 > 0, \quad \xi \in R, \tag{3.28}$$

for some positive constant C_1.

Proof. Notice that $\eta = e^{2\lambda_*^2\alpha - \lambda_* cr}$, $w(\xi) = e^{-2\lambda_*\xi}$, $\frac{w'(\xi)}{w(\xi)} = -2\lambda_*$ and $\frac{w(\xi+y+cr)}{w(\xi)} = e^{-2\lambda_*(y+cr)}$, as well as $0 \leqslant b'(\phi(\xi)) \leqslant p$ (see (3.18) in [16]), and $\int_R f_\alpha(y)\,dy = 1$. One may obtain

$$\begin{aligned}
B_{\mu,\eta,w}(\xi) &= 2c\lambda_* + 2d_m - 2D_m\lambda_*^2 - \varepsilon\eta \int_R b'\bigl(\phi(\xi - y - cr)\bigr) f_\alpha(y)\,dy \\
&\quad - \frac{\varepsilon b'(\phi(\xi))}{\eta} \int_R e^{-2\lambda_*(y+cr)} f_\alpha(y)\,dy \\
&\geqslant 2c\lambda_* + 2d_m - 2D_m\lambda_*^2 - \varepsilon\eta p \int_R f_\alpha(y)\,dy \\
&\quad - \frac{\varepsilon p}{\eta} e^{4\lambda_*^2\alpha - 2\lambda_* cr} \int_R \frac{1}{\sqrt{4\pi\alpha}} e^{-(\frac{y}{\sqrt{4\alpha}} + \sqrt{4\alpha}\lambda_*)^2}\,dy \\
&= 2c\lambda_* + 2d_m - 2D_m\lambda_*^2 - \varepsilon\eta p - \frac{\varepsilon p}{\eta} e^{4\lambda_*^2\alpha - 2\lambda_* cr} \\
&= 2\bigl[c\lambda_* + d_m - D_m\lambda_*^2 - \varepsilon p e^{2\lambda_*^2\alpha - \lambda_* cr}\bigr] \\
&= 2\bigl[c\lambda_* + d_m - D_m\lambda_*^2 - \varepsilon p e^{\lambda_*^2\alpha - \lambda_* cr} e^{\lambda_*^2\alpha}\bigr] \\
&= 2\bigl[\bigl(c\lambda_* + d_m - D_m\lambda_*^2\bigr) - \bigl(c_*\lambda_* + d_m - D_m\lambda_*^2\bigr) e^{\lambda_*^2\alpha}\bigr] \quad \bigl[\text{by (2.5)}\bigr] \\
&= 2\bigl(c_*\lambda_* + d_m - D_m\lambda_*^2\bigr) \biggl[\frac{c\lambda_* + d_m - D_m\lambda_*^2}{c_*\lambda_* + d_m - D_m\lambda_*^2} - e^{\lambda_*^2\alpha}\biggr] \\
&=: C_1 > 0 \quad \bigl[\text{by (2.8)}\bigr].
\end{aligned} \tag{3.29}$$

The lemma is proved. □

Lemma 3.4. *Let $\mu_1 > 0$ be the unique solution of the equation*

$$C_1 = 2\mu_1 + \varepsilon p \eta \bigl(e^{2\mu_1 r} - 1\bigr). \tag{3.30}$$

If $0 < \mu < \mu_1$, then

$$B_{\eta,\mu,w}(\xi) \geqslant C_2 > 0, \quad \xi \in (-\infty, \infty), \tag{3.31}$$

for some positive constant C_2.

Proof. Applying (3.28) to (3.22), and noting $0 \leqslant b'(\phi) \leqslant p$ for $v_- \leqslant \phi \leqslant v_+$, and $\eta = e^{2\lambda_*^2\alpha - \lambda_* cr}$, it can be verified that

$$B_{\eta,\mu,w}(\xi) \geqslant C_1 - 2\mu - \frac{\varepsilon}{\eta}\bigl(e^{2\mu r} - 1\bigr) b'\bigl(\phi(\xi)\bigr) \int_R \frac{w(\xi + y + cr)}{w(\xi)} f_\alpha(y)\,dy$$

$$\begin{aligned}
&\geqslant C_1 - 2\mu - \frac{\varepsilon p}{\eta}\left(e^{2\mu r}-1\right)\int_R e^{-2\lambda_*(y+cr)} f_\alpha(y)\,dy \\
&= C_1 - 2\mu - \frac{\varepsilon p}{\eta}\left(e^{2\mu r}-1\right)e^{4\lambda_*^2\alpha - 2\lambda_* cr}\int_R \frac{1}{\sqrt{4\pi\alpha}} e^{-(\frac{y}{\sqrt{4\alpha}}+\sqrt{4\alpha}\lambda_*)^2}\,dy \\
&= C_1 - 2\mu - \varepsilon p\eta\left(e^{2\mu r}-1\right) \\
&=: C_2 > 0, \quad \text{for } 0<\mu<\mu_1. \qquad (3.32)
\end{aligned}$$

This completes the proof of the lemma. □

Below we will derive the basic energy estimates which are crucial for our main stability result.

Applying (3.31) to (3.27), and dropping the positive term $\int_0^t\int_R e^{2\mu\tau} B_{\eta,\mu,w}(\xi)w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau$, one then immediately establishes the first energy estimate as follows.

Lemma 3.5. *It holds that*

$$e^{2\mu t}\left\|u(t)\right\|_{L_w^2}^2 \leqslant C\left(\left\|u_0(0)\right\|_{L_w^2}^2 + \int_{-r}^0 \left\|u_0(\tau)\right\|_{L_w^2}^2\,d\tau\right), \quad t\geqslant 0. \qquad (3.33)$$

Furthermore, differentiating Eq. (3.13) with respect to ξ, and multiplying it by $e^{2\mu t}w(\xi)u_\xi(t,\xi)$, and integrating the resultant equation over $R\times[0,t]$ with respect to ξ and t, then by using (3.33) in Lemma 3.5, one can obtain the second energy estimate as follows.

Lemma 3.6. *It holds that*

$$e^{2\mu t}\left\|u_\xi(t)\right\|_{L_w^2}^2 \leqslant C\left(\left\|u_0(0)\right\|_{H_w^1}^2 + \int_{-r}^0 \left\|u_0(\tau)\right\|_{H_w^1}^2\,d\tau\right), \quad t\geqslant 0. \qquad (3.34)$$

Thus, (3.33) and (3.34) imply the following lemma.

Lemma 3.7. *It holds that*

$$\left\|u(t)\right\|_{H_w^1}^2 \leqslant Ce^{-2\mu t}\left(\left\|u_0(0)\right\|_{H_w^1}^2 + \int_{-r}^0 \left\|u_0(\tau)\right\|_{H_w^1}^2\,d\tau\right), \quad t\geqslant 0. \qquad (3.35)$$

Notice that, $w(\xi)\to 0$ as $\xi\to\infty$, we cannot expect $H_w^1(R)\hookrightarrow C(R)$. However, for any interval $I=(-\infty,\bar\xi]$ for some large $\bar\xi\gg 1$, we may have the Sobolev's embedding result $H_w^1(I)\hookrightarrow C(I)$, which when combining with (3.35) gives the following L^∞-estimate.

Lemma 3.8. *It holds that*

$$\sup_{\xi\in I}\left|u(t,\xi)\right| \leqslant Ce^{-\mu t}\left(\left\|u_0(0)\right\|_{H_w^1}^2 + \int_{-r}^0 \left\|u_0(\tau)\right\|_{H_w^1}^2\,d\tau\right)^{1/2}, \quad t>0, \qquad (3.36)$$

for any interval $I=(-\infty,\bar\xi]$ with some large $\bar\xi\gg 1$.

Next, we need to extend the L^∞-convergence in (3.36) to the whole space $(-\infty, \infty)$. The key step is to prove the convergence at $\xi = \infty$.

Lemma 3.9. *It holds that*

$$\lim_{\xi \to \infty} \left| u(t, \xi) \right| \leqslant C e^{-\mu_2 t}, \quad t \geqslant 0, \tag{3.37}$$

where $\mu_2 := d_m - \varepsilon b'(v_+) > 0$.

Proof. As shown in (3.24), i.e., $Q(t - r, x - y) \leqslant 0$, (3.13) is reduced to

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial \xi} - D_m \frac{\partial^2 u}{\partial \xi^2} + d_m u - \varepsilon \int_{-\infty}^{\infty} b'\big(\phi(\xi - y - cr)\big) u(t - r, \xi - y - cr) f_\alpha(y)\, dy \leqslant 0. \tag{3.38}$$

Taking limits as $\xi \to \infty$, and noting that $u_\xi(t, \infty) = 0$, $u_{\xi\xi}(t, \infty) = 0$ due to the boundedness of $u(t, \xi)$ for all $\xi \in (-\infty, \infty)$, one immediately obtains

$$\frac{d}{dt} u(t, \infty) + d_m u(t, \infty) - \varepsilon b'(v_+) u(t - r, \infty) \int_{-\infty}^{\infty} f_\alpha(y)\, dy \leqslant 0.$$

Since $\int_R f_\alpha(y)\, dy = 1$, therefore

$$\frac{d}{dt} u(t, \infty) + d_m u(t, \infty) - \varepsilon b'(v_+) u(t - r, \infty) \leqslant 0. \tag{3.39}$$

Integrating (3.39) over $[0, t]$, one then has

$$u(t, \infty) + d_m \int_0^t u(\tau, \infty)\, d\tau - \varepsilon b'(v_+) \int_0^t u(\tau - r, \infty)\, d\tau \leqslant u_0(0, \infty). \tag{3.40}$$

By the change of variable $\tau - r \to \tau$ to the third term of (3.40), it follows that

$$\begin{aligned} \varepsilon b'(v_+) \int_0^t u(\tau - r, \infty)\, d\tau &= \varepsilon b'(v_+) \int_{-r}^{t-r} u(\tau, \infty)\, d\tau \\ &\leqslant \varepsilon b'(v_+) \int_0^t u(\tau, \infty)\, d\tau + \varepsilon b'(v_+) \int_{-r}^0 u_0(\tau, \infty)\, d\tau. \end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.40), we have

$$u(t, \infty) + \big[d_m - \varepsilon b'(v_+)\big] \int_0^t u(\tau, \infty)\, d\tau \leqslant C_3, \tag{3.42}$$

where

$$C_3 := u_0(0,\infty) + \varepsilon b'(v_+)\int_{-r}^{0} u_0(\tau,\infty)\,d\tau.$$

By a straightforward but tedious computation, one can check

$$\varepsilon b_1'(v_+) < d_m \quad \text{and} \quad \varepsilon b_2'(v_+) < d_m.$$

Thus, when $b(v) = b_1(v)$ or $b(v) = b_2(v)$, one always has

$$\varepsilon b'(v_+) < d_m.$$

Then, by the Gronwall's inequality, (3.42) yields

$$u(t,\infty) \leqslant Ce^{-\mu_2 t}, \tag{3.43}$$

with $\mu_2 = d_m - \varepsilon b'(v_+) > 0$. The proof is complete. □

Combining (3.37) and (3.38), and letting $0 < \mu < \min\{\mu_1, \mu_2\}$, one finally proves the L^∞-convergence for all $\xi \in R$.

Lemma 3.10. *It holds that*

$$\sup_{x\in R}\left|V^+(t,\xi) - \phi(x+ct)\right| = \sup_{\xi\in R}\left|u(t,\xi)\right| \leqslant Ce^{-\mu t}, \quad t \geqslant 0, \tag{3.44}$$

where $0 < \mu < \min\{\mu_1, \mu_2\}$.

Step 2: The convergence of $V^-(t,x)$ to $\phi(x+ct)$.

Let $\xi = x + ct$ and

$$u(t,\xi) = \phi(x+ct) - V^-(t,x), \qquad u_0(s,\xi) = \phi(x+cs) - V_0^-(s,x). \tag{3.45}$$

As shown in the above, we can similarly prove that $V^-(t,x)$ converges to $\phi(x+ct)$.

Lemma 3.11. *It holds that*

$$\sup_{x\in R}\left|V^-(t,x) - \phi(x+ct)\right| = \sup_{\xi\in R}\left|u(t,\xi)\right| \leqslant Ce^{-\mu t}, \quad t \geqslant 0, \tag{3.46}$$

where $0 < \mu < \min\{\mu_1, \mu_2\}$.

Step 3: The convergence of $v(t,x)$ to $\phi(x+ct)$.

Since the initial data satisfy

$$V_0^-(x,s) \leqslant v_0(x,s) \leqslant V_0^+(x,s),$$

by Lemma 3.2, it can be verified that the corresponding solutions of (1.1) and (1.2) satisfy

$$V^-(t,x) \leqslant v(t,x) \leqslant V^+(t,x), \quad (t,x) \in R_+ \times R.$$

Combining (3.44) in Lemma 3.10 and (3.46) in Lemma 3.11, we have the following stability result.

Table 1
Case studies: parameters and initial data.

Case	r	α	a	p	d_m	ε	D_m	$v_+ = \frac{1}{a}\ln\frac{p}{d}$	Initial data $v_0(s,x)$ for $s \in [-r,0]$
1	0.5	0.5	0.1	2.17	1	1	1	7.7472717	$\begin{cases} 0, & x \leqslant 0, \\ u_+, & x > 0 \end{cases}$
2	0.5	0.25	0.1	2.17	1	1	1	7.7472717	$\begin{cases} 0, & x \leqslant -20, \\ 40, & -20 < x \leqslant 20, \\ u_+, & x > 20 \end{cases}$
3	10	10	0.1	2.17	1	1	1	7.7472717	$\begin{cases} 0, & x \leqslant 0, \\ u_+, & x > 0 \end{cases}$
4	10	5	0.1	2.17	1	1	1	7.7472717	$\begin{cases} 0, & x \leqslant -20, \\ 40, & -20 < x \leqslant 20, \\ u_+, & x > 20 \end{cases}$

Lemma 3.12. *It holds that*

$$\sup_{x\in R}\left|v(t,x)-\phi(x+ct)\right| \leqslant Ce^{-\mu t}, \quad t \geqslant 0, \tag{3.47}$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

4. Numerical computations

In this section, we are going to carry out some numerical simulations, which will confirm our theoretical results. Although a number of numerical results were also presented by Liang and Wu in [12], the case of a large initial perturbation around the wavefront was not included.

The mathematical formulation given in Eqs. (1.1) and (1.2) is a nonlinear time-delayed partial differential equation with a nonlocal nonlinearity. The computational results reported in this section are based on the Crank–Nicholson scheme together with an approximation to the nonlocal linearity by Simpson's rule. It is noted that the nonlinear term is time-delayed; hence, the resulting finite-difference equations are linear. The advantage of this simple implicit scheme is that: (i) it is linear and, hence, no nonlinear solver (for example, an iterative method) is required to solve the resulting linear difference equations in each time step; (ii) the Crank–Nicholson scheme is unconditionally stable and, hence, a larger time-step can be allowed which makes our scheme more efficient when implemented.

In computation, the sizes of the time step and space step are chosen as $\Delta t = 0.04$ and $\Delta x = 0.08$. Although the original model assumes the spatial domain is the whole domain, a finite computational domain $(-L, L)$ is imposed. Here, we let $L = 800$, then the computational domain is sufficiently large so that numerical boundary effect is ignorable. In this section, let the birth rate function be $b_1(v) = pve^{-av}$ with $q = 1$. We report the numerical simulations for four test cases (see Table 1). For simplicity, we first choose $D_m = 1$, $\varepsilon = 1$ and $d_m = 1$; other parameters and the initial data for each case study are listed in Table 1. The final computed time is 120 for Cases 1 and 2, and 300 for Cases 3 and 4.

By using the Crank–Nicholson scheme, we obtain the following numerical results, which also demonstrate the stability of the traveling wavefronts. In Case 1, the delay time is chosen to be small ($r = 0.5$), and the initial data $v_0(s,x)$ is within $[v_-, v_+]$. The numerical results show that, after a short time the solution $v(t,x)$ of Eqs. (1.1) and (1.2) behaves as a traveling wave which propagates from right to left with a positive speed c (see Fig. 1). This numerically confirms the stability of the traveling wavefronts.

In Case 2, we still take the delay time small as in Case 1, but choose the initial data to exceed v_+ a lot in some points, namely, the initial perturbation around the traveling wavefront should be really large. The numerical computations in Fig. 2 demonstrate that the solution $v(t,x)$ asymptotically behaves like a traveling wavefront propagating from right to left with a positive speed c.

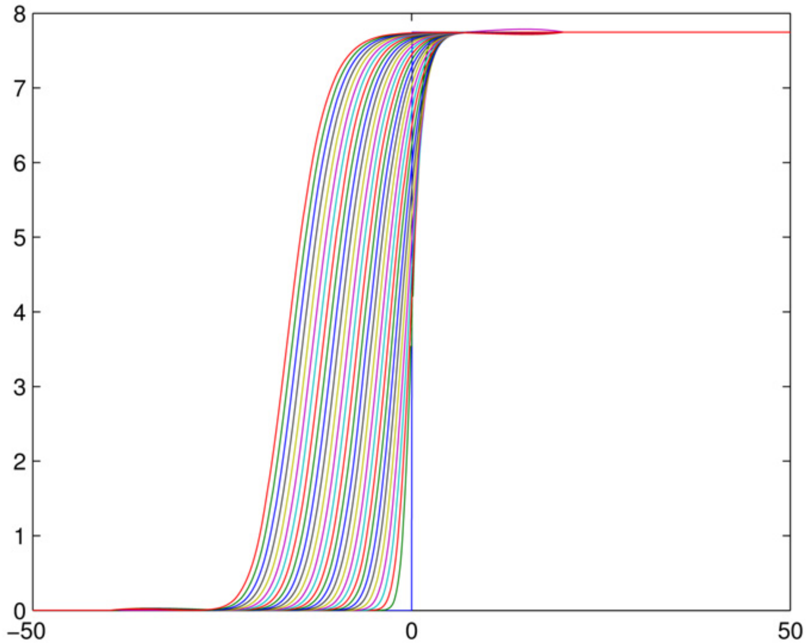

Fig. 1. Case 1: for the small delay time $r = 0.5$ and the initial data within $[v_-, v_+]$, the presented graphs are for the solution $v(t, x)$ at different time t, which behave like a stable traveling wavefront.

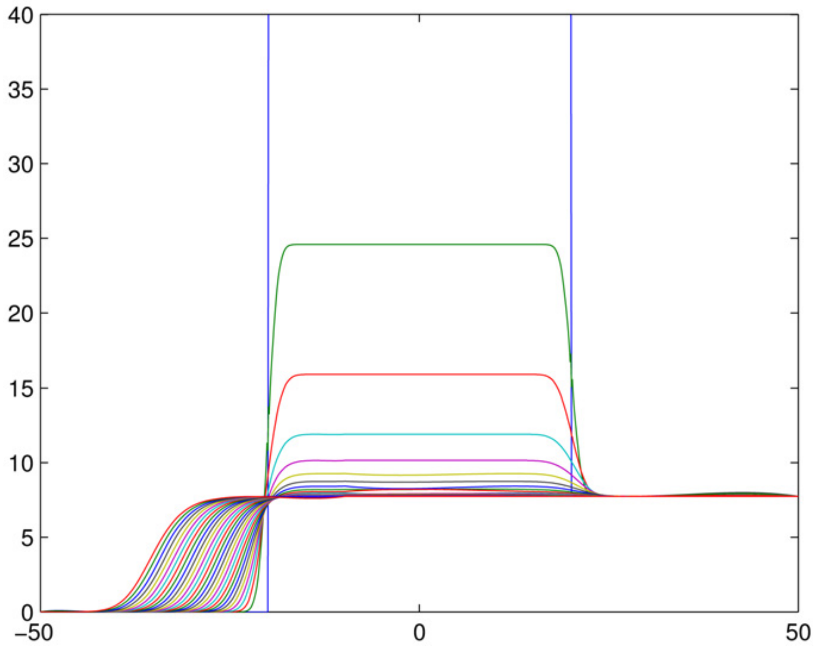

Fig. 2. Case 2: for the small delay time $r = 0.5$ and the initial data exceeding v_+ at some points, the presented graphs are for the solution $v(t, x)$ at different time t, which behave like a stable traveling wavefront.

Cases 3 and 4 are to show the stability of wavefronts with a large delay time. At this stage, we take $r = 10$, but the initial data $v_0(s, x)$ is selected as what we did in Cases 1 and 2 before, respectively. The numerical simulations presented in Figs. 3 and 4 show also that, after a short time, the solution $v(t, x)$ always behaves like a traveling wavefront which propagates from left to right with a positive speed c.

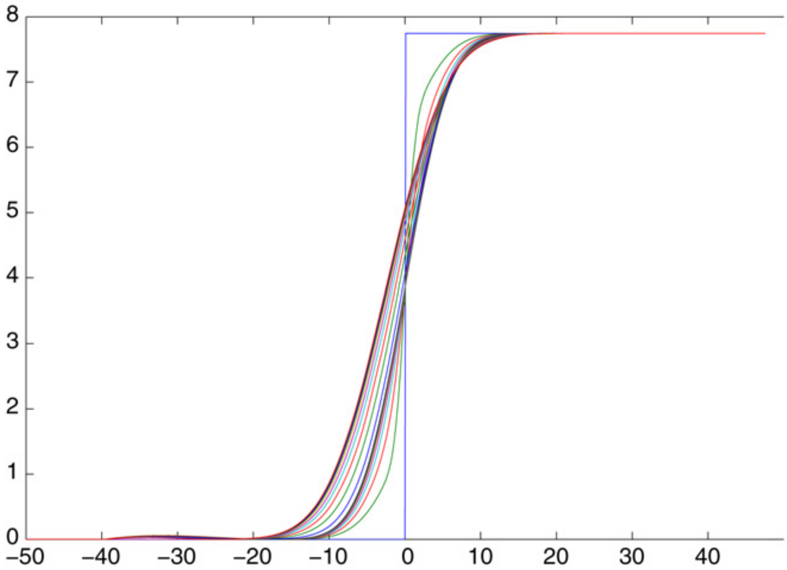

Fig. 3. Case 3: for the big delay time $r = 10$ and the initial data within $[v_-, v_+]$, the presented graphs are for the solution $v(t, x)$ at different time t, which behave like a stable traveling wavefront.

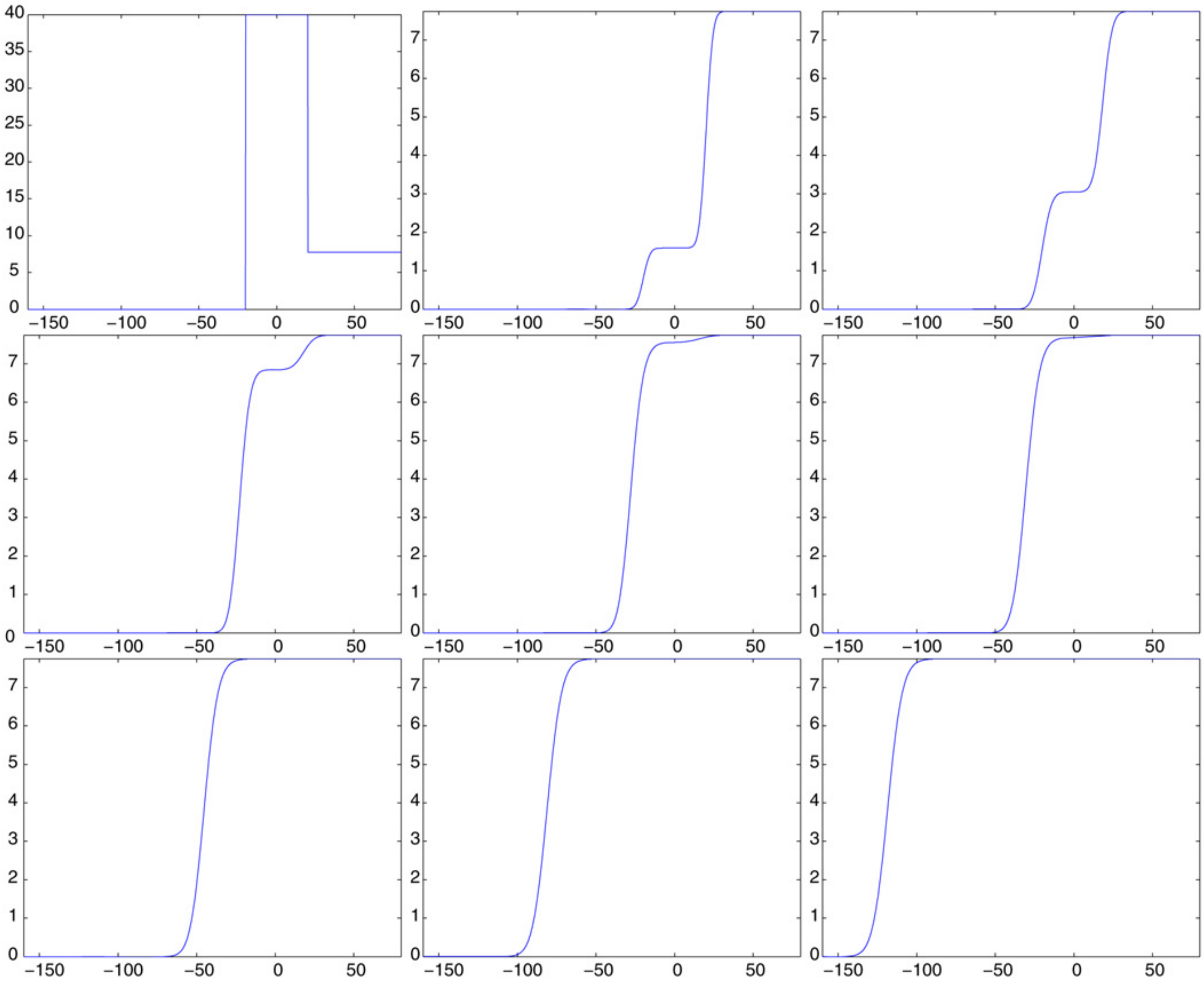

Fig. 4. Case 4: for the big delay time $r = 10$ and the initial data exceeding $[v_-, v_+]$, the presented graphs are for the solution $v(t, x)$ in different time t, which behave like a stable traveling wavefront.

5. Remark

For the local time-delayed reaction–diffusion equation

$$\frac{\partial v}{\partial t} - D_m \frac{\partial^2 v}{\partial x^2} + d_m v = \varepsilon b\big(v(t-r,x)\big), \quad (t,x) \in (0,\infty) \times R, \tag{5.1}$$

the stability has been proved in the first part [16] with a restriction $b'(v_+) \ll 1$, even so it is reasonable as explained in [16]. Such a condition is needed in the stability proof for $x \gg 1$. However, as we know, when $x \gg 1$, $v = v_+$ is the stable node of (5.1). This will be an advantage for the stability proof. By setting the weight function $w(x)$ as (2.7), we can prove a similar result as Lemma 3.9, which leads us to remove the condition $b(v_+) \ll 1$.

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