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Stability of strong travelling waves for a non-local time-delayed reaction–diffusion equation

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The paper is concerned with a non-local time-delayed reaction–diffusion equation. We prove the (time) asymptotic stability of a travelling wavefront without a smallness assumption on its wavelength, i.e. the so-called strong wavefront, by means of the (technical) weighted energy method, when the initial perturbation around the wave is small. The exponential convergent rate is also given. Selection of a suitable weight plays a key role in the proof.

1. Introduction

For population dynamics with age structure and diffusion, Metz and Diekmann [16] studied the governing model

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} - D(z)\frac{\partial^2 u}{\partial x^2} + d(z)u = 0, \tag{1.1}$$

where $u(t, z, x)$ denotes the population density of the species under consideration at time $t \geqslant 0$, age $z \geqslant 0$ and location $x \in \Omega$, and $D(z)$ and $d(z)$ are the diffusion rate and death rate of population at age z, respectively. We denote by $r > 0$ the maturation time for the species. Ω is the spatial domain where the species live and it can be bounded or unbounded. We take Ω to be $\mathbb{R} = (-\infty, \infty)$ in this paper.

Let $v(t, x)$ be the total mature population at time t and location x

$$v(t, x) = \int_r^\infty u(t, z, x)\,\mathrm{d}z, \tag{1.2}$$

and let $D_\mathrm{m}(z) = D(z)$ for $z \in [r, \infty)$ and $D_\mathrm{i}(z) = D(z)$ for $z \in [0, r)$ be the diffusion rates for the mature population and the immature population, respectively. In the event that the mature population is more effective in spatial diffusion than the immature population, it is then reasonable to assume that

$$\min_{z\in[r,\infty)} D_{\mathrm{m}}(z) \geqslant \max_{z\in[0,r]} D_{\mathrm{i}}(z). \tag{1.3}$$

We also denote by $d_{\mathrm{m}}(z) = d(z)$ for $z \in [r,\infty)$ and $d_{\mathrm{i}}(z) = d(z)$ for $z \in [0,r)$ the death rates for the mature population and the immature population, respectively. At age zero, $u(t,0,x)$ is the population density of the newborns. Since only matures can reproduce, we have

$$u(t,0,x) = b(v(t,x)), \tag{1.4}$$

where $b(\,\cdot)$ is the birth function.

When the diffusion and death rates for the mature population are independent of age, namely,

$$D_{\mathrm{m}}(z) = D_{\mathrm{m}} \quad \text{and} \quad d_{\mathrm{m}}(z) = d_{\mathrm{m}} \tag{1.5}$$

are constants for $z \in [r,\infty)$, So *et al.* [26], following the approach of Smith and Thieme [22] (see also [9, 27]), reduced equation (1.1) into a non-local time-delayed reaction–diffusion equation for $v(t,x)$ by the Fourier transform method,

$$\frac{\partial v}{\partial t} - D_{\mathrm{m}}\frac{\partial^2 v}{\partial x^2} + d_{\mathrm{m}} v = \varepsilon \int_{-\infty}^{\infty} b(v(t-r, x-y)) f_\alpha(y)\,\mathrm{d}y, \tag{1.6}$$

where

$$\varepsilon = \exp\biggl(-\int_0^r d_{\mathrm{i}}(z)\,\mathrm{d}z\biggr) \quad \text{and} \quad \alpha = \int_0^r D_{\mathrm{i}}(z)\,\mathrm{d}z \tag{1.7}$$

represent the impact of the death rate of the immature and the effect of the dispersal rate of the immature on the mature population, respectively. By (1.3), (1.5) and (1.7), we have

$$\alpha \leqslant r D_{\mathrm{m}}. \tag{1.8}$$

In (1.6), $f_\alpha(y)$ is the heat kernel function

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} \mathrm{e}^{-y^2/4\alpha} \quad \text{and} \quad \int_{-\infty}^{\infty} f_\alpha(y)\,\mathrm{d}y = 1. \tag{1.9}$$

In particular, when the birth function $b(v)$ is that used for Nicolson's blowflies [5, 15, 24–26], that is

$$b(v) = pv\mathrm{e}^{-av}, \tag{1.10}$$

where $p > 0$ and $a > 0$ are constants, the constant equilibria for equation (1.1) can be found by solving

$$d_{\mathrm{m}} v = \varepsilon p \int_{-\infty}^{\infty} v\mathrm{e}^{-av} f_\alpha(y)\,\mathrm{d}y.$$

By (1.9), this equation admits only two roots:

$$v_- = 0, \qquad v_+ = \frac{1}{a} \ln \frac{\varepsilon p}{d_{\mathrm{m}}}. \tag{1.11}$$

If $\varepsilon p/d_{\mathrm{m}} > 1$, then $v_- < v_+$. In [26], So *et al.* showed the existence of travelling waves $\phi(x+ct)$ connecting the two equilibria $v_\pm$ with speed c. For other models with non-local terms, the existence of travelling waves has been shown in [4,9,27,33] (see also the references therein).

Here, we are interested in the asymptotic stability for such waves. For the Cauchy problem for equation (1.6) with the birth function (1.10) and the initial data

$$v(t,x)|_{t=s} = v_0(s,x) \quad \text{for } s \in [-r,0],\ x \in \mathbb{R}, \tag{1.12}$$

where

$$v_0(s,x) \to v_\pm \quad \text{for all } s \in [-r,0] \text{ as } x \to \pm\infty,$$

we prove that the global solution $v(t,x)$ of (1.6), (1.10), (1.12) converges to the travelling wave $\phi(x+ct)$ asymptotically (in time), when the initial perturbation around the wave, that is, $|v_0(s,x) - \phi(x+cs)|$, $s \in [-r,0]$, is suitably small. The exponential convergence rate will also be obtained.

The study of the stability of travelling waves is interesting and usually (technically) difficult. For partial differential equations without time delays, including reaction–diffusion equations, travelling waves have been extensively studied; see, for example, the pioneering works [6, 20] and other more recent contributions [1–3, 7–14, 18–20, 30, 31], and the references therein (see also [28] and the survey papers for viscous equations of conservation laws by Matsumura [10] and the reaction–diffusion equations by Xin [32]). However, results for the time-delayed partial differential equations are very limited and incomplete. The first work related to this topic was done by Schaaf [21] on the linearized stability of the time-delayed Fisher–Kolmogorov–Petrovski–Piskunov equation by means of the spectral method. Later, Ogiwara and Matano [17] and Smith and Zhao [23] studied the nonlinear stability by the method of upper and lower solutions. See also [29]. More recently, Mei *et al.* [15] proved the nonlinear wave stability for the local equation with birth function (1.10), where, for the two steady states connected by the travelling wave, one of the equilibria is an unstable node. Such a case is different from the 'bistable' nodes studied in [23]. For the non-local case, Liang and Wu [9] studied theoretically the existence of the travelling waves for (1.6) with a different birth function, $b(v)$, and, furthermore, showed the wave approximations numerically.

Following [9, 15], we treat the non-local case here with nonlinearity (1.10), and prove theoretically the stability for the strong travelling waves. A wave is said to be weak if its wavelength is small, that is, $|v_+ - v_-| \ll 1$; otherwise, the wave is said to be strong. As is well known, one may prove the stability for the weak waves in certain cases, but one cannot usually prove it for the strong wave cases. As in [15], the approach adopted in this paper is still the weighted energy method. In the proof, a key role is played by the selection of a suitable weight function; see the key lemma (lemma 3.6), below.

The rest of the paper is organized as follows. In § 2, we state the result on the existence of travelling waves as given in [26]. After defining a suitable weight function, we state the theorem on wave stability. Section 3 is devoted to the proof of the stability theorem using the weighted energy method. The key step in the proof is to establish *a priori* estimates.

Before ending this section, we give some notation. Throughout the paper, $C > 0$ denotes a generic constant, while $C_i > 0$, $i = 0, 1, 2, \dots$, represents a specific constant. Let I be an interval; typically $I = \mathbb{R}$. $L^2(I)$ is the space of square integrable functions on an interval I, and $H^k(I)$, $k \geqslant 0$, is the Sobolev space of L^2-functions $f(x)$ defined on the interval I whose derivatives $\mathrm{d}^i f/\mathrm{d}x^i$, $i = 1, \dots, k$, also belong to $L^2(I)$. $L^2_w(I)$ represents the weighted L^2-space with weight $w(x) > 0$. Its norm is defined by

$$\|f\|_{L^2_w} = \left(\int_I w(x) f(x)^2 \,\mathrm{d}x \right)^{1/2}.$$

$H^k_w(I)$ is the weighted Sobolev space with the norm

$$\|f\|_{H^k_w} = \left(\sum_{i=0}^{k} \int_I w(x) \left| \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \right|^2 \mathrm{d}x \right)^{1/2}.$$

Let $T > 0$ and let $\mathcal{B}$ be a Banach space. We denote by $C^0([0,T];\mathcal{B})$ the space of $\mathcal{B}$-valued continuous functions on $[0,T]$, and by $L^2([0,T];\mathcal{B})$ the space of $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

2. Stability of strong travelling waves

A travelling wave of the type in equation (1.6) with the birth function (1.10) connecting with two constant steady states $v_\pm$ is a special solution of equation (1.6) of the form $\phi(x + ct)$ ($c > 0$ is the wave speed) satisfying the non-local delayed ordinary differential equation

$$\left.\begin{aligned} c\phi'(\xi) - D_\mathrm{m}\phi''(\xi) + d_\mathrm{m}\phi(\xi) &= \varepsilon p \int_{-\infty}^{\infty} \phi(\xi - cr - y)\mathrm{e}^{-a\phi(\xi - cr - y)} f_\alpha(y)\,\mathrm{d}y, \\ \phi(\pm\infty) &= v_\pm, \end{aligned}\right\} \tag{2.1}$$

where $\xi = x + ct$ and the prime denotes differentiation with respect to ξ. Using the upper and lower solution method, So *et al.* [26] proved the following result on the existence of monotone wavefronts $\phi(\xi)$ with $\phi'(\xi) > 0$.

PROPOSITION 2.1 (So *et al.* [26]). *If* $1 < \varepsilon p/d_\mathrm{m} \leqslant \mathrm{e}$, *then there exists a critical number* c^*,

$$0 < c^* < 2\sqrt{D_\mathrm{m}(\varepsilon p - d_\mathrm{m})}, \tag{2.2}$$

such that, for every $c > c^*$, *equation (1.6) has a travelling wavefront solution* $\phi(\xi)$ *connecting* $v_\pm$, *with* $\phi'(\xi) > 0$ *and* $v_- < \phi(\xi) < v_+$ *for all* $\xi \in (-\infty, \infty)$.

As is well known, in order to prove wave stability it is often necessary to restrict the wavelength to be sufficiently small (that is, $|v_+ - v_-| \ll 1$). Such a wave is called a weak wave. Here, we are interested in establishing the stability of a strong wave. For this, throughout the present paper, let us take $\varepsilon p/d_\mathrm{m}$ in proposition 2.1 ($1 < \varepsilon p/d_\mathrm{m} \leqslant \mathrm{e}$) to be e, that is,

$$\frac{\varepsilon p}{d_\mathrm{m}} = \mathrm{e}, \tag{2.3}$$

so that

$$v_+ = \frac{1}{a} \ln \frac{\varepsilon p}{d_\mathrm{m}} = \frac{1}{a}$$

is maximum, namely, the wave $\phi(\xi)$ connecting the two equilibria, $v_- = 0$ and $v_+ = 1/a$, is the strongest.

Let

$$\bar{\varepsilon} := \frac{(3 - \mathrm{e})d_\mathrm{m}}{2(ad_\mathrm{m} + aD_\mathrm{m} + \frac{3}{2}\mathrm{e}d_\mathrm{m})}. \tag{2.4}$$

We first have the following estimates.

LEMMA 2.2. *For a given strongest travelling wave $\phi(\xi)$, $\xi = x + ct$, there exists a number x_* such that, for $\xi > x_*$, the following inequalities hold:*

$$\left.\begin{aligned} \phi(\xi) &> \frac{1}{a} - \bar{\varepsilon}, \\ |\phi''(\xi)| &< \bar{\varepsilon}, \\ 0 < \mathrm{e}^{a\phi(\xi)}(1 - a\phi(\xi)) &< \bar{\varepsilon}. \end{aligned}\right\} \tag{2.5}$$

Proof. For the given strongest travelling wave $\phi(\xi)$, it is easy to see that $0 = v_- \leqslant \phi(\xi) \leqslant v_+ = 1/a$ because $\phi(\xi)$ is increasing. Since $\lim_{\xi\to+\infty}\phi(\xi) = v_+ = 1/a$, $\lim_{\xi\to+\infty}\phi''(\xi) = 0$ and $\lim_{\xi\to+\infty}(1 - a\phi(\xi)) = 0$, by the definition of limits, for the given $\bar{\varepsilon}$ there exists a number x_* such that when $\xi > x_*$ the following hold:

$$|\phi(\xi) - v_+| = \left|\phi(\xi) - \frac{1}{a}\right| < \bar{\varepsilon},$$
$$|\phi''(\xi)| < \bar{\varepsilon},$$
$$0 < \mathrm{e}^{a\phi(\xi)}(1 - a\phi(\xi)) < \bar{\varepsilon}.$$

These inequalities immediately imply (2.5). □

Now we define a weight function $w(\xi)$ as

$$w(\xi) = \begin{cases} \mathrm{e}^{-\beta(\xi - x_*)} = \mathrm{e}^{\beta|\xi - x_*|}, & \xi < x_*, \\ 1, & \xi \geqslant x_*, \end{cases} \tag{2.6}$$

where

$$\beta = \frac{c}{2D_\mathrm{m}}. \tag{2.7}$$

Our main result for the paper is the following.

THEOREM 2.3 (stability). *Consider the given strongest travelling wavefront $\phi(x + ct)$, where the speed $c > c_*$ satisfies*

$$c > 2\sqrt{D_\mathrm{m}(3\varepsilon p - 2d_\mathrm{m})}. \tag{2.8}$$

If

$$v_0(s, x) - \phi(x + cs) \in C^0([-r, 0]; H^1_w(\mathbb{R})), \tag{2.9}$$

where $w = w(x + cs)$, $s \in [-r, 0]$, is the weight function given in (2.6), then there exist positive constants δ_0 and μ, which are dependent only on the coefficients D_m, d_m, ε, p, a, r and the wave speed c, such that when $\|v_0(s, \cdot) - \phi(\,\cdot + cs)\|_{H^1_w} \leqslant \delta_0$ for

$s \in [-r, 0]$ the unique solution $v(t, x)$ of the Cauchy problem (1.6), (1.10), (1.12) exists globally, and it satisfies

$$v(t, x) - \phi(x + ct) \in C^0([0, \infty); H^1_w) \cap L^2([0, \infty); H^2_w)$$

and

$$\sup_{x\in\mathbb{R}} |v(t, x) - \phi(x + ct)| \leqslant C\mathrm{e}^{-\mu t}, \quad 0 \leqslant t < \infty, \tag{2.10}$$

where $C > 0$ is a constant dependent only on the initial perturbation $v_0(x, s) - \phi(x - cs)$.

REMARK 2.4. (i) Note that (2.2), (2.8) and $3\varepsilon p - 2d_{\mathrm{m}} > \varepsilon p - d_{\mathrm{m}}$ imply that

$$c > 2\sqrt{D_{\mathrm{m}}(3\varepsilon p - 2d_{\mathrm{m}})} > 2\sqrt{D_{\mathrm{m}}(\varepsilon p - d_{\mathrm{m}})} > c^*.$$

Thus, theorem 2.3 ensures that when the wave speed is not too close to c^* the strongest wave is asymptotically stable (in time). For speed c close to the critical point c^*, and in particular the case when $c = c^*$, the stability problem is still open.

(ii) By the definition of weight function (2.6) and the definition of the weighted Sobolev space $H^1_w(R)$, we have from (2.9) that

$$\sqrt{w(x + cs)}(v_0(s, x) - \phi(x + cs)) \in H^1(R), \quad s \in [-r, 0].$$

Thus, applying Sobolev's inequality, we obtain

$$\sqrt{w(x + cs)}(v_0(s, x) - \phi(x + cs)) \leqslant C\|v_0 - \phi\|_{H^1_w}, \quad s \in [-r, 0],$$

which in turn implies that

$$|v_0(s, x) - \phi(x + cs)| \sim w^{-1/2}(x + cs) \sim \mathrm{e}^{-\beta|x|/2} \quad \text{as } x \to -\infty.$$

(iii) Since c is large, by a straightforward but tedious computation we find that the convergence of the initial perturbation $|v_0(s, x) - \phi(x + ct)| \sim O(1)\exp\{-c|x|/4D_{\mathrm{m}}\}$ for $x \to -\infty$ is faster than the decay of the wavefront to $v_- = 0$ in the form $|\phi(x + cs) - v_-| = |\phi(x + cs)| \sim O(1)\mathrm{e}^{-\beta_-|x|}$ as $x \to -\infty$, where β_- is a positive constant satisfying $\beta_- < c/4D_{\mathrm{m}}$. So, as we show in theorem 2.3, it is not surprising that, when $|v_0(s, x) - \phi(x + cs)| \ll 1$, the solution $v(t, x)$ converges to $\phi(x + ct)$ and not to some shifted wave $\phi(x + ct + x_0)$ with a shift x_0. In fact, by another tedious computation, as shown in [11], we can formally show that $x_0 = 0$.

3. Proof of stability

This section is devoted to the proof of the stability result, theorem 2.3. Our proof relies on the weighted energy method.

Let $v(t, x)$ be the solution of the Cauchy problem (1.6), (1.10), (1.12), and let $\phi(x + ct)$ be the wavefront. Set

$$V(t, \xi) = v(t, x) - \phi(\xi), \quad \xi = x + ct.$$

The original problem (1.6), (1.10) and (1.12) can be reformulated as

$$\left.\begin{aligned} &V_t(t,\xi)+cV_\xi(t,\xi)-D_{\mathrm{m}}V_{\xi\xi}(t,\xi)+d_{\mathrm{m}}V(t,\xi)\\ &\qquad -\varepsilon\int_{\mathbb{R}}b'(\phi(\xi-y-cr))V(t-r,\xi-y-cr)f_\alpha(y)\,\mathrm{d}y\\ &\qquad\qquad =\varepsilon\int_{\mathbb{R}}Q(V(t-r,\xi-y-cr))f_\alpha(y)\,\mathrm{d}y,\quad (t,\xi)\in\mathbb{R}_+\times\mathbb{R},\\ &V(s,\xi)=v_0(s,\xi-cs)-\phi(\xi)=:V_0(s,\xi),\quad (s,\xi)\in[-r,0]\times\mathbb{R}. \end{aligned}\right\} \tag{3.1}$$

The nonlinear term $Q(V(t-r,\xi-y-cr))$ is given by

$$Q(V)=b(\phi+V)-b(\phi)-b'(\phi)V, \tag{3.2}$$

where $\phi=\phi(\xi-y-cr)$ and $V=V(t-r,\xi-y-cr)$. It suffices to prove the following stability result for equation (3.1).

THEOREM 3.1. *For the given strongest travelling wave $\phi(\xi)$, $\xi=x+ct$, with speed c satisfying (2.8), if $V_0(s,\xi)\in C^0([-r,0];H^1_w(\mathbb{R}))$, where $w(\xi)$ is the weight function defined in (2.6), then there exist positive constants δ_0 and μ such that when $\sup_{s\in[-r,0]}\|V_0(s)\|_{H^1_w}\leqslant\delta_0$, the solution $V(t,\xi)$ of the Cauchy problem (3.1) exists uniquely and globally, and satisfies*

$$V(t,\xi)\in C^0([0,\infty);H^1_w(\mathbb{R}))\cap L^2([0,\infty);H^2_w(\mathbb{R}))$$

and

$$\sup_{\xi\in\mathbb{R}}|V(t,\xi)|\leqslant C\mathrm{e}^{-\mu t},\quad 0\leqslant t\leqslant\infty. \tag{3.3}$$

Note that $\mu>0$ and $\delta_0>0$ are the same as those in theorem 2.3. We will prove theorem 3.1 based on the following two propositions: one local estimate and an *a priori* estimate by the continuity argument (see also [7, 11, 12, 14]).

For given constants $\tau\geqslant 0$ and $T>0$, we define the solution space by

$$X(\tau-r,T+\tau)=\{V\mid V(t,\xi)\in C^0([\tau-r,T+\tau];H^1_w(\mathbb{R}))\cap L^2([\tau-r,T+\tau];H^2_w(\mathbb{R}))\}$$

and

$$M_\tau(T):=\sup_{t\in[\tau-r,T+\tau]}\|V(t)\|_{H^1_w},$$

in particular, $M(T):=M_0(T)$ for $\tau=0$. For simplicity, we henceforth define $V(t)=V(t,\cdot)$. First we state the local estimate.

PROPOSITION 3.2 (local existence). *Consider the Cauchy problem with the initial time $\tau\geqslant 0$*

$$\left.\begin{aligned} &V_t(t,\xi)+cV_\xi(t,\xi)-D_{\mathrm{m}}V_{\xi\xi}(t,\xi)+d_{\mathrm{m}}V(t,\xi)\\ &\qquad -\varepsilon\int_{\mathbb{R}}b'(\phi(\xi-y-cr))V(t-r,\xi-y-cr)f_\alpha(y)\,\mathrm{d}y\\ &\qquad\qquad =\varepsilon\int_{\mathbb{R}}Q(V(t-r,\xi-y-cr))f_\alpha(y)\,\mathrm{d}y,\quad (t,\xi)\in(\tau,\infty)\times\mathbb{R},\\ &V(s,\xi)=v_0(s,\xi-cs)-\phi(\xi)=:V_\tau(s,\xi),\quad (s,\xi)\in[\tau-r,\tau]\times\mathbb{R}. \end{aligned}\right\} \tag{3.4}$$

If $V_\tau(s,\xi) \in H^1_w$ *and* $M_\tau(0) \leqslant \delta_1$ *for a given positive constant* δ_1*, then there exists a small* $t_0 = t_0(\delta_1) > 0$ *such that* $V(t,\xi) \in X(\tau - r, \tau + t_0)$ *and* $M_\tau(t_0) \leqslant \sqrt{2(1+r)}M_\tau(0)$.

The proof of proposition 3.2 can be given by the elementary energy method. We omit the details. Next, we state the *a priori* estimate.

Proposition 3.3 (*a priori* estimate). *Let* $V(t,\xi) \in X(-r,T)$ *be a local solution of (3.1) for a given constant* $T > 0$*. Then there exist positive constants* μ*,* δ_2 *and* $C_1 > 1$ *independent of* T *such that, when* $M(T) \leqslant \delta_2$*,*

$$\begin{aligned} \mathrm{e}^{2\mu t}\|V(t)\|^2_{H^1_w} &+ \int_0^t \mathrm{e}^{2\mu s}\|V(s)\|^2_{H^2_w}\,\mathrm{d}s \\ &\leqslant C_1\biggl(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V(s)\|^2_{H^1_w}\,\mathrm{d}s\biggr), \qquad 0 \leqslant t \leqslant T \end{aligned} \tag{3.5}$$

and

$$\|V(t)\|^2_{H^1_w} \leqslant C_1\biggl(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V_0(s)\|^2_{H^1_w}\,\mathrm{d}s\biggr)\mathrm{e}^{-2\mu t}, \quad 0 \leqslant t \leqslant T. \tag{3.6}$$

Remark 3.4. Positive constants μ, δ_2 and C_1, which depend only on the coefficients D_m, d_m, ε, p, a, r and the wave speed c, will be specified in (3.17), (3.40), and (3.43), below.

We postpone the proof of proposition 3.3 to the last part of this section. Now, based on propositions 3.2 and 3.3, we will prove theorem 3.1 using the continuation argument.

Proof of theorem 3.1. Recall that the constants δ_2, μ and C_1 from proposition 3.3 are independent of T. Let

$$\delta_1 = \max\{\sqrt{C_1(1+r)}M(0), \delta_2\}, \tag{3.7}$$

$$\delta_0 = \min\left\{\frac{\delta_2}{\sqrt{2(1+r)}}, \frac{\delta_2}{\sqrt{2C_1}(1+r)}\right\} \tag{3.8}$$

and

$$M(0) \leqslant \delta_0 < \delta_1. \tag{3.9}$$

By proposition 3.2, there exists $t_0 = t_0(\delta_1) > 0$ such that $V(t,\xi) \in X(-r,t_0)$ and

$$M(t_0) \leqslant \sqrt{2(1+r)}M(0) \leqslant \sqrt{2(1+r)}\delta_0 \leqslant \delta_2.$$

Thus, applying proposition 3.3 on the interval $[0,t_0]$, we obtain (3.6) for $t \in [0,t_0]$, and

$$\begin{aligned} \sup_{t\in[0,t_0]} \|V(t)\|_{H^1_w} &\leqslant \sup_{t\in[0,t_0]} \biggl\{C_1\biggl(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V_0(s)\|^2_{H^1_w}\,\mathrm{d}s\biggr)\biggr\}^{1/2}\mathrm{e}^{-\mu t} \\ &\leqslant \sqrt{C_1(1+r)}M(0) \leqslant \sqrt{C_1(1+r)}\delta_0 \\ &\leqslant \frac{\delta_2}{\sqrt{2(1+r)}}. \end{aligned} \tag{3.10}$$

Now consider the Cauchy problem (3.4) at the initial time $\tau = t_0$. Using (3.9), (3.10) and (3.7), we have

$$\begin{aligned} M_{t_0}(0) &= \sup_{s\in[t_0-r,t_0]} \|V(s)\|_{H^1_w} \\ &\leqslant \max\Big\{ \sup_{s\in[-r,0]} \|V(s)\|_{H^1_w}, \sup_{s\in[0,t_0]} \|V(s)\|_{H^1_w} \Big\} \\ &\leqslant \max\Big\{ M(0), \frac{\delta_2}{\sqrt{2(1+r)}} \Big\} \leqslant \delta_1. \end{aligned} \tag{3.11}$$

Applying proposition 3.2 again yields $V(t,\xi) \in X(-r, 2t_0)$ and

$$M_{t_0}(t_0) \leqslant \sqrt{2(1+r)} M_{t_0}(0).$$

On the other hand,

$$\begin{aligned} M_{t_0}(0) &= \sup_{t\in[t_0-r,t_0]} \|V(s)\|_{H^1_w} \\ &\leqslant \max\Big\{ \sup_{s\in[-r,0]} \|V(s)\|_{H^1_w}, \sup_{s\in[0,t_0]} \|V(s)\|_{H^1_w} \Big\} \\ &\leqslant \max\Big\{ \delta_0, \frac{\delta_2}{\sqrt{2(1+r)}} \Big\} \leqslant \frac{\delta_2}{\sqrt{2(1+r)}}. \end{aligned} \tag{3.12}$$

Furthermore, we have

$$M_{t_0}(t_0) \leqslant \sqrt{2(1+r)} M_{t_0}(0) \leqslant \delta_2.$$

Consequently,

$$\begin{aligned} M(2t_0) &= \sup_{s\in[-r,2t_0]} \|V(s)\|_{H^1_w} \\ &\leqslant \max\Big\{ \sup_{s\in[-r,0]} \|V(s)\|_{H^1_w}, \sup_{s\in[0,t_0-r]} \|V(s)\|_{H^1_w}, \sup_{s\in[t_0-r,2t_0]} \|V(s)\|_{H^1_w} \Big\} \\ &\leqslant \max\Big\{ \delta_0, \frac{\delta_2}{\sqrt{2(1+r)}}, \delta_2 \Big\} \leqslant \delta_2. \end{aligned} \tag{3.13}$$

We can apply proposition 3.3 to obtain (3.6) for $0 \leqslant t \leqslant 2t_0$ and

$$\begin{aligned} \sup_{t\in[0,2t_0]} \|V(t)\|_{H^1_w} &\leqslant \sup_{t\in[0,2t_0]} \Big\{ C_1\Big(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V_0(s)\|^2_{H^1_w}\,\mathrm{d}s \Big) \Big\}^{1/2} \mathrm{e}^{-\mu t} \\ &\leqslant \sqrt{C_1(1+r)} M(0) \leqslant \sqrt{C_1(1+r)}\delta_0 \leqslant \frac{\delta_2}{\sqrt{2(1+r)}}. \end{aligned} \tag{3.14}$$

Repeating the previous procedure, one can prove that $V(t,x) \in X(-r,\infty)$ and (3.6) for all $0 \leqslant t < \infty$. Also (3.3) follows immediately from (3.6). The proof is complete. □

REMARK 3.5. The proof of theorem 3.1 above corrected the mistake in the proof of theorem 3.1 in [15, pp. 586–587], where δ_0 and δ_1 were defined incorrectly.

Next, we will prove proposition 3.3. For this, we need the following important lemma.

Lemma 3.6 (key inequality). *Let $w(\xi)$ be the weight function given in (2.6) and define*

$$\begin{aligned} B_\mu(\xi) := & -c\frac{w'(\xi)}{w(\xi)} - D_{\mathrm{m}}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 + 2d_{\mathrm{m}} \\ & - 2\mu - \varepsilon\int_{\mathbb{R}} b'(\phi(\xi - y - cr))f_\alpha(y)\,\mathrm{d}y \\ & - \varepsilon \mathrm{e}^{2\mu r}\frac{b'(\phi(\xi))}{w(\xi)}\int_{\mathbb{R}} w(\xi + y + cr)f_\alpha(y)\,\mathrm{d}y. \end{aligned} \tag{3.15}$$

If (2.8) holds, then

$$B_\mu(\xi) \geqslant C_0(\mu) > 0 \quad \text{for all } \xi \in \mathbb{R}, \tag{3.16}$$

where

$$0 < \mu < \min\{\mu_1, \mu_2\} \tag{3.17}$$

and $\mu_1 > 0$ and $\mu_2 > 0$ are, respectively, the unique solutions to the following equations:

$$\frac{c^2}{4D_{\mathrm{m}}} - (3\varepsilon p - 2d_{\mathrm{m}}) - 2\mu_1 - 2\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu_1 r} - 1) = 0, \tag{3.18}$$

$$\tfrac{1}{2}(3 - \mathrm{e})d_{\mathrm{m}} - 2\mu_2 - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu_2 r} - 1) = 0, \tag{3.19}$$

and

$$C_0(\mu) := \min\{C_1(\mu), C_2(\mu)\}, \tag{3.20}$$

$$C_1(\mu) := \frac{c^2}{4D_{\mathrm{m}}} - (3\varepsilon p - 2d_{\mathrm{m}}) - 2\mu - 2\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r} - 1) > 0, \tag{3.21}$$

$$C_2(\mu) := \tfrac{1}{2}(3 - \mathrm{e})d_{\mathrm{m}} - 2\mu - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r} - 1) > 0. \tag{3.22}$$

Proof. We divide this into two cases.

Case 1 ($\xi < x_*$). Since $\phi(\xi)$ is increasing from $v_- = 0$ to $v_+ = 1/a$, we thus have $\phi'(\xi) > 0$ and $2 - a\phi(\xi) \geqslant 2 - av_+ = 2 - \ln \mathrm{e} = 1$. According to (1.10), i.e. $b(\phi) = p\phi\mathrm{e}^{-a\phi}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}b'(\phi(\xi)) = -pa(2 - a\phi(\xi))\mathrm{e}^{-a\phi(\xi)}\phi'(\xi) < 0.$$

Thus, $b'(\phi(\xi))$ is decreasing for $\xi \in (-\infty, \infty)$. This implies that

$$0 = b'(v_+) < b'(\phi(\xi)) < b'(v_-) = p. \tag{3.23}$$

Using (3.23), (1.9), and the facts that $\varepsilon p = d_{\mathrm{m}}\mathrm{e}$ from (2.3),

$$w(\xi) = \mathrm{e}^{-\beta(\xi - x_*)}, \quad \frac{w'(\xi)}{w(\xi)} = -\beta, \quad \mathrm{e}^{\beta(\xi - x_*)} < 1 \quad \text{for } \xi < x_*,$$

and that $cr - \alpha\beta = c(r - (\alpha/2D_{\mathrm{m}})) > 0$ from (2.7) and (1.8), which implies that $\mathrm{e}^{-\beta(cr-\alpha\beta)} < 1$, we obtain

$$
\begin{aligned}
B_\mu(\xi) &= c\beta - D_{\mathrm{m}}\beta^2 + 2d_{\mathrm{m}} - 2\mu - \varepsilon \int_{\mathbb{R}} b'(\phi(\xi - y - cr)) f_\alpha(y)\,\mathrm{d}y \\
&\quad - \varepsilon \mathrm{e}^{2\mu r} \frac{b'(\phi(\xi))}{\mathrm{e}^{-\beta(\xi - x_*)}} \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y)\,\mathrm{d}y \\
&\geqslant c\beta - D_{\mathrm{m}}\beta^2 + 2d_{\mathrm{m}} - 2\mu - \varepsilon p \int_{\mathbb{R}} f_\alpha(y)\,\mathrm{d}y \\
&\quad - \frac{\varepsilon \mathrm{e}^{2\mu r} p}{\mathrm{e}^{\beta(\xi - x_*)}} \biggl[\int_{-\infty}^{x_* - \xi - cr} \mathrm{e}^{-\beta(\xi + y + cr - x_*)} f_\alpha(y)\,\mathrm{d}y + \int_{x_* - \xi - cr}^{\infty} f_\alpha(y)\,\mathrm{d}y \biggr] \\
&= c\beta - D_{\mathrm{m}}\beta^2 + 2d_{\mathrm{m}} - 2\mu - \varepsilon p \\
&\quad - \varepsilon \mathrm{e}^{2\mu r} p \biggl[\int_{-\infty}^{x_* - \xi - cr} \mathrm{e}^{-\beta(y + cr)} f_\alpha(y)\,\mathrm{d}y + \int_{x_* - \xi - cr}^{\infty} \mathrm{e}^{\beta(\xi - x_*)} f_\alpha(y)\,\mathrm{d}y \biggr] \\
&\geqslant c\beta - D_{\mathrm{m}}\beta^2 + 2d_{\mathrm{m}} - 2\mu - \mathrm{e}d_{\mathrm{m}} \\
&\quad - \mathrm{e}d_{\mathrm{m}} \mathrm{e}^{2\mu r} \biggl[\int_{-\infty}^{x_* - \xi - cr} \mathrm{e}^{-\beta(y + cr)} f_\alpha(y)\,\mathrm{d}y + \int_{x_* - \xi - cr}^{\infty} f_\alpha(y)\,\mathrm{d}y \biggr] \\
&= c\beta - D_{\mathrm{m}}\beta^2 + 2d_{\mathrm{m}} - 2\mu - \mathrm{e}d_{\mathrm{m}} \\
&\quad - \mathrm{e}d_{\mathrm{m}} \mathrm{e}^{2\mu r} \biggl[\int_{-\infty}^{x_* - \xi - cr} \frac{1}{\sqrt{4\pi\alpha}} \exp\biggl(-\frac{y^2}{4\alpha} - \beta(y + cr) \biggr) \mathrm{d}y \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_{x_* - \xi - cr}^{\infty} f_\alpha(y)\,\mathrm{d}y \biggr] \\
&\geqslant c\beta - D_{\mathrm{m}}\beta^2 + 2d_{\mathrm{m}} - 2\mu - \mathrm{e}d_{\mathrm{m}} \\
&\quad - \mathrm{e}d_{\mathrm{m}} \mathrm{e}^{2\mu r} \biggl[\frac{\mathrm{e}^{-\beta(cr - \alpha\beta)}}{\sqrt{4\pi\alpha}} \int_{-\infty}^{\infty} \exp\biggl(-\frac{(y + 2\alpha\beta)^2}{4\alpha} \biggr) \mathrm{d}y + \int_{-\infty}^{\infty} f_\alpha(y)\,\mathrm{d}y \biggr] \\
&= c\beta - D_{\mathrm{m}}\beta^2 + (2 - e)\,\mathrm{d}_{\mathrm{m}} - 2\mu - \mathrm{e}d_{\mathrm{m}} \mathrm{e}^{2\mu r} [\mathrm{e}^{-\beta(cr - \alpha\beta)} + 1] \\
&\geqslant c\beta - D_{\mathrm{m}}\beta^2 + (2 - e)\,\mathrm{d}_{\mathrm{m}} - 2\mu - 2\mathrm{e}d_{\mathrm{m}} \mathrm{e}^{2\mu r} \\
&= \frac{c^2}{4D_{\mathrm{m}}} - (3e - 2)\,\mathrm{d}_{\mathrm{m}} - 2\mu - 2\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r} - 1) \\
&= \frac{c^2}{4D_{\mathrm{m}}} - (3\varepsilon p - 2d_{\mathrm{m}}) - 2\mu - 2\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r} - 1) \\
&=: C_1(\mu) > 0 \quad \text{for suitably small } \mu > 0,
\end{aligned}
\tag{3.24}
$$

where the last inequality follows from our sufficient condition (2.8), and $0 < \mu < \mu_1$, where $\mu_1 > 0$ is the unique solution of

$$
\frac{c^2}{4D_{\mathrm{m}}} - (3\varepsilon p - 2d_{\mathrm{m}}) - 2\mu_1 - 2\varepsilon d_{\mathrm{m}}(\mathrm{e}^{2\mu_1 r} - 1) = 0,
$$

so then we have

$$
C_1(\mu) = \frac{c^2}{4D_{\mathrm{m}}} - (3\varepsilon p - 2d_{\mathrm{m}}) - 2\mu - 2\varepsilon d_{\mathrm{m}}(\mathrm{e}^{2\mu r} - 1) > 0
$$

for $0 < \mu < \mu_1$.

Case 2 ($\xi \geqslant x_*$). In this case, $w(\xi) = 1$. Thus,

$$B_\mu(\xi) = 2d_{\mathrm{m}} - 2\mu - \varepsilon \int_{\mathbb{R}} b'(\phi(\xi - y - cr)) f_\alpha(y)\,\mathrm{d}y - \varepsilon \mathrm{e}^{2\mu r} b'(\phi(\xi)) \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y)\,\mathrm{d}y. \quad (3.25)$$

Note that

$$b'(\phi) = p(1 - a\phi)\mathrm{e}^{-a\phi} = p\mathrm{e}^{-a\phi} - pa\phi\mathrm{e}^{-a\phi} = p\mathrm{e}^{-a\phi} - ab(\phi),$$

$\phi(\xi) > 0$, $\phi'(\xi) > 0$ and

$$\int_{\mathbb{R}} f_\alpha(y)\,\mathrm{d}y = 1.$$

Applying equation (2.1) and the first and the second inequalities of (2.5), we can reduce the second term of the right-hand side of (3.25) as follows:

$$\begin{aligned}
-\varepsilon \int_{\mathbb{R}} b'(\phi(\xi - y - cr)) f_\alpha(y)\,\mathrm{d}y
&= -\varepsilon p \int_{\mathbb{R}} \mathrm{e}^{-a\phi(\xi - y - cr)} f_\alpha(y)\,\mathrm{d}y + a\varepsilon \int_{\mathbb{R}} b(\phi(\xi - y - cr)) f_\alpha(y)\,\mathrm{d}y \\
&= -\varepsilon p \int_{\mathbb{R}} \mathrm{e}^{-a\phi(\xi - y - cr)} f_\alpha(y)\,\mathrm{d}y + a[c\phi'(\xi) - D_{\mathrm{m}}\phi''(\xi) + d_{\mathrm{m}}\phi(\xi)] \\
&\geqslant -\varepsilon p \int_{\mathbb{R}} f_\alpha(y)\,\mathrm{d}y + a d_{\mathrm{m}}\phi(\xi) - aD_{\mathrm{m}}\phi''(\xi) \\
&= -\mathrm{e}d_{\mathrm{m}} \int_{\mathbb{R}} f_\alpha(y)\,\mathrm{d}y + a d_{\mathrm{m}}\phi(\xi) - aD_{\mathrm{m}}\phi''(\xi) \\
&= -\mathrm{e}d_{\mathrm{m}} + a d_{\mathrm{m}}\phi(\xi) - aD_{\mathrm{m}}\phi''(\xi) \\
&\geqslant -\mathrm{e}d_{\mathrm{m}} + d_{\mathrm{m}} - a d_{\mathrm{m}}\bar{\varepsilon} - aD_{\mathrm{m}}\bar{\varepsilon} \\
&= -(\mathrm{e} - 1)d_{\mathrm{m}} - (a d_{\mathrm{m}} + aD_{\mathrm{m}})\bar{\varepsilon}. \qquad (3.26)
\end{aligned}$$

On the other hand, applying the third inequality of (2.5) and noting that $w(\xi) = 1$ and $\mathrm{e}^{-\beta(\xi - x_*)} < 1$, we may furthermore estimate the third term of the right-hand side of (3.25) as follows:

$$\begin{aligned}
-\varepsilon \mathrm{e}^{2\mu r} b'(\phi(\xi)) \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y)\,\mathrm{d}y
&= -\varepsilon p \mathrm{e}^{2\mu r} \mathrm{e}^{a\phi(\xi)} (1 - a\phi(\xi)) \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y)\,\mathrm{d}y \\
&= -\mathrm{e}d_{\mathrm{m}} \mathrm{e}^{2\mu r} \mathrm{e}^{a\phi(\xi)} (1 - a\phi(\xi)) \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y)\,\mathrm{d}y \\
&\geqslant -\mathrm{e}d_{\mathrm{m}} \bar{\varepsilon} \mathrm{e}^{2\mu r} \left(\int_{-\infty}^{x_* - \xi - cr} + \int_{x_* - \xi - cr}^{+\infty} \right) w(\xi + y + cr) f_\alpha(y)\,\mathrm{d}y \\
&\geqslant -\mathrm{e}d_{\mathrm{m}} \bar{\varepsilon} \mathrm{e}^{2\mu r} \left[\int_{-\infty}^{x_* - \xi - cr} \mathrm{e}^{-\beta(\xi + y + cr - x_*)} f_\alpha(y)\,\mathrm{d}y + \int_{x_* - \xi - cr}^{\infty} f_\alpha(y)\,\mathrm{d}y \right]
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{e}d_{\mathrm{m}}\bar{\varepsilon}\mathrm{e}^{2\mu r}\biggl[\frac{\mathrm{e}^{-\beta(cr-\alpha\beta)}\mathrm{e}^{-\beta(\xi-x_*)}}{\sqrt{4\pi\alpha}} \\
&\qquad\qquad \times \int_{-\infty}^{x_*-\xi-cr} \mathrm{e}^{-(y+2\alpha\beta)^2/4\alpha}\,\mathrm{d}y + \int_{x_*-\xi-cr}^{\infty} f_\alpha(y)\,\mathrm{d}y\biggr] \\
&\geqslant -\mathrm{e}d_{\mathrm{m}}\bar{\varepsilon}\mathrm{e}^{2\mu r}\biggl[\frac{\mathrm{e}^{-\beta(cr-\alpha\beta)}}{\sqrt{4\pi\alpha}}\int_{-\infty}^{0} \mathrm{e}^{-(y+2\alpha\beta)^2/4\alpha}\,\mathrm{d}y + \int_{-\infty}^{\infty} f_\alpha(y)\,\mathrm{d}y\biggr] \\
&= -\mathrm{e}d_{\mathrm{m}}\bar{\varepsilon}\mathrm{e}^{2\mu r}[\tfrac{1}{2}\mathrm{e}^{-\beta(cr-\alpha\beta)}+1] \\
&\geqslant -\mathrm{e}d_{\mathrm{m}}\bar{\varepsilon}\mathrm{e}^{2\mu r}[\tfrac{1}{2}+1] \\
&= -\tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}\bar{\varepsilon}\mathrm{e}^{2\mu r}. \qquad (3.27)
\end{aligned}
$$

Substituting (3.26) and (3.27) into (3.25), and noting (2.4), we obtain

$$
\begin{aligned}
B_\mu(\xi) &\geqslant 2d_{\mathrm{m}} - 2\mu - (\mathrm{e}-1)\,\mathrm{d}_{\mathrm{m}} - (ad_{\mathrm{m}} + aD_{\mathrm{m}})\bar{\varepsilon} - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}\bar{\varepsilon}\mathrm{e}^{2\mu r} \\
&= (3-\mathrm{e})d_{\mathrm{m}} - (ad_{\mathrm{m}} + aD_{\mathrm{m}} + \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}})\bar{\varepsilon} - 2\mu - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r}-1) \\
&= \tfrac{1}{2}(3-\mathrm{e})d_{\mathrm{m}} - 2\mu - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r}-1) \\
&=: C_2(\mu) > 0 \quad \text{for some suitably small } \mu > 0, \qquad (3.28)
\end{aligned}
$$

where we select μ such that $0 < \mu < \mu_2$. Here $\mu_2 > 0$ is the unique solution to the following equation

$$\tfrac{1}{2}(3-\mathrm{e})d_{\mathrm{m}} - 2\mu_2 - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu_2 r}-1) = 0,$$

and we always have

$$C_2(\mu) = \tfrac{1}{2}(3-\mathrm{e})d_{\mathrm{m}} - 2\mu - \tfrac{3}{2}\mathrm{e}d_{\mathrm{m}}(\mathrm{e}^{2\mu r}-1) > 0$$

for $\mu < \mu_2$.

Now, we take

$$0 < \mu < \min\{\mu_1, \mu_2\}$$

then we have that both (3.24) and (3.28) hold, which leads to (3.16). The proof is complete. □

Finally, we prove proposition 3.3.

Proof of proposition 3.3. Let $w(\xi)$ be a weight function which will be specified later. Multiplying equation (3.1) by $\mathrm{e}^{2\mu t}w(\xi)V(t,\xi)$ for $0 < \mu \leqslant \mu_0$, we have

$$
\begin{aligned}
&\{\tfrac{1}{2}\mathrm{e}^{2\mu t}wV^2\}_t + \{(\tfrac{1}{2}cwV^2 - D_{\mathrm{m}}wVV_\xi)\mathrm{e}^{2\mu t}\}_\xi + D_{\mathrm{m}}\mathrm{e}^{2\mu t}wV_\xi^2 \\
&\qquad + D_{\mathrm{m}}\mathrm{e}^{2\mu t}w'V_\xi V + \biggl\{-\frac{c}{2}\frac{w'}{w} + d_{\mathrm{m}} - \mu\biggr\}\mathrm{e}^{2\mu t}wV^2 \\
&\qquad\qquad - \varepsilon\mathrm{e}^{2\mu t}wV\int_{\mathbb{R}} b'(\phi(\xi-y-cr))V(t-r,\xi-y-cr)f_\alpha(y)\,\mathrm{d}y \\
&\qquad\qquad\qquad = \varepsilon\mathrm{e}^{2\mu t}wV\int_{\mathbb{R}} Q(V(t-r,\xi-y-cr))f_\alpha(y)\,\mathrm{d}y, \qquad (3.29)
\end{aligned}
$$

where $w = w(\xi)$, $V = V(t,\xi)$. Using the Cauchy–Schwarz inequality $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$, we obtain

$$|D_{\mathrm{m}} \mathrm{e}^{2\mu t} w'(\xi) V_\xi(t,\xi) V(t,\xi)| \leqslant \frac{D_{\mathrm{m}}}{2} \mathrm{e}^{2\mu t} w V_\xi^2 + \frac{D_{\mathrm{m}}}{2} \left(\frac{w'}{w}\right)^2 \mathrm{e}^{2\mu t} w V^2.$$

Substituting this into (3.29) and integrating the resulting inequality over $[0,t] \times \mathbb{R}$, we have

$$\begin{aligned}
&\mathrm{e}^{2\mu t}\|V(t)\|_{L_w^2}^2 + D_{\mathrm{m}} \int_0^t \mathrm{e}^{2\mu s} \|V_\xi(s)\|_{L_w^2}^2 \, \mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}} \left\{ -c\frac{w'(\xi)}{w(\xi)} + 2d_{\mathrm{m}} - 2\mu - D_{\mathrm{m}} \left(\frac{w'(\xi)}{w(\xi)}\right)^2 \right\} \mathrm{e}^{2\mu s} w(\xi) V(s,\xi)^2 \, \mathrm{d}\xi \, \mathrm{d}s \\
&\quad - 2\varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} b'(\phi(\xi - y - cr)) w(\xi) \\
&\qquad \times V(s,\xi) V(s-r, \xi - y - cr) f_\alpha(y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s \\
&\qquad\quad \leqslant \|v_0(0)\|_{L_w^2}^2 + 2\varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} w(\xi) \\
&\qquad\qquad \times V(s,\xi) Q(V(s-r, \xi - y - cr)) f_\alpha(y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s.
\end{aligned} \tag{3.30}$$

Now, using the change of variables $y \mapsto y$, $\xi - y - cr \mapsto \xi$, $s - r \mapsto s$, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} w(\xi) b'(\phi(\xi - y - cr)) V^2(s-r, \xi - y - cr) f_\alpha(y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s \\
&\quad = \int_{-r}^{t-r} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu(s+r)} w(\xi + y + cr) b'(\phi(\xi)) V^2(s,\xi) f_\alpha(y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s \\
&\quad = \mathrm{e}^{2\mu r} \int_{-r}^{t-r} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} \left[\frac{b'(\phi(\xi))}{w(\xi)} \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y) \, \mathrm{d}y \right] w(\xi) V^2(s,\xi) \, \mathrm{d}\xi \, \mathrm{d}s.
\end{aligned} \tag{3.31}$$

Once again, using the Cauchy–Schwarz inequality, and noting (3.31) and (3.23), then we can estimate the delay term on the left-hand side of (3.30) as follows:

$$\begin{aligned}
&2\varepsilon \left| \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} b'(\phi(\xi - y - cr)) w(\xi) V(s,\xi) V(s-r, \xi - y - cr) f_\alpha(y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s \right| \\
&\quad \leqslant \varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} w(\xi) b'(\phi(\xi - y - cr)) \\
&\qquad \times [V^2(s,\xi) + V^2(s-r, \xi - y - cr)] f_\alpha(y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s \\
&\quad = \varepsilon \int_0^t \int_{\mathbb{R}} \mathrm{e}^{2\mu s} w(\xi) \left[\int_{\mathbb{R}} b'(\phi(\xi - y - cr)) f_\alpha(y) \, \mathrm{d}y \right] V^2(s,\xi) \, \mathrm{d}\xi \, \mathrm{d}s \\
&\qquad + \varepsilon \mathrm{e}^{2\mu r} \int_{-r}^{t-r} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} \left[\frac{b'(\phi(\xi))}{w(\xi)} \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y) \, \mathrm{d}y \right] w(\xi) V^2(s,\xi) \, \mathrm{d}\xi \, \mathrm{d}s \\
&\quad \leqslant \varepsilon \int_0^t \int_{\mathbb{R}} \mathrm{e}^{2\mu s} \left[\int_{\mathbb{R}} b'(\phi(\xi - y - cr)) f_\alpha(y) \, \mathrm{d}y \right] w(\xi) V^2(s,\xi) \, \mathrm{d}\xi \, \mathrm{d}s
\end{aligned}$$

$$+ \varepsilon e^{2\mu r} \int_0^t \int_{\mathbb{R}} e^{2\mu s} \left[\frac{b'(\phi(\xi))}{w(\xi)} \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y) \, dy \right] w(\xi) V^2(s, \xi) \, d\xi \, ds$$
$$+ \varepsilon e^{2\mu r} \int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s} \left[\frac{b'(\phi(\xi))}{w(\xi)} \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y) \, dy \right] w(\xi) V_0^2(s, \xi) \, d\xi \, ds. \tag{3.32}$$

Substituting (3.32) into (3.30) yields

$$e^{2\mu t} \|V(t)\|_{L_w^2}^2 + D_m \int_0^t e^{2\mu s} \|V_\xi(s)\|_{L_w^2}^2 \, ds + \int_0^t \int_{-\infty}^{\infty} B_\mu(\xi) e^{2\mu s} w(\xi) V^2(s, \xi) \, d\xi \, ds$$
$$\leqslant \|V_0(0)\|_{L_w^2}^2$$
$$+ \varepsilon e^{2\mu r} \int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s} \left[\frac{b'(\phi(\xi))}{w(\xi)} \int_{\mathbb{R}} w(\xi + y + cr) f_\alpha(y) \, dy \right] w(\xi) V_0^2(s, \xi) \, d\xi \, ds$$
$$+ 2\varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} w(\xi) e^{2\mu s} V(s, \xi) Q(V(s - r, \xi - y - cr)) f_\alpha(y) \, dy \, d\xi \, ds, \tag{3.33}$$

where $B_\mu(\xi)$ is defined in (3.15). We need to select a suitable weight function, $w(\xi)$, so that $B_\mu(\xi) > 0$ for all $\xi \in \mathbb{R}$. The choice of $w(\xi)$ is, of course, not unique. One possibility is

$$w(\xi) = \begin{cases} e^{-\beta(\xi - x_*)}, & \xi < x_*, \\ 1, & \xi \geqslant x_*, \end{cases}$$

as in (2.6) with $\beta = c/2D_m$. According to lemma 3.6, $B_\mu(\xi) \geqslant C_0(\mu) > 0$ for $0 < \mu \leqslant \min\{\mu_1, \mu_2\}$, where the positive constants μ_1, μ_2 and $C_0(\mu)$ are defined in (3.18), (3.19) and (3.20), respectively. Then, by using (3.23) and

$$e^{2\mu r} \int_{\mathbb{R}} \frac{w(\xi + y + cr)}{w(\xi)} f_\alpha(y) \, dy \leqslant C \quad \text{for all } \xi \in \mathbb{R}, \tag{3.34}$$

which can be proved analogously to lemma 3.6, we can reduce (3.33) to

$$e^{2\mu t} \|V(t)\|_{L_w^2}^2 + D_m \int_0^t e^{2\mu s} \|V_\xi(s)\|_{L_w^2}^2 \, ds + C_0(\mu) \int_0^t e^{2\mu s} \|V(s)\|_{L_w^2}^2 \, ds$$
$$\leqslant \|V_0(0)\|_{L_w^2}^2 + \varepsilon C \int_{-r}^0 \|V_0(s)\|_{L_w^2}^2 \, ds$$
$$+ 2\varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\mu s} w(\xi) V(s, \xi) Q(V(s - r, \xi - y - cr)) f_\alpha(y) \, dy \, d\xi \, ds. \tag{3.35}$$

Next, we will estimate the nonlinear term on the right-hand side of (3.35). From (3.2), by Taylor's formula, we first have $Q(V) = O(V^2)$ as $V \to 0$, i.e.

$$|Q(V(t - r, \xi - y - cr))| \sim C_5 |V(t - r, \xi - y - cr)|^2 \tag{3.36}$$

for some positive constant C_5. Then, by the standard Sobolev embedding inequality $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ and the modified embedding inequality $H_w^1(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ for

$w(\xi) > 0$ defined as in (2.6) (see [13]), we obtain

$$|V(t,\xi)| \leqslant \sup_{\xi\in\mathbb{R}} |V(t,\xi)| \leqslant C_6\|V(t,\cdot)\|_{H^1} \leqslant C_6\|V(t,\cdot)\|_{H^1_w} \leqslant C_6 M(t), \tag{3.37}$$

where $C_6 > 0$ is the embedding constant. Applying (3.34), (3.36) and (3.37) and making the change of variables $y \mapsto y$, $\xi - y - cr \mapsto \xi$, $s - cr \mapsto s$, we then obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} w(\xi) V(s,\xi) Q(V(s-r,\xi-y-cr)) f_\alpha(y)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad \leqslant C_7 M(t) \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu s} w(\xi) |V(s-r,\xi-y-cr)|^2 f_\alpha(y)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad = C_7 M(t) \int_{-r}^{t-r} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\mu(s+r)} w(\xi+y+cr) |V(s,\xi)|^2 f_\alpha(y)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad \leqslant C_7 M(t) \biggl\{ \int_0^t \int_{\mathbb{R}} \mathrm{e}^{2\mu(s+r)} w(\xi) |V(s,\xi)|^2 \biggl(\int_{\mathbb{R}} \frac{w(\xi+y+cr)}{w(\xi)} f_\alpha(y)\,\mathrm{d}y \biggr) \mathrm{d}\xi\,\mathrm{d}s \\
&\qquad\qquad + \int_{-r}^0 \int_{\mathbb{R}} \mathrm{e}^{2\mu(s+r)} w(\xi) |V_0(s,\xi)|^2 \biggl(\int_{\mathbb{R}} \frac{w(\xi+y+cr)}{w(\xi)} f_\alpha(y)\,\mathrm{d}y \biggr) \mathrm{d}\xi\,\mathrm{d}s \biggr\} \\
&\quad \leqslant C_8 M(t) \biggl\{ \int_0^t \mathrm{e}^{2\mu s} \|V(s)\|^2_{L^2_w}\,\mathrm{d}s + \int_{-r}^0 \mathrm{e}^{2\mu s} \|V_0(s)\|^2_{L^2_w}\,\mathrm{d}s \biggr\}
\end{aligned} \tag{3.38}$$

for some positive constants C_7 and C_8. Substituting (3.38) into (3.35), we finally obtain

$$\begin{aligned}
\mathrm{e}^{2\mu t}\|V(t)\|^2_{L^2_w} + D_{\mathrm{m}} \int_0^t \mathrm{e}^{2\mu s} \|V_\xi(s)\|^2_{L^2_w}\,\mathrm{d}s + [C_0(\mu) - C_8 M(t)] \int_0^t \mathrm{e}^{2\mu s} \|V(s)\|^2_{L^2_w}\,\mathrm{d}s \\
\leqslant \|V_0(0)\|^2_{L^2_w} + C_9[1 + M(t)] \int_{-r}^0 \|V_0(s)\|^2_{L^2_w}\,\mathrm{d}s
\end{aligned} \tag{3.39}$$

for some constant $C_9 > 0$.

One can find a positive constant δ_2 such that

$$C_0(\mu) - C_8\delta_2 > 0 \text{ or, equivalently, } \delta_2 < \frac{C_0(\mu)}{C_8}. \tag{3.40}$$

Clearly, δ_2 depends only on the coefficients D_{m}, d_{m}, ε, p, a, r and the wave speed c, because μ depends on these parameters (see (3.17)–(3.19)). When $M(T) \leqslant \delta_2$, i.e.

$$C_0(\mu) - C_8 M(T) \geqslant C_0(\mu) - C_8\delta_2 > 0,$$

we have

$$\mathrm{e}^{2\mu t}\|V(t)\|^2_{L^2_w} + D_{\mathrm{m}} \int_0^t \mathrm{e}^{2\mu s} \|V_\xi(s)\|^2_{L^2_w}\,\mathrm{d}s \leqslant \|V_0(0)\|^2_{L^2_w} + C_{10} \int_{-r}^0 \|V_0(s)\|^2_{L^2_w}\,\mathrm{d}s \tag{3.41}$$

for some positive constant C_{10}.

Similarly, by differentiating (3.1) with respect to ξ, multiplying the resultant equation by $\mathrm{e}^{2\mu t} w(\xi) V_\xi(t,\xi)$, and then integrating it over $[0,t] \times \mathbb{R}$ for $t \leqslant T$, using the basic energy estimate (3.41), we finally have

$$e^{2\mu t}\|V_\xi(t)\|^2_{L^2_w} + D_m \int_0^t e^{2\mu s}\|V_{\xi\xi}(s)\|^2_{L^2_w}\,ds \leqslant C_{11}\biggl(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V_0(s)\|^2_{H^1_w}\,ds\biggr) \tag{3.42}$$

for some positive constant C_{11}, provided that $M(T) \leqslant \delta_2$. We omit the detail.

Combining (3.41) and (3.42), we obtain

$$e^{2\mu t}\|V_\xi(t)\|^2_{H^1_w} + D_m \int_0^t e^{2\mu s}\|V_\xi(s)\|^2_{H^1_w}\,ds \leqslant C_1\biggl(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V_0(s)\|^2_{H^1_w}\,ds\biggr) \tag{3.43}$$

for some absolute constant $C_1 = \max\{1 + C_{11}, C_{10} + C_{11}\} > 0$ that is independent of T and $V(t, x)$. Finally, from (3.43), we automatically reach

$$\|V(t)\|^2_{H^1_w} \leqslant C_1\biggl(\|V_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|V_0(s)\|^2_{H^1_w}\,ds\biggr)e^{-2\mu t}, \quad 0 \leqslant t \leqslant T.$$

The proof is complete. □

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