SPHERICAL AVERAGES OF GAUSSIAN FREE FIELDS

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Abstract. In this note we study the properties of the spherical average (circle average if dim=2) of Gaussian free fields as a stochastic process parametrized by the radius. In particular, we give explicit formulas for the covariance of the Gaussian family which consists of the spherical averages as well as certain functionals of the spherical averages. We further prove the Markov property for various processes involving the spherical averages. These results are useful in the study of point-wise approximation of generic element of Gaussian free fields.

1. Introduction: Abstract Wiener Space and Gaussian Free Fields

The theory of Abstract Wiener Space (AWS), first introduced by Gross [3], provides an analytical foundation to construct and study Gaussian measures in infinite dimensions. To be specific, given a real separable Banach space E, a *non-degenerate centered Gaussian measure* $\mathcal{W}$ *on* E is a Borel probability measure such that for every $x^* \in E^*\backslash\{0\}$, the functional $x \in E \mapsto \langle x, x^*\rangle \in \mathbb{R}$ has non-degenerate centered Gaussian distribution under $\mathcal{W}$, where E^* is the space of bounded linear functionals on E, and $\langle \cdot, x^*\rangle$ is the action of $x^* \in E^*$ on E. Further assume H is a real separable Hilbert space which is continuously embedded in E as a dense subspace. Then E^* can also be continuously and densely embedded into H, and for any $x^* \in E^*$, there exists a unique $h_{x^*} \in H$ such that $\langle h, x^*\rangle = (h, h_{x^*})_H$ for all $h \in H$. Under this setting if the Gaussian measure $\mathcal{W}$ on E has the following Fourier transform:

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(i\left\langle \cdot, x^*\right\rangle\right)\right] = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right) \text{ for all } x^* \in E^*,$$

or equivalently, if $\langle \cdot, x^*\rangle$ under $\mathcal{W}$ is a centered Gaussian random variable with variance $\|h_{x^*}\|_H^2$ for every $x^* \in E^*$, then the triple $(H, E, \mathcal{W})$ is called an *Abstract Wiener Space*. Moreover, since $\{h_{x^*} : x^* \in E^*\}$ is dense in H, the mapping

$$\mathcal{I} : \, h_{x^*} \in H \mapsto \mathcal{I}\left(h_{x^*}\right) := \langle \cdot, x^*\rangle \in L^2\left(\mathcal{W}\right)$$

can be uniquely extended as a linear isometry between H and $L^2\left(\mathcal{W}\right)$. The extended isometry, also denoted by $\mathcal{I}$, is called the *Paley-Wiener map* and its images $\{\mathcal{I}\left(h\right) : h \in H\}$, known as the *Paley-Wiener integrals*, form a centered Gaussian family whose covariance is given by

$$\mathbb{E}^{\mathcal{W}}\left[\mathcal{I}\left(h\right)\mathcal{I}\left(g\right)\right] = \left(h, g\right)_H \text{ for all } h, g \in H.$$

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It is clear that although $\mathcal{W}$ is a measure on E, it is the inner product of H that fully determines the covariance structure of $\mathcal{W}$. H is known as the *Cameron-Martin space*. In fact, the theory of AWS says that given any separable Hilbert space H, one can always find E and $\mathcal{W}$ such that the triple $(H, E, \mathcal{W})$ forms an AWS. On the other hand, given a separable Banach space E, a non-degenerate centered Gaussian measure $\mathcal{W}$ on E must exist in the form of an AWS. For further discussions on the construction and the properties of AWS, we refer to [3], [4] and §8 of [5].

We now apply the general theory of AWS to study Gaussian measures on function or generalized function spaces. To be specific, given $s \in \mathbb{R}$ and $\nu \in \mathbb{N}$, $\nu \geq 2$, consider the following inner product on $C_c^\infty(\mathbb{R}^\nu)$, the space of compactly supported smooth functions on $\mathbb{R}^\nu$: for every $\phi, \psi \in C_c^\infty(\mathbb{R}^\nu)$,

$$\begin{aligned}(\phi, \psi)_s &:= \left((I-\Delta)^s \phi, \psi\right)_{L^2(\mathbb{R}^\nu)} \\ &= \frac{1}{(2\pi)^\nu} \int_{\mathbb{R}^\nu} \left(1+|\xi|^2\right)^s \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} d\xi,\end{aligned}$$

where $\hat{\cdot}$ denotes the Fourier transform. The closure of $C_c^\infty(\mathbb{R}^\nu)$ under $(\cdot,\cdot)_s$ is the Sobolev space $H^s := H^s(\mathbb{R}^\nu)$ which will be taken as the Cameron-Martin space. According to above, there exists a separable Banach space $\Theta^s := \Theta^s(\mathbb{R}^\nu)$ and the Gaussian measure $\mathcal{W}^s := \mathcal{W}^s(\mathbb{R}^\nu)$ on Θ^s such that the triple $(H^s, \Theta^s, \mathcal{W}^s)$ forms an AWS, to which we refer as the *dim-ν order-s Gaussian free field* (GFF). It's clear that the covariance of such a GFF is characterized by the kernel of the operator $(I-\Delta)^s$ on $\mathbb{R}^\nu$.

For some special values of (ν, s) we have rather explicit formulations of the abstract theory introduced above. For example, when $s = \frac{\nu+1}{2}$, it's proven that $\Theta^{\frac{\nu+1}{2}}$ can be taken as

$$\Theta^{\frac{\nu+1}{2}} := \left\{\theta \in C(\mathbb{R}^\nu) : \lim_{|x|\to\infty} \frac{|\theta(x)|}{\log(e+|x|)} = 0\right\},$$

equipped with the norm

$$\|\theta\|_{\Theta^{\frac{\nu+1}{2}}} := \left\|\frac{\theta}{\log(e+|\cdot|)}\right\|_{\mathrm{u}} = \sup_{x\in\mathbb{R}^\nu} \frac{|\theta(x)|}{\log(e+|x|)}.$$

So the dim-ν order-$\frac{\nu+1}{2}$ GFF actually consists of continuous functions. By the Riesz representation theorem, every $\lambda \in \left(\Theta^{\frac{\nu+1}{2}}\right)^\star$ is characterized as a Borel measure on $\mathbb{R}^\nu$ with the property that

$$\int_{\mathbb{R}^\nu} \log(e+|x|)\, |\lambda|(dx) < \infty.$$

In addition, if we identify the dual space of $H^{\frac{\nu+1}{2}}$ with $H^{-\frac{\nu+1}{2}}$, then λ can also be treated as an element in $H^{-\frac{\nu+1}{2}}$ and

$$\begin{aligned}\|\lambda\|^2_{-\frac{\nu+1}{2}} &= \frac{1}{(2\pi)^\nu} \int_{\mathbb{R}^\nu} \left(1+|\xi|^2\right)^{-\frac{\nu+1}{2}} \left|\hat{\lambda}(\xi)\right|^2 d\xi \\ &= \frac{\pi^{\frac{1-\nu}{2}}}{\Gamma\left(\frac{\nu+1}{2}\right)} \iint_{\mathbb{R}^\nu\times\mathbb{R}^\nu} \exp(-|x-y|)\, \lambda(dx)\, \lambda(dy).\end{aligned}$$

This further implies that the map

$$\lambda \mapsto h_\lambda := \left((I-\Delta)^{-\frac{\nu+1}{2}} \lambda\right) = \frac{\pi^{\frac{1-\nu}{2}}}{\Gamma\left(\frac{\nu+1}{2}\right)} \int_{\mathbb{R}^\nu} \exp(-|x-y|)\, \lambda(dy), \tag{1.1}$$

gives the unique element $h_\lambda \in H^{\frac{\nu+1}{2}}$ such that the action of $\lambda \in \left(\Theta^{\frac{\nu+1}{2}}\right)^*$ restricted on $H^{\frac{\nu+1}{2}}$, denoted by $\langle \cdot, \lambda \rangle \restriction_{H^{\frac{\nu+1}{2}}}$, coincides with $(\cdot, h_\lambda)_{\frac{\nu+1}{2}}$ on $H^{\frac{\nu+1}{2}}$. Moreover, being an isometry between $H^{-\frac{\nu+1}{2}}$ and $H^{\frac{\nu+1}{2}}$, the map (1.1) naturally extends to $\lambda \in H^{-\frac{\nu+1}{2}}$ and we still denote the image by h_λ. Therefore, all the Paley-Wiener integrals, written as $\left\{\mathcal{I}(h_\lambda) : \lambda \in H^{-\frac{\nu+1}{2}}\right\}$, forms a centered Gaussian family with the covariance

$$\mathbb{E}^{\mathcal{W}^{\frac{\nu+1}{2}}}\left[\mathcal{I}(h_\lambda)\mathcal{I}(h_\eta)\right] = (h_\lambda, h_\eta)_{\frac{\nu+1}{2}} = (\lambda, \eta)_{-\frac{\nu+1}{2}}$$

for every $\lambda, \eta \in H^{-\frac{\nu+1}{2}}$.

Finally, for general $s_1, s_2 \in \mathbb{R}$, H^{s_2} is the isometric image of H^{s_1} under the Bessel-type operator $(I-\Delta)^{\frac{s_1-s_2}{2}}$. Therefore, if

$$\Theta^s := (I-\Delta)^{\frac{\nu+1}{4}-\frac{s}{2}}\,\Theta^{\frac{\nu+1}{2}},$$

and

$$\mathcal{W}^s := \left((I-\Delta)^{-\frac{\nu+1}{4}+\frac{s}{2}}\right)_\star \mathcal{W}^{\frac{\nu+1}{2}},$$

then the triple $(H^s, \Theta^s, \mathcal{W}^s)$ forms the dim-ν order-s GFF. The formulations above will follow accordingly by the action of $(I-\Delta)^{\frac{\nu+1}{4}-\frac{s}{2}}$. It's also clear from this aspect that with fixed dimension, the larger the order s is, the more regular the GFF is; on the other hands, when s is fixed, the higher the dimension is, the more singular the GFF becomes. In most of the cases that of interest to us, a generic element of GFF is only a tempered distribution which is not defined pointwise.

2. Spherical Averages of Gaussian Free Fields

Throughout this note, we assume $s \geq 1$ and $\nu \geq 2$. As shown in the previous section, for most (ν, s), if θ is a generic element of the dim-ν order-s GFF, i.e., θ is sampled from Θ^s under $\mathcal{W}^s$, θ is only a generalized function and "$\theta(x)$" may not be defined. Hence, to study the behavior of θ, we will need some proper approximations of pointwise value of θ. To achieve this, we consider the "average" of θ over a sphere (circle in the case when $\nu = 2$) centered at x with radius $t > 0$. To make this precise, we need to introduce some notations. Let $S(x,t)$ be the $\nu - 1$ dimensional sphere centered at $x \in \mathbb{R}^\nu$ with radius $t > 0$, σ_t^x the surface measure on $S(x,t)$, $\alpha_\nu(t) := \frac{\nu\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}+1\right)}t^{\nu-1}$ the surface area of $S(x,t)$ with $\alpha_\nu := \alpha_\nu(1)$, and finally $\bar{\sigma}_t^x := \frac{\sigma_t^x}{\alpha_\nu(t)}$ the "average" over $S(x,t)$. We simply omit "x" when x is the origin. Under this setting, we have the following simple fact about $\bar{\sigma}_t^x$ (proof is omitted).

Lemma 1. *For every $x \in \mathbb{R}^\nu$ and $t > 0$, $\bar{\sigma}_t^x \in H^{-s}(\mathbb{R}^\nu)$ and its Fourier transform is given by*

$$\widehat{\bar{\sigma}_t^x}(\xi) = \frac{(2\pi)^{\frac{\nu}{2}}}{\alpha_\nu} e^{i(x,\xi)_{\mathbb{R}^\nu}} \cdot (t|\xi|)^{\frac{2-\nu}{2}} J_{\frac{\nu-2}{2}}(t|\xi|)$$

for every $\xi \in \mathbb{R}^\nu$, where J_μ is the standard Bessel function of the first kind.

Since $\bar{\sigma}_t^x \in H^{-s}(\mathbb{R}^\nu)$, we can "apply" $\bar{\sigma}_t^x$ to the GFF element θ in the sense of the Paley-Wiener integral. Namely $X_t^x(\theta) := \mathcal{I}\left(h_{\bar{\sigma}_t^x}\right)(\theta)$ is well-defined for $\mathcal{W}$-a.e. $\theta \in \Theta^s$ as a Gaussian random variable, which, heuristically speaking, gives the spherical average of θ. Based on the previous lemma, we make the following observation.

Lemma 2. $\{X_t^x : x \in \mathbb{R}^\nu,\, t > 0\}$ *is a two-parameter centered Gaussian family under* $\mathcal{W}^s$ *with covariance given by*

$$\begin{aligned} C^s(x,t;\, y,r) &:= \mathbb{E}^{\mathcal{W}^s}\left[X_t^x X_r^y\right] = \left(\bar{\sigma}_t^x,\, \bar{\sigma}_r^y\right)_{-s} \\ &= \frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu}\left(1+|\xi|^2\right)^{-s}\widehat{\bar{\sigma}_t^x}(\xi)\,\overline{\widehat{\bar{\sigma}_r^y}(\xi)}d\xi \\ &= \frac{A_\nu}{(tr\,|x-y|)^{\frac{\nu-2}{2}}}\int_0^\infty \frac{\tau^{2-\frac{\nu}{2}}J_{\frac{\nu-2}{2}}(t\tau)\,J_{\frac{\nu-2}{2}}(r\tau)\,J_{\frac{\nu-2}{2}}(|x-y|\,\tau)}{(1+\tau^2)^s}d\tau, \end{aligned} \tag{2.1}$$

where $A_\nu := \left(\frac{2}{\pi}\right)^{\frac{\nu}{2}}\frac{\Gamma^2\left(1+\frac{\nu}{2}\right)}{\nu^2}$. *In particular, when* $x = y$, *i.e., in the concentric case,*

$$C^s(t,r) := C^s(x,t;\, x,r) = \frac{1}{\alpha_\nu}(tr)^{\frac{2-\nu}{2}}\int_0^\infty \frac{\tau J_{\frac{\nu-2}{2}}(t\tau)\,J_{\frac{\nu-2}{2}}(r\tau)}{(1+\tau^2)^s}d\tau. \tag{2.2}$$

These results follow from Lemma 1, computations in polar coordinates as well as integral representations of Bessel functions [6]. Details are omitted.

Since the kernel of $(I-\Delta)^s$ is stationary, i.e., invariant under translation, when studying the behavior of the concentric spherical averages X_t^x's, we may assume x is the origin and omit "x" in the expressions. In fact, we have rather explicit formulas for the covariance function of the concentric family when $s \in \mathbb{N}$, $s \geq 1$. We now discuss the cases $s = 1$ and $s \geq 2$ separately.

2.1. When $s = 1$.

The first result of this subsection is the following:

Theorem 3. *Following the same notation as above, let* $C^1(t,r)$ *be as in (2.2) with* $s = 1$, *then for every* $t, r > 0$,

$$C^1(t,r) = \frac{1}{\alpha_\nu}(tr)^{\frac{2-\nu}{2}} I_{\frac{\nu-2}{2}}(t\wedge r)\, K_{\frac{\nu-2}{2}}(t\vee r),$$

where I_μ *and* K_μ *are modified Bessel functions (with pure imaginary argument).*

Proof. This result can be derived from the formula (5) in §13.53 in [6]. For the same of completeness, we hereby give another proof. In fact, we are going to prove the general result for the operator $p^2 I - \Delta$ with $p \in \mathbb{R}$, which will be useful in the next subsection. Set $\zeta_t := \left(p^2 I - \Delta\right)^{-1}\bar{\sigma}_t$ and

$$\begin{aligned} \widehat{\zeta_t}(\xi) &= \left(p^2+|\xi|^2\right)^{-1}\widehat{\bar{\sigma}_t}(\xi) \\ &= \frac{(2\pi)^{\frac{\nu}{2}}}{\alpha_\nu}(t\,|\xi|)^{\frac{2-\nu}{2}}\left(p^2+|\xi|^2\right)^{-1}J_{\frac{\nu-2}{2}}(t\,|\xi|). \end{aligned}$$

Notice that $\zeta_t(u)$ is actually a function in $u \in \mathbb{R}^\nu$ and

$$\begin{aligned} \zeta_t(u) &= \frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu} e^{-i(u,\xi)_{\mathbb{R}^\nu}}\cdot\widehat{\zeta_t}(\xi)\,d\xi \\ &= \frac{1}{\alpha_\nu(2\pi)^{\frac{\nu}{2}}}\int_{\mathbb{R}^\nu} e^{-i(u,\xi)_{\mathbb{R}^\nu}}\cdot\frac{(t\,|\xi|)^{\frac{2-\nu}{2}}}{p^2+|\xi|^2}J_{\frac{\nu-2}{2}}(t\,|\xi|)\,d\xi. \end{aligned}$$

Switching to the polar coordinates and making use of the fact that

$$J_{\frac{\nu-2}{2}}\left(|u|\,\tau\right)=\frac{\alpha_{\nu-1}\left(|u|\,\tau\right)^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}}\int_0^{\pi}e^{i|u|\tau\cos\varphi}\sin^{\nu-2}\varphi d\varphi,$$

we have that

$$\zeta_t\left(u\right)=\frac{1}{\alpha_\nu}\left(t\,|u|\right)^{\frac{2-\nu}{2}}\int_0^{\infty}\frac{\tau J_{\frac{\nu-2}{2}}\left(|u|\,\tau\right)J_{\frac{\nu-2}{2}}\left(t\tau\right)}{p^2+\tau^2}d\tau.$$

If we replace $(I-\Delta)$ by $\left(p^2I-\Delta\right)$ in deriving (2.2) and set $s=1$, then expression for the covariance will become

$$C_p^1\left(t,r\right)=\frac{1}{\alpha_\nu}\left(tr\right)^{\frac{2-\nu}{2}}\int_0^{\infty}\frac{\tau J_{\frac{\nu-2}{2}}\left(t\tau\right)J_{\frac{\nu-2}{2}}\left(r\tau\right)}{p^2+\tau^2}d\tau. \tag{2.3}$$

It follows from above that $\zeta_t\left(u\right)=C_p^1\left(t,|u|\right)$ for every $u\in\mathbb{R}^\nu$, and hence

$$\left(p^2-\Delta\right)C_p^1\left(t,|\cdot|\right)=\left(p^2-\Delta\right)\zeta_t=\bar{\sigma}_t$$

in the sense of tempered distribution. Since $\bar{\sigma}_t$ is a measure supported on the sphere $S\left(0,t\right)$ and $C_p^1\left(t,|\cdot|\right)$ is radially symmetric function on $\mathbb{R}^\nu$, then in the radial component we get the following equation:

$$\left(p^2-\partial_r^2-\frac{\nu-1}{r}\partial_r\right)C_p^1\left(t,r\right)=0,$$

wherever $0<r<t$ with fixed $t>0$. The solution to this equation is of the form

$$C_1\left(t\right)\frac{K_{\frac{\nu-2}{2}}\left(pr\right)}{r^{\frac{\nu-2}{2}}}+C_2\left(t\right)\frac{I_{\frac{\nu-2}{2}}\left(pr\right)}{r^{\frac{\nu-2}{2}}},$$

where C_1 and C_2 are functions only depending on t. To determine C_1 and C_2, let's examine (2.3) more carefully. First by the basic properties of Bessel functions,

$$\sup_{u>0}\left|J_{\frac{\nu-2}{2}}\left(u\right)\right|\le K_\nu\cdot\min\left(1,u^{-\frac{1}{2}}\right)\text{ for some }K_\nu>0$$

and $\lim_{u\to0+}J_{\frac{\nu-2}{2}}\left(u\right)=0$. Therefore, (2.3) implies that for any fixed $t>0$, $r^{\frac{\nu-2}{2}}\cdot C_p^1\left(t,r\right)\to0$ as $r\downarrow0$. However, $\lim_{r\to0}K_{\frac{\nu-2}{2}}\left(pr\right)=\infty$ and $\lim_{r\to0}I_{\frac{\nu-2}{2}}\left(pr\right)$ exists. Thus we can conclude that $C_1\left(t\right)\equiv0$ and $C_p^1\left(t,\,r\right)=C_2\left(t\right)\frac{I_{\frac{\nu-2}{2}}(pr)}{r^{\frac{\nu-2}{2}}}$.

Next, since $C_p^1\left(t,r\right)=C_p^1\left(r,t\right)$, we apply exactly the same argument to get the equation in t, i.e.,

$$\left(p^2-\partial_t^2-\frac{\nu-1}{t}\partial_t\right)C_p^1\left(t,r\right)=0$$

for all $t>r$ with $r>0$ fixed, whence $C_2\left(t\right)$ must be of the form

$$C_2\left(t\right)=C_3\frac{K_{\frac{\nu-2}{2}}\left(pt\right)}{t^{\frac{\nu-2}{2}}}+C_4\frac{I_{\frac{\nu-2}{2}}\left(pt\right)}{t^{\frac{\nu-2}{2}}}\text{ for some constants }C_3,\,C_4.$$

This time we notice that $\sup_{t>0}\left|t^{\frac{\nu-2}{2}}C_p^1\left(t,r\right)\right|<\infty$ but $\lim_{t\to\infty}I_{\frac{\nu-2}{2}}\left(pt\right)=\infty$ and $\lim_{t\to\infty}K_{\frac{\nu-2}{2}}\left(pt\right)=0$, which implies that $C_4=0$ and

$$C_p^1\left(t,r\right)=C_3\frac{K_{\frac{\nu-2}{2}}\left(pt\right)}{t^{\frac{\nu-2}{2}}}\frac{I_{\frac{\nu-2}{2}}\left(pr\right)}{r^{\frac{\nu-2}{2}}}, \tag{2.4}$$

for all $t > r > 0$. Since $C_p^1(t,r)$ is clearly a continuous function in t and r, we have that

$$C_p^1(t,r) = C_p^1(r,t) = C_3 (tr)^{\frac{2-\nu}{2}} K_{\frac{\nu-2}{2}}(p(t\vee r)) I_{\frac{\nu-2}{2}}(p(t\wedge r))$$

for any $t, r > 0$. The only thing left is to determine C_3. To this end, we consider $C_p^1(r,r)$ given by (2.3) notice that

$$\lim_{r\to 0+} r^{\nu-2}\cdot C_p^1(r,r) = \frac{1}{\alpha_\nu}\int_0^\infty \frac{J^2_{\frac{\nu-2}{2}}(u)}{u}du = \frac{1}{\alpha_\nu}\cdot\frac{1}{\nu-2}.$$

The second equation is due to a well known integral identity of Bessel function (formula (1), §13.42, [6]), which can be easily verified by invoking the series expansion of $J_{\frac{\nu-2}{2}}$. On the other hand, $\lim_{r\to 0+} K_{\frac{\nu-2}{2}}(pr)\, I_{\frac{\nu-2}{2}}(pr) = \frac{1}{\nu-2}$ by a simple analysis of the asymptotics of $K_{\frac{\nu-2}{2}}$ and $I_{\frac{\nu-2}{2}}$ near zero. Therefore, we have that $C_3 = \frac{1}{\alpha_\nu}$. This completes the proof. □

One direct result from the previous theorem is the following.

Corollary 4. *Let* $\left\{X_t := X_t^O : t > 0\right\}$ *be as introduced above, then*

$$\mathbb{E}^{\mathcal{W}^1}\left[\left|t^{\frac{\nu-1}{2}}X_t - r^{\frac{\nu-1}{2}}X_r\right|^2\right] \le |t-r| \text{ for every } t,r>0. \tag{2.5}$$

In particular, $\{X_t : t > 0\}$ *is* $\mathcal{W}^1$*-a.e. continuous.*

Proof. The second assertion follows directly from (2.5) and Kolmogorov's continuity theorem. To prove (2.5), let's assume $t \ge r > 0$. Then $\mathbb{E}^{\mathcal{W}^1}\left[\left|t^{\frac{\nu-1}{2}}X_t - r^{\frac{\nu-1}{2}}X_r\right|^2\right]$ can be written as

$$\begin{aligned} & t^{\nu-1}C^1(t,t) + r^{\nu-1}C^1(r,r) - 2t^{\frac{\nu-1}{2}}r^{\frac{\nu-1}{2}}C^1(t,r) \\ = \;& \sqrt{t}K_{\frac{\nu-2}{2}}(t)\left(\sqrt{t}I_{\frac{\nu-2}{2}}(t) - \sqrt{r}I_{\frac{\nu-2}{2}}(r)\right) + \sqrt{r}I_{\frac{\nu-2}{2}}(r)\left(\sqrt{r}K_{\frac{\nu-2}{2}}(r) - \sqrt{t}K_{\frac{\nu-2}{2}}(t)\right) \\ = \;& \int_r^t \left[\sqrt{t}K_{\frac{\nu-2}{2}}(t)\cdot\left(DI_{\frac{\nu-2}{2}}\right)(u) - \sqrt{r}I_{\frac{\nu-2}{2}}(r)\cdot\left(DK_{\frac{\nu-2}{2}}\right)(u)\right]du, \end{aligned}$$

where

$$\left(DI_{\frac{\nu-2}{2}}\right)(u) := \frac{d}{du}\left(\sqrt{u}I_{\frac{\nu-2}{2}}(u)\right) = \frac{I_{\frac{\nu-2}{2}}(u)}{2\sqrt{u}} + \frac{\sqrt{u}}{2}\left(I_{\frac{\nu-4}{2}}(u) + I_{\frac{\nu}{2}}(u)\right)$$

and

$$\left(DK_{\frac{\nu-2}{2}}\right)(u) := \frac{d}{du}\left(\sqrt{u}K_{\frac{\nu-2}{2}}(u)\right) = \frac{K_{\frac{\nu-2}{2}}(u)}{2\sqrt{u}} - \frac{\sqrt{u}}{2}\left(K_{\frac{\nu-4}{2}}(u) + K_{\frac{\nu}{2}}(u)\right).$$

One can easily verify that as $u \in (0,\infty)$ increases, $\sqrt{u}K_{\frac{\nu-2}{2}}(u)$ decreases and $\sqrt{u}I_{\frac{\nu-2}{2}}(u)$ increases. Therefore, $DI_{\frac{\nu-2}{2}} \ge 0$, $DK_{\frac{\nu-2}{2}} \le 0$, $\sqrt{t}K_{\frac{\nu-2}{2}}(t) \le \sqrt{u}K_{\frac{\nu-2}{2}}(u)$ and $\sqrt{r}I_{\frac{\nu-2}{2}}(r) \le \sqrt{u}I_{\frac{\nu-2}{2}}(u)$ whenever $r \le u \le t$. Putting everything together, we have that

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{W}^1}\left[\left|t^{\frac{\nu-1}{2}}X_t - r^{\frac{\nu-1}{2}}X_r\right|^2\right]\\
\leq\ & \int_r^t \left[\sqrt{u}K_{\frac{\nu-2}{2}}(u)\left(DI_{\frac{\nu-2}{2}}\right)(u) - \sqrt{u}I_{\frac{\nu-2}{2}}(u)\left(DK_{\frac{\nu-2}{2}}\right)(u)\right]du\\
=\ & \frac{1}{2}\int_r^t u\left(I_{\frac{\nu-4}{2}}(u)K_{\frac{\nu-2}{2}}(u) + I_{\frac{\nu-2}{2}}(u)K_{\frac{\nu-4}{2}}(u) + I_{\frac{\nu}{2}}(u)K_{\frac{\nu-2}{2}}(u) + I_{\frac{\nu-2}{2}}(u)K_{\frac{\nu}{2}}(u)\right)du\\
=\ & t-r.
\end{aligned}
$$

The last equality is due to the fact that for every $\mu \in \mathbb{R}$,

$$I_{\mu-1}(u)K_\mu(u) + I_\mu(u)K_{\mu-1}(u) = \frac{1}{u}. \tag{2.6}$$

□

With all these preparations, the following theorem becomes an almost immediate result.

Theorem 5. *$\{X_t : t > 0\}$ is a $\mathcal{W}^1$-a.e. continuous Gaussian Markov process, and its transition probability density is given by*

$$p(x, r;\, y, t) = \frac{1}{\sqrt{2\pi\sigma^2(r,t)}}\exp\left\{-\frac{[y - m(r,t)x]^2}{2\sigma^2(r,t)}\right\},$$

where $0 < r \leq t$, $x, y \in \mathbb{R}$, $m(r,t) := \left(\frac{t}{r}\right)^{\frac{2-\nu}{2}}\frac{K_{\frac{\nu-2}{2}}(t)}{K_{\frac{\nu-2}{2}}(r)}$, and

$$\sigma^2(r,t) := t^{\nu-2}K_{\frac{\nu-2}{2}}(t)I_{\frac{\nu-2}{2}}(t) - t^{\nu-2}K^2_{\frac{\nu-2}{2}}(t)\frac{I_{\frac{\nu-2}{2}}(r)}{K_{\frac{\nu-2}{2}}(r)}.$$

The Kolmogorov backward equation associated with X_t's is given by

$$\partial_r p(x, r;\, y, t) + \frac{r^{\nu-3}}{2}\partial_x^2 p(x, r;\, y, t) + \frac{K_{\frac{\nu}{2}}(r)}{K_{\frac{\nu-2}{2}}(r)}\partial_x p(x, r;\, y, t) = 0.$$

Proof. To complete the first assertion, we only need to show that $\{X_t : t > 0\}$ is Markovian. X_t is a Gaussian process, and it's well known that a Gaussian process is Markovian if and only if its covariance function has the following property ():

$$\mathbb{E}^{\mathcal{W}^1}[X_t X_w]\,\mathbb{E}^{\mathcal{W}^1}[X_r X_r] = \mathbb{E}^{\mathcal{W}^1}[X_t X_r]\,\mathbb{E}^{\mathcal{W}^1}[X_r X_w] \tag{2.7}$$

whenever $t \geq r \geq w > 0$. But this is a direct consequence of Theorem 3. The second statement follows from the standard property of conditional expectation of Gaussian random variables. Namely, $\mathbb{E}^{\mathcal{W}^1}[X_t|X_r]$ is again a Gaussian random variable with mean value $m(r,t)X_r = \frac{C^1(t,r)}{C^1(r,r)}X_r$ and variance

$$\sigma^2(r,t) = \frac{C^1(t,t)C^1(r,r) - \left(C^1(t,r)\right)^2}{C^1(r,r)}.$$

One can directly verify that $p(x, r;\, y, t)$ satisfies the given equation. In general (§3.3 in [2]), the associated Kolmogorov backward equation is given by

$$\partial_r p(x, r;\, y, t) = \frac{\partial_r \sigma^2(r,t)}{2m^2(r,t)}\partial_x^2 p(x, r;\, y, t) + \frac{\partial_r m(r,t)}{m(r,t)}\partial_x p(x, r;\, y, t).$$

The rest follows from the observations that

$$\partial_r\sigma^2(r,t) = -r^{\nu-3}m^2(r,t) \text{ and } \partial_r m(r,t) = -\frac{K_{\frac{\nu}{2}}(r)}{K_{\frac{\nu-2}{2}}(r)}m(r,t).$$

□

So far we have been focusing on the spherical averages of the GFF. One would naturally wonder whether the properties we derived above also hold for the averages of θ over concentric solid balls $\{B(t) := B(O,t) : t > 0\}$. To this end, let μ_t be the tempered distribution on $\mathbb{R}^\nu$ such that for any $\phi \in C_c^\infty(\mathbb{R}^\nu)$,

$$\langle \phi,\, \mu_t \rangle = \frac{1}{\beta_\nu(t)} \int_{B(t)} \phi(x)\, dx,$$

where $\beta_\nu(t) := \frac{\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2}+1)} t^\nu$ is the volume of dim-ν ball with radius $t > 0$. Set $\beta_\nu := \beta_\nu(1)$. We have that

$$\widehat{\mu_t}(\xi) = \frac{(2\pi)^{\frac{\nu}{2}}}{\beta_\nu}(t|\xi|)^{-\frac{\nu}{2}} J_{\frac{\nu}{2}}(t|\xi|).$$

It's easy to check that $\mu_t \in H^{-1}(\mathbb{R}^\nu)$ for every $t > 0$, and in fact

$$\frac{d}{dt}(\beta_v(t)\widehat{\mu_t})(\xi) = \alpha_\nu(t)\widehat{\bar{\sigma}_t}(\xi) \text{ for every } \xi \in \mathbb{R}^\nu.$$

Define the Gaussian process $Y_t := \mathcal{I}(h_{\mu_t})$ for $t > 0$, which corresponds to the averages over concentric balls.

In order to study the behavior of Y_t, we will need to know the covariance of the family $\{Y_t : t > 0\}$. In fact, we will compute Y_t's covariance through the bigger family which consists of X_t and Y_t. To this end, first consider $F(r,t) := \mathbb{E}^{\mathcal{W}^1}[Y_r X_t]$. Clearly, $F(r,t)$ is continuous in t and C^1 in r. Moreover, based on the observations above, one can verify that

$$\frac{d}{dr}(\beta_\nu(r)F(r,t)) = \frac{d}{dr}\left(\beta_\nu(r)(\mu_r,\bar{\sigma}_t)_{-1}\right) = \alpha_\nu(r)(\bar{\sigma}_r,\bar{\sigma}_t)_{-1}.$$

On one hand,

$$\beta_\nu(r)F(r,t) = t^{\frac{2-\nu}{2}} r^{\frac{\nu}{2}} \int_0^\infty \frac{J_{\frac{\nu}{2}}(r\tau)J_{\frac{\nu-2}{2}}(t\tau)}{1+\tau^2} d\tau, \tag{2.8}$$

and on the other hand,

$$\alpha_\nu(r)(\bar{\sigma}_r,\bar{\sigma}_t)_{-1} = r^{\frac{\nu}{2}} t^{\frac{2-\nu}{2}} I_{\frac{\nu-2}{2}}(t\wedge r) K_{\frac{\nu-2}{2}}(t\vee r). \tag{2.9}$$

Let's fix $t > 0$ and first assume $0 < r \le t$. (2.8) implies that $\lim_{r\to 0+}\beta_\nu(r)F(r,t) = 0$. Combining with (2.9), we have that, when $0 < r \le t$,

$$\begin{aligned} \beta_\nu(r)F(r,t) &= \left(\int_0^r \rho^{\frac{\nu}{2}} I_{\frac{\nu-2}{2}}(\rho)\, d\rho\right) t^{\frac{2-\nu}{2}} K_{\frac{\nu-2}{2}}(t) \\ &= r^{\frac{\nu}{2}} I_{\frac{\nu}{2}}(r)\, t^{\frac{2-\nu}{2}} K_{\frac{\nu-2}{2}}(t). \end{aligned}$$

When $r \geq t$, making use of the continuity of $F(r,t)$ in r, we have that

$$\begin{aligned}\beta_\nu(r)F(r,t) &= tI_{\frac{\nu}{2}}(t)K_{\frac{\nu-2}{2}}(t) + \left(\int_t^r \rho^{\frac{\nu}{2}}K_{\frac{\nu-2}{2}}(\rho)\,d\rho\right)t^{\frac{2-\nu}{2}}I_{\frac{\nu-2}{2}}(t).\\ &= tI_{\frac{\nu}{2}}(t)K_{\frac{\nu-2}{2}}(t) + \left(-\rho^{\frac{\nu}{2}}K_{\frac{\nu}{2}}(r)\,t^{\frac{2-\nu}{2}}I_{\frac{\nu-2}{2}}(t)\right)\Big|_t^r\\ &= 1 - r^{\frac{\nu}{2}}K_{\frac{\nu}{2}}(r)\,t^{\frac{\nu-2}{2}}I_{\frac{\nu-2}{2}}(t).\end{aligned}$$

The last equality again makes use of (2.6). Therefore,

$$F(r,t) = \begin{cases}\frac{1}{\beta_\nu}\cdot r^{-\frac{\nu}{2}}t^{\frac{2-\nu}{2}}I_{\frac{\nu}{2}}(r)K_{\frac{\nu-2}{2}}(t), & \text{if } 0<r\leq t,\\ \frac{1}{\beta_\nu}\left(r^{-\nu} - r^{-\frac{\nu}{2}}t^{\frac{2-\nu}{2}}K_{\frac{\nu}{2}}(r)I_{\frac{\nu-2}{2}}(t)\right), & \text{if } r>t.\end{cases} \tag{2.10}$$

Next, set $G(r,t) := \mathbb{E}^{\mathcal{W}^1}[Y_rY_t]$, and we will just repeat the procedure above to get the explicit formula for $G(r,t)$. This time, $\frac{d}{dt}(\beta_\nu(t)G(r,t)) = \alpha_\nu(t)F(r,t)$, and with fixed $r>0$, $\lim_{t\to 0+}\beta_\nu(t)G(r,t) = 0$. Hence, when $0<t\leq r$,

$$\begin{aligned}G(r,t) &= \frac{1}{\beta_\nu(t)}\frac{\alpha_\nu}{\beta_\nu}\int_0^t\left(\rho^{\nu-1}r^{-\nu} - r^{-\frac{\nu}{2}}K_{\frac{\nu}{2}}(r)\,\rho^{\frac{\nu}{2}}I_{\frac{\nu-2}{2}}(\rho)\right)d\rho\\ &= \frac{1}{\beta_\nu}\left(\frac{1}{r^\nu} - \nu r^{-\frac{\nu}{2}}K_{\frac{\nu}{2}}(r)\,t^{-\frac{\nu}{2}}I_{\frac{\nu}{2}}(t)\right).\end{aligned}$$

Similarly, since $G(r,t)$ is continuous in t, we have that

$$G(r,t) = \frac{1}{\beta_\nu}\left(\frac{1}{t^\nu} - \nu r^{-\frac{\nu}{2}}I_{\frac{\nu}{2}}(r)\,t^{-\frac{\nu}{2}}K_{\frac{\nu}{2}}(t)\right)$$

for $t>r>0$. We are now ready to state the following theorem.

Theorem 6. *Let $\{X_t : t>0\}$ and $\{Y_t : t>0\}$ be the Gaussian processes as defined above, then $\mathcal{W}^1$-a.e.*

$$Y_t = \frac{1}{\beta_\nu(t)}\int_0^t \alpha_\nu(\tau)\,X_\tau d\tau.$$

Moreover, $\{Y_t : t>0\}$ has the covariance function

$$G(r,t) = \mathbb{E}^{\mathcal{W}^1}[Y_rY_t] = \frac{\nu}{\beta_\nu}\left(\frac{1}{\nu(t\vee r)^\nu} - (tr)^{-\frac{\nu}{2}}I_{\frac{\nu}{2}}(t\wedge r)K_{\frac{\nu}{2}}(t\vee r)\right),$$

and $\{Y_t : t>0\}$ is not Markovian. However, if $\mathbf{V}_t := (X_t, Y_t)$, then $\{\mathbf{V}_t : t>0\}$ under $\mathcal{W}^1$ is a 2-vector valued a.e. continuous Gaussian Markov process.

Proof. Recall that $\{X_t : t>0\}$ is continuous $\mathcal{W}^1$-a.e., so $\int_0^t\alpha_\nu(\tau)X_\tau d\tau$ is well defined as a Gaussian random variable. With all the computational results above, one can easily conclude the first assertion by verifying that

$$\mathbb{E}^{\mathcal{W}^1}\left[\left|Y_t - \frac{1}{\beta_\nu(t)}\int_0^t\alpha_\nu(\tau)X_\tau d\tau\right|^2\right] = 0.$$

Next, $G(r,t)$, the covariance of $\{Y_t : t>0\}$, clearly violates the condition (2.7), so $\{Y_t : t>0\}$ is not Markovian.

As for vector valued Gaussian process $\{\mathbf{V}_t : t > 0\}$, it's also known that to prove the Markov property for $\{\mathbf{V}_t : t > 0\}$, we only need to show that the corresponding covariance matrix $\mathbf{C}_{2\times 2}(t,r) := \left(\mathbb{E}^{\mathcal{W}^1}\left[(\mathbf{V}_t)_i (\mathbf{V}_r)_j\right]\right)$ has the following property:

$$\mathbf{C}_{2\times 2}(t,w) = \mathbf{C}_{2\times 2}(t,r)\,\mathbf{C}^{-1}_{2\times 2}(r,r)\,\mathbf{C}_{2\times 2}(r,w) \tag{2.11}$$

whenever $t \geq r \geq w > 0$. Furthermore, it would be sufficient if we can show that $\mathbf{C}_{2\times 2}(t,r) = \mathbf{A}_{2\times 2}(t\vee r)\,\mathbf{B}_{2\times 2}(t\wedge r)$ for some non-degenerate 2×2 matrices $\mathbf{A}_{2\times 2}$ and $\mathbf{B}_{2\times 2}$. Let's assume that $t \geq r > 0$. In fact we have that

$$\begin{aligned}
\mathbf{C}_{2\times 2}(t,r) &= \begin{pmatrix} C^1(t,r) & F(r,t) \\ F(t,r) & G(r,t) \end{pmatrix} \\
&= \frac{1}{\beta_\nu}\begin{pmatrix} \frac{1}{\nu} r^{\frac{2-\nu}{2}} t^{\frac{2-\nu}{2}} I_{\frac{\nu-2}{2}}(r) K_{\frac{\nu-2}{2}}(t) & r^{-\frac{\nu}{2}} t^{\frac{2-\nu}{2}} I_{\frac{\nu}{2}}(r) K_{\frac{\nu-2}{2}}(t) \\ t^{-\nu} - t^{-\frac{\nu}{2}} r^{\frac{2-\nu}{2}} K_{\frac{\nu}{2}}(t) I_{\frac{\nu-2}{2}}(r), & t^{-\nu} - \nu r^{-\frac{\nu}{2}} I_{\frac{\nu}{2}}(r) t^{-\frac{\nu}{2}} K_{\frac{\nu}{2}}(t) \end{pmatrix} \\
&= \frac{1}{\beta_\nu}\begin{pmatrix} t^{\frac{2-\nu}{2}} K_{\frac{\nu-2}{2}}(t) & 0 \\ -\nu t^{-\frac{\nu}{2}} K_{\frac{\nu}{2}}(t) & t^{-\nu} \end{pmatrix}\begin{pmatrix} \frac{1}{\nu} r^{\frac{2-\nu}{2}} I_{\frac{\nu-2}{2}}(r) & r^{-\frac{\nu}{2}} I_{\frac{\nu}{2}}(r) \\ 1 & 1 \end{pmatrix}.
\end{aligned}$$

The continuity of $\mathbf{V}_t$ simply follows from the continuity of X_t and the first assertion. □

2.2. **When $s \geq 2$ and $s \in \mathbb{N}$.** In this subsection, we are going to discuss the spherical averages of dim-ν order-s GFF for general $s \in \mathbb{N}$ and $s \geq 2$. First, we make the remark that the condition (2.11) applies to any k-vector valued Gaussian process with $k \in \mathbb{N}$. Namely, if $\{\mathbf{W}_t = (W_{t,1}, W_{t,2}, \ldots, W_{t,k}) : t > 0\}$is a k-vector valued Gaussian process on Θ^s under $\mathcal{W}^s$, with the covariance matrix $\mathbf{C}_{k\times k}(t,r) := \left(\mathbb{E}^{\mathcal{W}^s}\left[(\mathbf{W}_t)_i (\mathbf{W}_r)_j\right]\right)$. Then, $\{\mathbf{W}_t : t > 0\}$ is Markovian if and only if for any $t \geq r \geq w > 0$,

$$\mathbf{C}_{k\times k}(t,w) = \mathbf{C}_{k\times k}(t,r)\,\mathbf{C}^{-1}_{k\times k}(r,r)\,\mathbf{C}_{k\times k}(r,w). \tag{2.12}$$

Again we consider the spherical average $\bar{\sigma}_t$. When $s \geq 2$ and $s \in \mathbb{N}$, we notice that that $\bar{\sigma}_t^{(k)} := \left(\frac{d}{dt}\right)^k \bar{\sigma}_t$ with $k \in \{0, 1, \ldots, s-1\}$, as tempered distributions, are all elements of H^{-s}. Consequently, for every $k \in \{0, 1, \ldots, s-1\}$, $\left\{X_t^{(k)} := \mathcal{I}\left(h_{\bar{\sigma}_t^{(k)}}\right) : t > 0\right\}$ forms a Gaussian family under $\mathcal{W}^s$. We identify $X_t^{(0)}$ as X_t. The following theorem says that although X_t fails to be Markovian when $s \geq 2$, by putting $X_t, X_t^{(1)}, \ldots, X_t^{(s-1)}$ together, we can restore the Markov property.

Theorem 7. *Set $\mathbf{W}_t := \left(X_t, X_t^{(1)}, \ldots, X_t^{(s-1)}\right)$, then $\{\mathbf{W}_t : t > 0\}$ is a s-vector valued a.e. continuous Gaussian Markov process on Θ^s under $\mathcal{W}^s$.*

Proof. Recall the notations

$$C^s(t,r) = \mathbb{E}^{\mathcal{W}^s}[X_t X_r] \text{ and } \mathbf{C}_{s\times s}(t,r) = \left(\mathbb{E}^{\mathcal{W}^s}\left[(\mathbf{W}_t)_i (\mathbf{W}_r)_j\right]\right).$$

It's immediate from the definition of $X_t^{(k)}$ for $k \in \{0, 1, \ldots, s-1\}$ that

$$(\mathbf{C}_{s\times s}(t,r))_{ij} = \left(\frac{d}{dt}\right)^{i-1}\left(\frac{d}{dr}\right)^{j-1} C^s(t,r).$$

Therefore, to understand $\mathbf{C}_{s\times s}(t,r)$, it comes down to computing $C^s(t,r)$. Recall from (2.2) , (2.3) and (2.4) that

$$C^s(t,r)=\frac{1}{\alpha_\nu}(tr)^{\frac{2-\nu}{2}}\int_0^\infty\frac{\tau J_{\frac{\nu-2}{2}}(t\tau)J_{\frac{\nu-2}{2}}(r\tau)}{(1+\tau^2)^s}d\tau,$$

$$\begin{aligned}C_p^1(t,r) &= \frac{1}{\alpha_\nu}(tr)^{\frac{2-\nu}{2}}\int_0^\infty\frac{\tau J_{\frac{\nu-2}{2}}(t\tau)J_{\frac{\nu-2}{2}}(r\tau)}{p^2+\tau^2}d\tau\\ &= \frac{1}{\alpha_\nu}(tr)^{\frac{2-\nu}{2}}K_{\frac{\nu-2}{2}}(p(t\vee r))I_{\frac{\nu-2}{2}}(p(t\wedge r)),\end{aligned}$$

from which it follows that

$$\begin{aligned}C^s(t,r) &= \frac{1}{s!}\left(-\frac{1}{2p}\frac{d}{dp}\right)_{p=1}^{s-1}C_p^1(t,r)\\ &= \frac{1}{s!}\left(-\frac{1}{2p}\frac{d}{dp}\right)_{p=1}^{s-1}\left(\frac{1}{\alpha_\nu}(tr)^{\frac{2-\nu}{2}}K_{\frac{\nu-2}{2}}(p(t\vee r))I_{\frac{\nu-2}{2}}(p(t\wedge r))\right).\end{aligned}$$

Now let's assume that $t\geq r>0$. Then it's obvious that $C^s(t,r)$ must be of the form

$$C^s(t,r)=\sum_{k=0}^{s-1}a_k(t)\,b_k(r),$$

where a_k and b_k are functions only depending on t and r respectively. Therefore, $\mathbf{C}_{s\times s}(t,r)=\mathbf{A}_{s\times s}(t)\,\mathbf{B}_{s\times s}(r)$, where $\mathbf{A}_{s\times s}(t)$ and $\mathbf{B}_{s\times s}(r)$ are $s\times s$ matrices, and $(\mathbf{A}_{s\times s}(t))_{ij}=\left(\frac{d}{dt}\right)^{i-1}a_{j-1}(t)$ and $(\mathbf{B}_{s\times s}(r))_{ij}=\left(\frac{d}{dr}\right)^{j-1}b_{i-1}(r)$. Hence, $\mathbf{C}_{s\times s}(t,r)$ satisfied the condition (2.12).

Finally, to establish the continuity of $X_t^{(k)}$, we notice that the covariance of $\left\{X_t^{(k)}:t>0\right\}$ is in fact

$$\begin{aligned}C^{s,(k)}(t,r) &:= \left(\frac{d}{dt}\right)^k\left(\frac{d}{dr}\right)^kC^s(t,r)\\ &= \frac{1}{\alpha_\nu}\int_0^\infty\frac{\tau^{\nu-1+2k}\left(D^kJ_{\frac{\nu-2}{2}}\right)(t\tau)\left(D^kJ_{\frac{\nu-2}{2}}\right)(r\tau)}{(1+\tau^2)^s}d\tau\end{aligned}$$

where

$$\left(D^kJ_{\frac{\nu-2}{2}}\right)(u)=\left(\frac{d}{du}\right)^k\left(u^{\frac{2-\nu}{2}}\cdot J_{\frac{\nu-2}{2}}(u)\right).$$

By the basic identities of the derivatives of $J_{\frac{\nu-2}{2}}$, we recognize that $\left(D^kJ_{\frac{\nu-2}{2}}\right)(u)$ can be written as a finite sum of the terms in the form of

$$c_{j,l}\cdot u^{\frac{2-\nu}{2}-l+j}\cdot J_{\frac{\nu-2}{2}+l}(u)\ \text{ where } 0\leq l\leq k,\ 0\leq j\leq l \text{ and } c_{j,l}\in\mathbb{R}.$$

We will need an estimate on $\left[\left(D^kJ_{\frac{\nu-2}{2}}\right)(t\tau)-\left(D^kJ_{\frac{\nu-2}{2}}\right)(r\tau)\right]^2$ in terms of $|t-r|$. By invoking the series expansion of $J_{\frac{\nu-2}{2}+l}$, it's not hard to see that the "worst" term in that difference will occur when $j=0$, and even in that situation (assuming $0<r\le t$) we have that

$$c_{0,l}^{2}\left(\frac{J_{\frac{\nu-2}{2}+l}(t\tau)}{(t\tau)^{\frac{\nu-2}{2}+l}}-\frac{J_{\frac{\nu-2}{2}+l}(r\tau)}{(r\tau)^{\frac{\nu-2}{2}+l}}\right)^{2} = c_{0,l}^{2}\tau^{2-\nu-2l}\left(\int_{r}^{t}\rho^{\frac{2-\nu}{2}-l}J_{\frac{\nu}{2}+l}(\tau\rho)\,d\rho\right)^{2}$$

$$\le C\tau^{2-\nu-2l}\cdot r^{\nu-2+2l}\cdot(t-r)\cdot\min\left\{1,\frac{1}{\sqrt{t\tau}}\right\}.$$

for every $0\le l\le k$. Finally we see that

$$\mathbb{E}^{\mathcal{W}^{s}}\left[\left|X_{t}^{(k)}-X_{r}^{(k)}\right|^{2}\right] = \frac{1}{\alpha_{\nu}}\int_{0}^{\infty}\frac{\tau^{\nu-1+2k}\left[\left(D^{k}J_{\frac{\nu-2}{2}}\right)(t\tau)-\left(D^{k}J_{\frac{\nu-2}{2}}\right)(r\tau)\right]^{2}}{(1+\tau^{2})^{s}}d\tau$$

is always bounded by $C\,r^{\nu-2+2k}\,t^{-\frac{1}{2}}\,(t-r)$ so long as $0\le k\le s-1$. It follows immediately from Kolmogorov's continuity theorem that $X_{t}^{(k)}$ is $\mathcal{W}^{s}$-a.e. continuous on $t\in(0,\infty)$. □

2.3. **Non-Concentric Spherical Averages.** For the Gaussian family consisting of non-concentric spherical averages, it is also possible to obtain the explicit formulas for the covariance in some situations. However, the technicality is rather heavy. Again, we need various integral formulas for Bessel functions, for which we refer to pp 429-430 of [6] for technical details. The main results are the following:

Theorem 8. *Let $s=1$, and $\{X_{t}^{x}: x\in\mathbb{R}^{\nu},\ t>0\}$ the Gaussian family as defined at the beginning of §2 with covariance*

$$C^{1}(x,t;\,y,r)=\mathbb{E}^{\mathcal{W}^{1}}\left[X_{t}^{x}X_{r}^{y}\right].$$

If $t\ge|x-y|+r$, then

$$C^{1}(x,t;\,y,r)=\frac{A_{\nu}}{(tr\,|x-y|)^{\frac{\nu-2}{2}}}K_{\frac{\nu-2}{2}}(t)\,I_{\frac{\nu-2}{2}}(r)\,I_{\frac{\nu-2}{2}}(|x-y|),$$

and if $|x-y|>t+r$, then

$$C^{1}(x,t;\,y,r)=\frac{A_{\nu}}{(tr\,|x-y|)^{\frac{\nu-2}{2}}}K_{\frac{\nu-2}{2}}(|x-y|)\,I_{\frac{\nu-2}{2}}(r)\,I_{\frac{\nu-2}{2}}(t),$$

where $A_{\nu}:=\left(\frac{2}{\pi}\right)^{\frac{\nu}{2}}\frac{\Gamma^{2}\left(1+\frac{\nu}{2}\right)}{\nu^{2}}$.

For the general $s\ge2$ and $s\in\mathbb{N}$, we also have a result that is analogous to the concentric case. We will need the following remark.

Remark 9. Define $C_{p}^{1}(x,t;\,y,r)$ similarly as $C_{p}^{1}(t,r)$, i.e., consider the operator $\left(p^{2}I-\Delta\right)$ instead of $(I-\Delta)$. Then, if $t\ge|x-y|+r$, then

$$C_{p}^{1}(x,t;\,y,r)=\frac{A_{\nu}}{(ptr\,|x-y|)^{\frac{\nu-2}{2}}}K_{\frac{\nu-2}{2}}(pt)\,I_{\frac{\nu-2}{2}}(pr)\,I_{\frac{\nu-2}{2}}(p\,|x-y|);$$

and if $|x-y|>t+r$, then

$$C_{p}^{1}(x,t;\,y,r)=\frac{A_{\nu}}{(ptr\,|x-y|)^{\frac{\nu-2}{2}}}K_{\frac{\nu-2}{2}}(p\,|x-y|)\,I_{\frac{\nu-2}{2}}(pr)\,I_{\frac{\nu-2}{2}}(pt).$$

Theorem 10. *Let $s \geq 2$ and $s \in \mathbb{N}$. The covariance of the Gaussian family $\{X_t^x : x \in \mathbb{R}^\nu,\, t > 0\}$ is given by*

$$C^s(x,t;\, y,r) = \frac{1}{s!}\left(-\frac{1}{2p}\frac{d}{dp}\right)^{s-1}_{p=1} C_p^1(x,t;\, y,r).$$

Therefore, either when $t > |x-y| + r$ or when $|x-y| > t + r$, one can derive the explicit formulas for $C^s(x,t;\, y,r)$ based on the previous remark.

In particular, in either case, there exist vector-valued functions $\mathbf{V} : (0,\infty) \to \mathbb{R}^s$ and $\mathbf{U} : (0,\infty) \to \mathbb{R}^s$, and matrix-valued function $\mathbf{A}_{s\times s} : (0,\infty) \to \mathbb{R}^s \times \mathbb{R}^s$ such that

$$C^s(x,t;\, y,r) = \mathbf{V}^\top(t)\,\mathbf{A}(|x-y|)\,\mathbf{U}(r).$$

Set $X_t^{x,(k)} := \left(\frac{d}{dt}\right)^{(k)} X_t^x$. Then $X_t^{x,(k)}$ is well-defined as a Gaussian random variable on Θ^s under $\mathcal{W}^s$ for every $k = 0, 1, \cdots, s-1$. Identify $X_t^{x,(0)}$ as X_t^x, and denote $\mathbf{W}_t^x := \left(X_t^{x,(0)}, \cdots, X_t^{x,(s-1)}\right)$. Then $\{\mathbf{W}_t^x : x \in \mathbb{R}^\nu,\, t > 0\}$ forms a vector-valued two-parameter Gaussian family under $\mathcal{W}^s$ with the covariance matrix

$$\mathbf{C}_{s\times s}(x,t;\, y,r) = \left(\mathbb{E}^{\mathcal{W}^s}\left[(\mathbf{W}_t^x)_i\,(\mathbf{W}_r^y)_j\right]\right).$$

Then, either when $t > |x-y| + r$ or when $|x-y| > t + r$, we have that

$$\mathbf{C}_{s\times s}(x,t;\, y,r) = \mathbf{V}_{s\times s}(t)\,\mathbf{A}(|x-y|)\,\mathbf{U}_{s\times s}(r)$$

where $\mathbf{V}_{s\times s}(t)$ and $\mathbf{U}_{s\times s}(r)$ are $s \times s$ matrices such that

$$(\mathbf{V}_{s\times s}(t))_{ij} = \left(\frac{d}{dt}\right)^{i-1}(\mathbf{V}_j(t))$$

and

$$(\mathbf{U}_{s\times s}(r))_{ij} = \left(\frac{d}{dr}\right)^{y-1}(\mathbf{U}_i(r)).$$

A detailed treatement in the case when $s = 2$ and $\nu = 4$ can be found in the appendix of [1].

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