# Sheaf Cohomology

Gabriel Chênevert
Payman Kassaei

September 24, 2003

In this lecture, we define the cohomology groups of a topological space $X$ with coefficients in a sheaf of abelian groups $\mathcal{F}$ on $X$ in terms of the derived functors of the global section functor $\Gamma(X, \cdot\,)$. Then we introduce Čech cohomology with respect to an open covering of $X$, which permits to make explicit calculations, and discuss under which conditions it can be used to compute sheaf cohomology.

## 1 Derived functors

We first need to review some homological algebra in order to be able to define sheaf cohomology using the derived functors of the global sections functor.

Let $\mathscr{A}$ be an *abelian category*, that is, roughly, an additive category in which there exist well-behaved kernels and cokernels for each morphism, so that, for example, the notion of an exact sequence in $\mathscr{A}$ makes sense.

If $X$ is a fixed object in $\mathscr{A}$ and Ab denotes the category of abelian groups, then we have a contravariant functor

$$\mathrm{Hom}(\,\cdot\,, X) : \mathscr{A} \longrightarrow \mathrm{Ab}.$$

It is readily seen to be *left exact*, that is, for any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathscr{A}$, the sequence

$$0 \longrightarrow \mathrm{Hom}(C, X) \xrightarrow{g^*} \mathrm{Hom}(B, X) \xrightarrow{f^*} \mathrm{Hom}(A, X)$$

is exact in *Ab*.

**Definition 1.1.** An object $I$ of $\mathscr{A}$ is said to be *injective* if the functor $\mathrm{Hom}(\,\cdot\,, I)$ is exact.

Since $\mathrm{Hom}(\,\cdot\,, I)$ is always left exact, we see that an object $I$ of $\mathscr{A}$ is injective if and only if for each exact sequence $0 \to A \to B$ and morphism $A \to I$, there exists a morphism $B \to I$ making the following diagram commute.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

That is, morphisms to an injective defined on a sub-object $A$ can always be extended to the whole object $B$.

**Definition 1.2.** An abelian category $\mathscr{A}$ is said to have *enough injectives* if each object of $\mathscr{A}$ can be embedded in an injective object.

This is equivalent to saying that each object $A$ of $\mathscr{A}$ admits an *injective resolution*, that is, a long exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

where each $I^i$ is injective. To see this, first embed $A$ in an injective $I^0$, then embed the cokernel of the inclusion $\varepsilon : A \to I^0$ in an injective $I^1$, and take for $I^0 \to I^1$ the composite $I^0 \to \mathrm{Coker}\,\varepsilon \to I^1$, and so on.

**Definition 1.3.** A (*cochain*) *complex* $A^\bullet$ in an abelian category $\mathscr{A}$ is a collection of objects $A^i$ of $\mathscr{A}$, $i \in \mathbb{Z}$, together with morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all $i$. The maps $d^i$ are called the *differentials* or *coboundary maps* of the complex $A$.

To any complex $A^\bullet$ in $\mathscr{A}$ one can associate a complex $H^\bullet(A^\bullet)$ with zero differential, called the *cohomology* of $A^\bullet$, by defining for each $i \in \mathbb{Z}$

$$H^i(A^\bullet) := \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$$

(the condition that $d^i \circ d^{i-1} = 0$ ensures that this definition makes sense).

A morphism of complexes $f : A^\bullet \to B^\bullet$ is a collection of maps $f^i : A^i \to B^i$ which commute with the differentials, i.e. that make the following diagram commutative.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d^{i-1}} & A^i & \xrightarrow{d^i} & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \longrightarrow & B^{i-1} & \xrightarrow[d^{i-1}]{} & B^i & \xrightarrow[d^i]{} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

Any such morphism induces a morphism

$$H^\bullet(f) : H^\bullet(A^\bullet) \to H^\bullet(B^\bullet)$$

on the cohomology, defined on $H^i(A^\bullet)$ by

$$H^i(f)(a + \operatorname{Im} d^i) := f^i(a) + \operatorname{Im} d^i.$$

Thus, we may think of $H^\bullet$ as a functor on the category of complexes in $\mathscr{A}$. One of the main properties of this functor is the following.

**Proposition 1.4.** *Let $\mathscr{A}$ be an abelian category and*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

*a short exact sequence of complexes in $\mathscr{A}$. Then we get a natural* long exact sequence

$$0 \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet) \to \dots$$

*in cohomology.*

*Proof.* Use the "snake lemma" (e.g. [W, Th. 1.3.1]). □

**Definition 1.5.** Two morphisms of complexes $f, g : A^\bullet \to B^\bullet$ are *homotopic* if there exists a $k : A^\bullet \to B^\bullet$ of degree $-1$ such that $f - g = dk + kd$.

Note that $k$ is simply a collection of morphisms from $A^i \to B^{i-1}$ for each integer $i$. It is direct to see that if $f$ and $g$ are homotopic, they induce the same morphism on the cohomology, i.e. $H^\bullet(f) = H^\bullet(g)$.

Now, let's write an injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

as $0 \to A \to I^\bullet$, where we think of $I^\bullet$ as a complex which is 0 in negative degrees.

Let $\mathscr{A}$ be an abelian category with enough injectives and $F : \mathscr{A} \to \mathscr{B}$ an (additive) covariant left exact functor.

Choose an injective resolution $0 \to A \to I^\bullet$ of each object $A$ of $\mathscr{A}$. Then forget $A$ itself to retain only the complex $I^\bullet$. If we apply $F$ we still get a complex $F(I^\bullet)$, so that we can define

$$R^iF(A) := H^i(F(I^\bullet)).$$

The key point which makes this definition work is the following lemma, which relies on the property of being injective.

**Lemma 1.6.** *Let* $0 \to B \to I^\bullet$ *be an injective resolution of an object* $B$ *and* $0 \to A \to J^\bullet$ *any resolution of* $A$*. Then any morphism* $f : A \to B$ *induces a morphism of complexes* $f^\bullet : J^\bullet \to I^\bullet$*, which is unique up to homotopy.*

This implies, first of all, that any two injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$ of the same object $A$ are homotopy equivalent, thus it is also the case for the complexes $F(I^\bullet)$ and $F(J^\bullet)$, which guarantees that

$$H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$$

for all $i$. This means that $R^iF(A)$ is well defined (up to canonical isomorphism).

Second, any morphism $f : A \to B$ in $\mathscr{A}$ induces a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$ between injective resolutions, so that we get a morphism

$$R^iF(f) := H^i(F(f^\bullet)) : R^iF(A) \longrightarrow R^iF(B)$$

at the level of cohomology.

**Proposition 1.7.** *If* $\mathscr{A}$ *has enough injectives and* $F : \mathscr{A} \to \mathscr{B}$ *is a left exact functor, then for each* $i \geq 0$

$$R^iF : \mathscr{A} \longrightarrow \mathscr{B}$$

*as defined above is an additive functor, called the* $i^{\text{th}}$ right derived functor *of* $F$*. Moreover,* $R^0F \cong F$*, and each short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*in* $\mathscr{A}$ *induces a natural long exact sequence*

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots$$

*in* $\mathscr{B}$*.*

*Proof.* The fact that each $R^iF$ is an additive functor follows from Lemma 1.6.

To verify that $R^0F \cong F$, choose an injective resolution

$$0 \longrightarrow A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots$$

of $A$. Since $F$ is left exact, we know that

$$0 \longrightarrow F(A) \xrightarrow{F(\varepsilon)} F(I^0) \xrightarrow{F(d^0)} F(I^1)$$

is still exact. Thus we find that

$$R^0F(A) = \operatorname{Ker} F(d^0)/\operatorname{Im} 0 = \operatorname{Ker} F(d^0) = \operatorname{Im} F(\varepsilon) \cong F(A).$$

Similarly, if $f$ is a morphism in $\mathscr{A}$, it is easy to verify that $R^0F(f)$ corresponds to $F(f)$ via this isomorphism.

Now, let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be any short exact sequence in $\mathscr{A}$. Choose injective resolutions $0 \to A \to I^\bullet$ and $0 \to C \to K^\bullet$. Then, the "horseshoe lemma" (see e.g. [W, Th. 2.2.8]) implies that there exists an injective resolution $0 \to B \to J^\bullet$ of $B$ so that our short exact sequence induces a short exact sequence of complexes

$$0 \longrightarrow I^\bullet \longrightarrow J^\bullet \longrightarrow K^\bullet \longrightarrow 0.$$

Applying $F$, this sequence remains exact (because the $I^i$'s are injective), and the short exact sequence of complexes

$$0 \longrightarrow F(I^\bullet) \longrightarrow F(J^\bullet) \longrightarrow F(K^\bullet) \longrightarrow 0$$

induces in cohomology the appropriate long exact sequence using Proposition 1.4 and the fact that $R^0F \cong F$ just proved.

□

**Proposition 1.8.** *If $F : \mathscr{A} \to \mathscr{B}$ is any left exact functor and $I$ is injective, then $R^iF(I) = 0$ for all $i > 0$.*

*Proof.* Use the injective resolution $0 \to I \to I \to 0$ to compute $R^iF(I)$.

□

**Remark 1.9.** The collection of functors $\{R^iF\}$ forms what is called a *$\delta$-functor*. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A (covariant) $\delta$-functor from $\mathcal{A}$ to $\mathcal{B}$ is a collection of (covariant) functors $\{F^i\}$ for each $i \geq 0$ such that a short exact sequences of objects (functorially) gives rise to a long exact sequence as in Proposition 1.7. (The name $\delta$-functor relates to the connecting homomorphisms in the long exact sequence, which are often denoted by $\delta$).

A $\delta$-functor $\{F^i\}$ is *universal* if for any other $\delta$-functor $\{G^i\}$, any morphism $F^0 \to G^0$ can be uniquely completed to morphisms $F^i \to G^i$, commuting with $\delta$. Therefore, there is at most one universal $\delta$-functor which in degree 0 is a given functor.

If the covariant $\delta$-functor $\{F^i\}$ has the property that for any object $A$ of $\mathcal{A}$ there is some monomorphism $u : A \to M$ such that $F^i(u) = 0$ for all $i > 0$, then $\{F^i\}$ is universal. This implies that the right derived functors of a left exact covariant functor form a universal $\delta$-functor (since one can take $u$ to be the injection of $A$ in an injective object, as $\mathcal{A}$ has enough injectives). In particular, if $\mathcal{A}$ has enough injectives, and $\{G^i\} : \mathcal{A} \to \mathcal{B}$ is a universal $\delta$-functor such that $G^0$ is left exact, then we have $G^i = R^iG^0$.

**Remark 1.10.** If $F$ is exact, then $R^iF(A) = 0$ for any $A$ and $i > 0$. Indeed, if we choose an injective resolution $0 \to A \to I^\bullet$, the complex $F(I^\bullet)$ is exact in degrees $i > 0$ since $F$ is exact, thus $R^iF(A) = H^i(F(I^\bullet)) = 0$ for $i > 0$.

**Definition 1.11.** An object $A$ of $\mathscr{A}$ is *$F$-acyclic* if $R^iF(A) = 0$ for all $i > 0$.

In particular, it implies that if $0 \to A \to B \to C \to 0$ is exact and $A$ is $F$-acyclic, then $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.

We've just seen that injectives are $F$-acyclic for any left exact functor $F$. We know that we can compute the derived functors $R^iF$ using injective resolutions, but it is sometimes useful to use resolutions which are more adapted to $F$.

**Proposition 1.12.** *If $0 \to A \to J^\bullet$ is an $F$-acyclic resolution of $A$ then there exists a natural isomorphism $R^iF(A) \cong H^i(F(J^\bullet))$.*

*Proof.* The case $i = 0$ follows from the left exactness of $F$. For $i > 0$ this can be proved by induction using a "dimension shifting" argument. The long exact sequence of the resolution $0 \to A \to J^\bullet$ can be decomposed into a short exact sequence $0 \to A \to J^0 \to B \to 0$, and a resolution $0 \to B \to J^1 \to J^2 \to ...$ which we denote by $0 \to B \to K^\bullet$. Writing the corresponding long exact sequence for the first short exact sequence, and using the $F$-acyclicity of each $J_i$ we get natural isomorphisms $R^{i-1}F(B) \cong R^iF(A)$ for all $i \geq 1$. By induction hypothesis

$$R^iF(A) \cong R^{i-1}F(B) \cong H^{i-1}(F(K^\bullet)) \cong H^i(F(J^\bullet)).$$

Now assume $i = 1$. From the above short exact sequence we get an exact sequence $0 \to F(A) \to F(J^0) \to F(B) \to R^1F(A)$. On the other hand, by left exactness of $F$ we know $\ker(F(J^1) \to F(J^2)) = F(B)$. Therefore we have natural isomorphisms

$$\begin{aligned} H^1(J^\bullet) &\cong \ker(F(J^1) \to F(J^2))/\mathrm{Im}\,(F(J^0) \to F(J^1)) \\ &\cong F(B)/\mathrm{Im}\,(F(J^0) \to F(B)) \\ &\cong R^1F(A). \end{aligned}$$

This completes the proof. ☐

# 2 Sheaf cohomology as derived functor

We want to define the cohomology groups of a sheaf by taking the derived functors of the global sections functor. In order to be able to do this, we must first ensure that the global sections functor $\Gamma$ is left exact and that the category $\mathrm{Ab}(X)$ has enough injectives. we will do so more generally for any ringed space.

Let $(X, \mathcal{O}_X)$ be a *ringed space* (that is, $\mathcal{O}_X$ is a sheaf of rings on the topological space $X$), and consider the abelian category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules on $X$.

**Remark 2.1.** Note that if we take $\mathcal{O}_X$ to be the constant sheaf $\underline{\mathbb{Z}}$, then $\mathrm{Mod}(X)$ becomes just the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on $X$.

Let $R$ denote $\Gamma(X, \mathcal{O}_X)$.

**Proposition 2.2.** *The functor of global sections* $\Gamma(X, \cdot) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}_R$ *is left exact.*

*Proof.* Recall that in the category $\mathrm{Mod}(X)$ of sheaves, kernels are defined as ordinary kernels in the category of presheaves, but images are defined as the sheafification of images as presheaves.

Let $0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \longrightarrow 0$ be a short exact sequence of $\mathcal{O}_X$-modules. We prove that

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{f_X} \mathcal{G}(X) \xrightarrow{g_X} \mathcal{H}(X)$$

is exact. At the first place we have $\mathrm{Ker}\, f_X = (\mathrm{Ker}\, f)(X) = 0$ since $\mathrm{Ker}\, f = 0$. In the middle, it is clear that $\mathrm{Im}\,(f_X) \subset \mathrm{Ker}\,(g_X)$. Any section $s$ in $\mathcal{G}(X)$ which maps to 0 in $\mathcal{H}(X)$ is by definition locally a section of $\mathcal{F}$ on $X$. But $\mathcal{F}$ is a sheaf on $X$ and hence $s$ is a section in $\mathcal{F}(X)$.

□

**Remark 2.3.** On the category of *presheaves*, the global sections functor is exact, thus it cannot yield any interesting derived functor.

Recall that if $f : X \to Y$ is a continuous map of ringed spaces, then we get two functors

$$f_* : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(Y) \quad \text{and} \quad f^* : \mathrm{Mod}(Y) \longrightarrow \mathrm{Mod}(X).$$

If $\mathcal{F} \in \mathrm{Mod}(X)$, its *direct image* $f_*(\mathcal{F})$, defined by

$$f_*(\mathcal{F})(U) := \mathcal{F}(f^{-1}(U)),$$

has a natural $\mathcal{O}_Y$-module structure.

On the other hand, if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then we define its *inverse image*

$$f^*(\mathcal{G}) := f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X,$$

where $f^{-1}$ is the inverse image functor at the level of sheaves of abelian groups, defined by

$$f^{-1}(\mathcal{H})(U) := \varinjlim_{V \supseteq f(U)} \mathcal{H}(V).$$

One of the main properties of these functors is that they form an *adjoint pair*, that is, for each $\mathcal{O}_X$-module $\mathcal{F}$ and $\mathcal{O}_Y$-module $\mathcal{G}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

In particular, corresponding to the identity maps

$$f^*\mathcal{G} \longrightarrow f^*\mathcal{G} \quad \text{and} \quad f_*\mathcal{F} \longrightarrow f_*\mathcal{F},$$

there exists natural maps

$$\mathcal{G} \longrightarrow f_*f^*\mathcal{G} \quad \text{and} \quad f^*f_*\mathcal{F} \longrightarrow \mathcal{F}.$$

**Remark 2.4.** We can give an alternate proof of Proposition 2.2 using the fact that if $f : X \to \{*\}$ is the unique map from $X$ to a space consisting of a single point, by identifying $\mathrm{Mod}(\{*\})$ with $\mathrm{Mod}_R$, we see that $f_* = \Gamma(X, \cdot)$ and that $f^*$ is the constant sheaf functor.

The result then follows using the general fact that a right adjoint functor is always left exact (see [W, Th. 2.6.1]).

**Proposition 2.5.** *If $R$ is a commutative ring with identity, then the category $\mathrm{Mod}_R$ of $R$-modules has enough injectives.*

For a proof, see e.g. [G, Theorem 1.2.2].

**Proposition 2.6.** *If $(X, \mathcal{O}_X)$ is a ringed space, then the category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules has enough injectives.*

*Proof.* Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For any point $x \in X$ we let $\mathcal{F}_x$ denote the stalk of $\mathcal{F}$ at $x$, and fix an inclusion $\mathcal{F}_x \hookrightarrow I_x$ of $\mathcal{F}_x$ in an injective $\mathcal{O}_{X,x}$-module. The idea is to construct a sheaf of discontinuous sections whose stalks are $\mathcal{F}_x$. Let $\mathcal{I}$ be the sheaf defined by $\mathcal{I}(U) = \Pi_{x \in U} I_x$. Then there is clearly an injection of $\mathcal{F}$ into $\mathcal{I}$ via the above fixed inclusions. We need to show that $\mathcal{I}$ is injective. To give a morphism from a sheaf $\mathcal{A}$ to $\mathcal{I}$ is equivalent to giving a collection of morphisms from $\mathcal{A}_x$ to $I_x$ for every $x \in X$. Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be an inclusion of sheaves, and $f : \mathcal{A} \to \mathcal{I}$ a morphism. Then every $f_x : \mathcal{A}_x \to I_x$ can be extended to a morphism $g_x : \mathcal{B}_x \to I_x$, since $I_x$ is injective. The collection of the morphisms $g_x$ then give a morphism $g : \mathcal{B} \to \mathcal{I}$ which extends $f$. This proves the injectivity of $\mathcal{I}$. □

**Remark 2.7.** This proof can be reformulated in a purely formal way using the formalism of the functors $f_*$, $f^*$. If $x \in X$, let $j_x : \{x\} \to X$ denote the inclusion. Then the sheaf $\mathcal{I}$ constructed above is just $\prod_{x \in X} (j_x)_* I_x$, so that for any $\mathcal{O}_X$-module $\mathcal{G}$, we have, using the adjointness property,

$$\mathrm{Hom}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}(\mathcal{G}, (j_x)_* I_x) = \prod_{x \in X} \mathrm{Hom}((j_x)^* \mathcal{G}, I_x) = \prod_{x \in X} \mathrm{Hom}(\mathcal{G}_x, I_x).$$

This implies that $\mathcal{I}$ is injective, since the functor $\mathrm{Hom}(\,\cdot\,, \mathcal{I})$ is a composition of exact functors, and that $\mathcal{F}$ embeds in $\mathcal{I}$.

**Corollary 2.8.** *The category* $\mathrm{Ab}(X)$ *has enough injectives.*

Now we define the derived-functor sheaf cohomology.

**Definition 2.9.** Let $\Gamma = \Gamma(X, \cdot\,) : \mathrm{Ab}(X) \to \mathrm{Ab}$ be the global sections functor. Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. Then for each $i \geq 0$, the *i-th derived functor cohomology group of $F$* is defined as $H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F})$.

**Remark 2.10.** By Remark 1.9 the collection of functors $H^i(X, \cdot\,)$ forms a universal $\delta$-functor on $\mathrm{Ab}(X)$. This can be used, for instance, to prove equivalence of cohomology theories. If $\underline{H}^i(X, \cdot\,)$ is another cohomology theory such that $\{\underline{H}^i\}$ form a universal $\delta$-functor, and such that $\underline{H}^0 = \Gamma$, then we have the equality $H^i(X, \mathcal{F}) = \underline{H}^i(X, \mathcal{F})$ for all sheaves $\mathcal{F}$. Variations of this argument are quite useful in proving theoretical results about sheaf cohomology.

As an immediate result of the above definition, if $\mathcal{I}$ is an injective object in $\mathrm{Ab}(X)$, then $H^i(X, \mathcal{I}) = 0$ for all $i > 0$.

If $(X, \mathcal{O}_X)$ is a ringed space, a similar definition of cohomology functors could be given for $\mathrm{Mod}(X)$. In what follows we will show that this gives the same cohomology groups as above (by forgetting the $\mathcal{O}_X$-module structure), though it endows the cohomology groups with extra structure.

**Definition 2.11.** A sheaf of abelian groups $\mathcal{F}$ on a topological space $X$ is called *flasque* (or *flabby*) if for any inclusion of open sets $U \subseteq V$, the restriction map

$\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Equivalently, a sheaf $\mathcal{F}$ is flasque if and only if for all open sets $U$ of $X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. That means that any *local* section of $\mathcal{F}$ can be extended to a *global* section. First we show that that flasque sheaves have no cohomology except in dimension 0. First a couple of lemmas.

**Lemma 2.12.** *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence of sheaves and* $\mathcal{F}'$ *is flasque, then for each open* $U$ *of* $X$*, the sequence* $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ *is exact.*

*Proof.* We can assume $U = X$. Let $s'' \in \mathcal{F}''(X)$. We need to show that it can be represented by a section of $\mathcal{F}$ on $X$. Consider all pairs $(V, s)$ such that $s$ is a section of $\mathcal{F}$ on an open $V$ which represents $s''$, with the natural ordering. Let $(V_0, s_0)$ be a maximal element. If $V_0$ is not $X$, we can find a nonempty open $V_1$ which doesn't lie inside $V_0$ and a section $s_1$ in $\mathcal{F}(V_1)$ which represents $s''$. On $V_1 \cap V_0$, the section $s_1$ differs from $s_0$ by an element of $\mathcal{F}'(V_1 \cap V_0)$ which by flasqueness of $\mathcal{F}'$ can be extended to $V_1$. So we can modify $s_1$ (using this extended section) to agree with $s_0$ on $U_0 \cap U_1$. This contradicts the maximality of $(V_0, s_0)$ and proves our claim.

□

**Lemma 2.13.** *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence of sheaves and* $\mathcal{F}'$ *and* $\mathcal{F}$ *are flasque sheaves, then so is* $\mathcal{F}''$*. In other words, the quotient of a flasque sheaf by a flasque subsheaf is a flasque sheaf.*

*Proof.* Let $U$ be an open subset of $X$. By the above lemma, any section $s''$ in $\mathcal{F}''(U)$ is represented by a section $s$ in $\mathcal{F}(U)$ which can be extended to the whole space $X$. □

**Lemma 2.14.** *Let* $(X, \mathcal{O}_X)$ *is a ringed space. Any injective* $\mathcal{O}_X$*-module is flasque.*

*Proof.* Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. For any open $U$, let $\mathcal{O}_U$ denote the subsheaf of $\mathcal{O}_X$,which is the extension by zero of $\mathcal{O}_X$ outside $U$. Any section $s$ in $\mathcal{I}(U)$ gives a morphism from $\mathcal{O}_U$ to $\mathcal{I}$, which by injectivity of $\mathcal{I}$ can be extended to a morphism from $\mathcal{O}_X$ to $\mathcal{I}$. This morphism gives a section of $\mathcal{I}(X)$ extending $s$. □

**Proposition 2.15.** *If $\mathcal{F}$ is a flasque sheaf, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$, i.e. flasque sheaves are acyclic for the global sections functor $\Gamma(X, \cdot)$.*

*Proof.* We embed $\mathcal{F}$ in an injective object $\mathcal{I} \in \mathrm{Ab}(X)$ and let $\mathcal{G}$ be the quotient. This gives a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

By the last Lemma above, $\mathcal{I}$ is injective and hence flasque. By the second Lemma, since both $\mathcal{F}$ and $\mathcal{I}$ are flasque so is $\mathcal{G}$. By the first Lemma

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to 0$$

is exact. Since $\mathcal{I}$ is injective in $\mathrm{Ab}(X)$, we know $H^i(X, \mathcal{I}) = 0$ for all $i > 0$. Putting these facts together, and writing the long exact sequence of cohomology, we get that $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G})$ for all $i \geq 2$. Since $\mathcal{G}$ is also flasque, we get the result by induction on $i$.

□

**Remark 2.16.** Since every sheaf has an injective resolution, and any injective sheaf is flasque, we know there are flasque resolutions for any sheaf. But more explicitly, one can use an argument as in Proposition 2.6 to embed $\mathcal{F}$ in the sheaf of discontinuous sections of $\mathcal{F}$. This is the sheaf whose sections on $U$ are given by $\Pi_{x \in U} \mathcal{F}_x$. It is easy to see that this is a flasque sheaf which contains $\mathcal{F}$ as a subsheaf. Using this one gets a *canonical flasque resolution* for any sheaf $\mathcal{F}$. Godement [G] used this flasque resolution to define cohomology groups of $\mathcal{F}$.

Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Now we can show that if one calculates the images of $\mathcal{F}$ under the derived functors of the global section functor on the category $\mathrm{Mod}(X)$, one will get the same groups as the cohomology groups of $\mathcal{F}$ (considered as a sheaf of abelian groups on $X$).

**Proposition 2.17.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X, \cdot)$ from $\mathrm{Mod}(X)$ to $\mathrm{Ab}(X)$ coincide with the cohomology functors $H^i(X, \cdot)$.*

*Proof.* To calculate the derived functors of $\Gamma(X, \cdot)$ on $\mathrm{Mod}(X)$, we use a resolution by injective $\mathcal{O}_X$-modules which we proved are flasque sheaves, and hence acyclic (for the functor $\Gamma(X, \cdot) : \mathrm{Ab}(X) \to \mathrm{Ab}$). By Proposition 2.15 this resolution will give us the usual cohomology functors. In other words, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Mod}(X) & \xrightarrow{R^i\Gamma} & \mathrm{Mod}(X) \\ {\scriptstyle \text{forget}}\downarrow & & \downarrow{\scriptstyle \text{forget}} \\ \mathrm{Ab}(X) & \xrightarrow[R^i\Gamma]{} & \mathrm{Ab}(X) \end{array}$$

where forget is the forgetful functor. □

Note that when $\mathcal{F}$ is an $\mathcal{O}_X$-module, the cohomology groups will inherit a $\Gamma(X, \mathcal{O}_X)$-module structure.

**Remark 2.18.** The above Proposition can be proved by an argument as in Remark 2.10. If one denotes by $\underline{H}^i$ the cohomology functors on $\mathrm{Mod}(X)$ calculated by injective resolutions in $\mathrm{Mod}(X)$, then one gets a universal $\delta$-functor which in degree 0 is $\Gamma$. Using the above results one can verify the universality of the $\delta$-functor $\{H^i\}$ (calculated by injective resolutions in $\mathrm{Ab}(X)$, but restricted to $\mathrm{Mod}(X)$). Indeed for every $\mathcal{F}$ in $\mathrm{Mod}(X)$, there is an embedding $u : \mathcal{F} \to \mathcal{M}$, where $\mathcal{M}$ is a flasque $\mathcal{O}_X$-module. The flasqueness of $\mathcal{M}$ implies that $H^i(X, \mathcal{M}) = 0$ for all $i > 0$ and hence $H^i(u) = 0$ for all $i > 0$. This shows that $\{H^i\}$ is a universal $\delta$-functor which in degree 0 is just $\Gamma$. Therefore $H^i = \underline{H}^i$.

We close this section by mentioning two vanishing theorems. Proofs can be found in [H].

**Theorem 2.19 (Grothendieck).** *If $X$ is a noetherian topological space of dimension $n$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and any sheaf of abelian groups $\mathcal{F}$.*

**Theorem 2.20 (Serre).** *Let $X$ be a noetherian scheme. Then $X$ is affine if and only if for every quasi-coherent sheaf $\mathcal{F}$ on $X$, we have $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

# 3 Čech cohomology

Let $X$ be a topological space and $\mathscr{U} = (U_j)_{j \in J}$ an open covering of $X$. If $\sigma$ is a finite subset of the index set $J$, put $U_\sigma := \bigcap_{j \in \sigma} U_j$. We then define

$$C^i(\mathscr{U}, \mathcal{F}) := \prod_{|\sigma| = i+1} \mathcal{F}(U_\sigma).$$

For $\alpha \in C^i(\mathscr{U}, \mathcal{F})$ and $|\sigma| = i+1$, write $\alpha(\sigma)$ for the $\sigma^{\text{th}}$ component of $\alpha$.

Now, fix arbitrarily a well-ordering $\leq$ of the index set $J$. If $\sigma$ is a finite subset of $J$ of $i+1$ elements, order its elements as $j_0 < j_1 < \ldots < j_i$ and define $\sigma_k := \sigma \setminus \{j_k\}$ for $k = 0, \ldots, i$. Now we can define a differential $d^i : C^i(\mathscr{U}, \mathcal{F}) \to C^{i+1}(\mathscr{U}, \mathcal{F})$ by

$$(d^i\alpha)(\sigma) := \sum_{k=0}^{i+1} (-1)^k \alpha(\sigma_k)|_{U_\sigma}$$

**Lemma 3.1.** $d^{i+1} \circ d^i = 0$.

*Proof.* Let $\alpha \in C^i(\mathscr{U}, \mathcal{F})$. We wish to prove that $d^2\alpha = 0 \in C^{i+2}(\mathscr{U}, \mathcal{F})$, so let $\sigma$ be a finite subset of $J$ of $i+2$ elements. Using the definition of $d$, we get

$$(d^2\alpha)(\sigma) = \sum_{k=0}^{i+2} (-1)^k \left( \sum_{l=0}^{i+1} (-1)^l \alpha(\sigma_{k,l})|_{U_{\sigma_k}} \right)\bigg|_{U_\sigma} = \sum_{k=0}^{i+2} \sum_{l=0}^{i+1} (-1)^{k+l} \alpha(\sigma_{k,l})|_{U_\sigma}.$$

If $\sigma = \{j_0, \ldots, j_{i+2}\}$ with $j_0 < \ldots < j_{i+2}$ and we fix $j_k < j_l$, then, putting $\tau := \sigma \setminus \{j_k, j_l\}$, we see that $\alpha(\tau)|_{U_k}$ appears exactly two times in the last sum, with $\tau = \sigma_{k,l-1}$ and $\tau = \sigma_{l,k}$, with opposed signs, so that they cancel each other, giving

$$(d^2\alpha)(\sigma) = 0.$$

□

So, for any sheaf $\mathcal{F}$ on $X$ and any open cover $\mathscr{U}$, we get a complex $C^\bullet(\mathscr{U}, \mathcal{F})$. We define

$$\check{H}^i(\mathscr{U}, \mathcal{F}) := H^i(C^\bullet(\mathscr{U}, \mathcal{F})),$$

the $i^{\text{th}}$ *Čech cohomology group of $\mathcal{F}$ with respect to the covering $\mathscr{U}$.*

This construction is functorial in $\mathcal{F}$, because a morphism $\mathcal{F} \to \mathcal{G}$ between sheaves on $X$ induces a morphism $C^\bullet(\mathscr{U}, \mathcal{F}) \to C^\bullet(\mathscr{U}, \mathcal{G})$ for each $i$, which is easily seen to commute with the differential. Thus we get a morphism

$$\check{H}^\bullet(\mathscr{U}, \mathcal{F}) \longrightarrow \check{H}^\bullet(\mathscr{U}, \mathcal{G}).$$

**Remark 3.2.** It is important to mention that Čech cohomology does *not* take short exact sequences of sheaves to long exact sequences of cohomology groups in general. For instance, If we take on $X$ the open covering $\mathscr{U}$ containing only the open set $X$, then we easily see that

$$\check{H}^i(\mathscr{U}, \mathcal{F}) = \begin{cases} \Gamma(X, \mathcal{F}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Thus the existence of long exact sequence in Čech cohomology would imply that the global sections functor $\Gamma(X, \cdot\,)$ is exact, which is certainly not always the case.

# 4 Čech vs. derived functor cohomology

We will now compare the Čech cohomology and derived functor cohomology of a sheaf $\mathcal{F}$ on a topological space $X$. We will see that in some cases we are able to conclude that these two cohomologies coincide.

We will begin by establishing that they always coincide in degree 0.

**Lemma 4.1.** *For any sheaf $\mathcal{F}$ on $X$ and open covering $\mathscr{U}$ of $X$, there is a natural isomorphism $\check{H}^0(\mathscr{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.*

*Proof.* Remark that $\check{H}^0(\mathscr{U}, \mathcal{F}) = \operatorname{Ker} d^0$ where $d^0$ is the coboundary map $C^0(\mathscr{U}, \mathcal{F}) \to C^1(\mathscr{U}, \mathcal{F})$. For any $\alpha = (\alpha_j)_{j \in J} \in C^0(\mathscr{U}, \mathcal{F})$, we find that

$$(d^0\alpha)_{jj'} = \alpha_j|_{U_j \cap U_{j'}} - \alpha_{j'}|_{U_j \cap U_{j'}},$$

so that $d^0\alpha = 0$ if and only if $\alpha_j$ and $\alpha_{j'}$ coincide on $U_j \cap U_{j'}$ for all pair of indices $j, j' \in J$.

The fact that $\mathcal{F}$ is a sheaf implies that the map

$$\Gamma(X, \mathcal{F}) \longrightarrow \operatorname{Ker} d^0$$

sending a global section $\alpha$ of $\mathcal{F}$ to $(\alpha|_{U_j})_{j \in J}$ is bijective.

□

In order to compare Čech cohomology with derived functor cohomology, we will need to consider first a sheafified version of the Čech complex.

For every open set $U$ of $X$, let $\iota_U : U \hookrightarrow X$ denote the inclusion of $U$ in $X$. Define

$$\mathcal{C}^i(\mathscr{U}, \mathcal{F}) := \prod_{|\sigma|=i+1} (\iota_{U_\sigma})_*(\mathcal{F}|_{U_\sigma}) \tag{1}$$

which is a sheaf on $X$, together with differentials $d^i$ just as above. Then, by definition, we have

$$\Gamma(X, \mathcal{C}^\bullet(\mathscr{U}, \mathcal{F})) = C^\bullet(\mathscr{U}, \mathcal{F}).$$

Let $\varepsilon : \mathcal{F} \to \mathcal{C}^0(\mathscr{U}, \mathcal{F})$ be the product over $j \in J$ of the canonical maps

$$\mathcal{F} \longrightarrow (\iota_{U_j})_*(\iota_{U_j})^*(\mathcal{F}) = (\iota_{U_j})_*(\mathcal{F}|_{U_j}).$$

**Proposition 4.2.** $0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^\bullet(\mathscr{U}, \mathcal{F})$ *is a resolution of* $\mathcal{F}$*.*

*Proof.* The facts that $\varepsilon$ is injective and that $\operatorname{Im} f = \operatorname{Ker} d^0$ follow directly from $\mathcal{F}$ being a sheaf.

It remains to be shown that the proposed sequence is exact in degrees $i > 0$. For this, it suffices to work at the level of stalks.

Let us consider

$$\mathcal{C}^i(\mathscr{U},\mathcal{F})_x \xrightarrow{d_x^i} \mathcal{C}^{i+1}(\mathscr{U},\mathcal{F})_x \xrightarrow{d_x^{i+1}} \mathcal{C}^{i+2}(\mathscr{U},\mathcal{F})_x$$

and prove that $\operatorname{Ker} d_x^{i+1} \subseteq \operatorname{Im} d_x^i$ (we already know that the reverse inclusion holds).

So take a germ $\alpha_x \in \operatorname{Ker} d_x^{i+1}$. It can be represented by an element $\alpha \in \mathcal{C}^{i+1}(\mathscr{U},\mathcal{F})(V)$, where $V$ is an open set which can be chosen to lie entirely inside one of the open sets $U_j$ of the covering $\mathscr{U}$.

If $\sigma$ is any finite subset of $J$, since $V \subseteq U_j$, we remark that

$$V \cap U_\sigma = V \cap U_{\sigma \cup \{j\}}.$$

Thus, we may define an element $\beta \in \mathcal{C}^i(\mathscr{U},\mathcal{F})(V)$ by the formula

$$\beta(\sigma) := \alpha(\sigma \cup \{j\})$$

(where we understand that $\alpha(\sigma \cup \{j\}) = 0$ if $j$ is already contained in $\sigma$).

Then we can check, using the fact that $d_x^{i+1}(\alpha_x) = 0$, that $d_x^i(\beta_x) = \alpha_x$, thus concluding the proof.

□

If follows from Lemma 1.6 that if $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ is any injective resolution of $\mathcal{F}$, then the identity map on $\mathcal{F}$ lifts to a unique (up to homotopy) morphism of complexes

$$\mathcal{C}^\bullet(\mathscr{U},\mathcal{F}) \to \mathcal{I}^\bullet$$

which in turn induces a canonical morphism

$$\check{H}^\bullet(\mathscr{U},\mathcal{F}) \longrightarrow H^\bullet(X,\mathcal{F})$$

from Čech cohomology to sheaf cohomology, which enables us to compare the two cohomologies.

When we will now state some results stating sufficient conditions for these canonical morphisms to actually be isomorphisms, thus enabling us to calculate sheaf cohomology via Čech cohomology.

**Lemma 4.3.** *If a sheaf $\mathcal{F}$ on $X$ is flasque, then $\check{H}^i(\mathscr{U},\mathcal{F}) = 0$ for $i > 0$ (thus in this case, using the preceding lemma, Čech and derived functor cohomology coincide).*

*Proof.* If $\mathcal{F}$ is flasque, then each sheaf $\mathcal{C}^i(\mathscr{U},\mathcal{F})$ is flasque according to 1, since "flasqueness" is preserved under restriction, applying $\iota_*$ and taking products (this is easy to check).

But then it means that $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathscr{U},\mathcal{F})$ is a flasque resolution of $\mathcal{F}$, so it can used to compute derived functor cohomology by Proposition 2.15. Thus, we get

$$\check{H}^i(\mathscr{U},\mathcal{F}) \cong H^i(X,\mathcal{F}) = 0$$

for $i > 0$ since $\mathcal{F}$ is flasque (Proposition 2.15).

□

**Theorem 4.4 (Leray).** *Let $X$ be a topological space, $\mathcal{F}$ a sheaf of abelian groups on $X$, and $\mathscr{U}$ an open cover of $X$. Assume that for any finite intersection $V := U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets in the covering $\mathscr{U}$ and $i > 0$, we have $H^i(V, \mathcal{F}|_V) = 0$. Then the natural maps*

$$\check{H}^i(\mathscr{U}, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$$

*are isomorphisms for all $i$.*

*Proof.* We proceed by induction on the degree $i$. For $i = 0$ we already know that the result is true thanks to Lemma 4.1.

Now, embed $\mathcal{F}$ in a flasque sheaf $\mathcal{G}$ and let $\mathcal{H}$ be the quotient of $\mathcal{F}$ by $\mathcal{G}$, so that we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

For each finite set $\sigma \subset J$, the sequence

$$0 \longrightarrow \mathcal{F}(U_\sigma) \longrightarrow \mathcal{G}(U_\sigma) \longrightarrow \mathcal{H}(U_\sigma) \longrightarrow 0 \tag{2}$$

is exact, using the long exact sequence for $H^\bullet(U_\sigma, \cdot\,)$ and the hypothesis that $H^1(U_\sigma, \mathcal{F}) = 0$.

Taking products, we find that the corresponding short sequence of Čech complexes

$$0 \longrightarrow C^\bullet(\mathscr{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathscr{U}, \mathcal{G}) \longrightarrow C^\bullet(\mathscr{U}, \mathcal{H}) \longrightarrow 0$$

is exact, so that we get a long exact sequence in Čech cohomology. The fragment

$$\ldots \to \check{H}^i(\mathscr{U}, \mathcal{G}) \to \check{H}^i(\mathscr{U}, \mathcal{H}) \to \check{H}^{i+1}(\mathscr{U}, \mathcal{F}) \to \check{H}^{i+1}(\mathscr{U}, \mathcal{G}) \to \ldots$$

of this long exact sequence, together with the fact that $\check{H}^i(\mathscr{U}, \mathcal{G}) = 0$ for all $i > 0$ ($\mathcal{G}$ is flasque), tells us that

$$\check{H}^i(\mathscr{U}, \mathcal{H}) \cong \check{H}^{i+1}(\mathscr{U}, \mathcal{F})$$

for each $i > 0$.

Writing the long exact sequence of the derived functor cohomology corresponding to the above short exact sequence, and using the hypothesis, we get $H^i(U_\sigma, \mathcal{H}) = 0$ for each finite intersection $U_\sigma$ and $i > 0$. Thus, the induction hypothesis applies to $\mathcal{H}$, so that the following commutative diagram tells us that the natural maps for $\mathcal{F}$ are isomorphisms for $2 \leq i \leq n$.

$$\begin{array}{ccc} \check{H}^i(\mathscr{U}, \mathcal{H}) & \xrightarrow{\cong} & \check{H}^{i+1}(\mathscr{U}, \mathcal{F}) \\ \cong\downarrow & & \downarrow \\ H^i(X, \mathcal{H}) & \xrightarrow{\cong} & H^{i+1}(X, \mathcal{F}) \end{array}$$

It remains to be shown that the natural map is an isomorphism for $i = 1$. But this follows easily by chasing in the following diagram obtained by comparing the two long exact sequences (or by the Five Lemma, if you wish).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & \Gamma(X,\mathcal{H}) & \longrightarrow & \check{H}^1(X,\mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & \Gamma(X,\mathcal{H}) & \longrightarrow & H^1(X,\mathcal{F}) & \longrightarrow & 0 \end{array}$$

□

**Corollary 4.5.** *Let $X$ be a noetherian separated scheme, $\mathscr{U}$ an open affine cover of $X$ and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Then for all $i > 0$, the natural maps*

$$\check{H}^i(\mathscr{U},\mathcal{F}) \longrightarrow H^i(X,\mathcal{F})$$

*are isomorphisms.*

*Proof.* Since $X$ is a noetherian separated scheme, finite intersection of open sets in the covering $\mathscr{U}$ are affine, so $\mathcal{F}$, being quasi-coherent, has no cohomology along them (Theorem 2.20). Thus the previous theorem applies.

□

If $\mathscr{V}$ is a refinement of $\mathscr{U}$, we get a *refinement map*

$$C^\bullet(\mathscr{U},\mathcal{F}) \longrightarrow C^\bullet(\mathscr{V},\mathcal{F})$$

which induces a map

$$\check{H}^\bullet(\mathscr{U},\mathcal{F}) \longrightarrow \check{H}^\bullet(\mathscr{V},\mathcal{F})$$

on Čech cohomology.

The coverings of $X$ form a partially ordered set under refinement, so we can define

$$\check{H}^\bullet(X,\mathcal{F}) := \varinjlim_{\mathscr{U}} \check{H}^\bullet(\mathscr{U},\mathcal{F}).$$

The refinement maps are compatible with the canonical maps to derived functor sheaf cohomology, so that, by the universal property of the direct limit, we get a canonical map

$$\check{H}^\bullet(X,\mathcal{F}) \longrightarrow H^\bullet(X,\mathcal{F}).$$

The following two comparison theorems may be found in [G].

**Theorem 4.6 (Cartan).** *Let $X$ be a topological space, $\mathcal{F}$ a sheaf of abelian groups on $X$ and $\mathscr{U}$ an open covering of $X$ closed under the operation of taking* *finite intersections and containing arbitrarily small open sets. Suppose furthermore that we have* $\check{H}^i(U, \mathcal{F}) = 0$ *for all* $U \in \mathscr{U}$ *and* $i > 0$. *Then the natural morphisms*

$$\check{H}^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$$

*are isomorphisms for all* $i$.

**Theorem 4.7.** *If* $X$ *is a topological space and* $\mathcal{F}$ *a sheaf on* $X$, *then the canonical morphism*

$$\check{H}^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$$

*is bijective if* $i = 0$ *or* $1$ *and injective if* $i = 2$.

# 5 Examples

**Example 5.1.** In this example we compute explicitly $H^\bullet(\mathbb{P}^1, \Omega)$, where $\Omega$ is the sheaf of differentials on $\mathbb{P}^1$. For this, by Leray's Theorem, we can use Čech cohomology with respect to the open covering $\mathscr{U}$ of $\mathbb{P}^1$ consisting of the two affine open sets $U := \{x_1 \neq 0\}$ and $V := \{x_0 \neq 0\}$ where $x_0$ and $x_1$ are the two homogeneous coordinates on $\mathbb{P}^1$. Use the affine coordinate $x := x_0/x_1$ on $U$ and $y := x_1/x_0$ on $V$, so that we have $y = 1/x$ on $U \cap V$.

The Čech complex is

$$0 \longrightarrow C^0(\mathscr{U}, \Omega) \xrightarrow{d} C^1(\mathscr{U}, \Omega) \longrightarrow 0$$

where

$$C^0(\mathscr{U}, \Omega) = \Gamma(U, \Omega) \times \Gamma(V, \Omega) = k[x]dx \times k[y]dy,$$
$$C^1(\mathscr{U}, \Omega) = \Gamma(U \cap V, \Omega) = k[x, x^{-1}]dx,$$

and the differential $d$ is given by

$$d(f(x)dx, g(y)dy) = \left( f(x) + \frac{1}{x^2} g\left(\frac{1}{x}\right)\right) dx.$$

This allows one to compute explicitly that $H^0(\mathbb{P}^1, \Omega) = 0$ and that $H^1(\mathbb{P}^1, \Omega)$ is a one-dimensional vector space over $k$ spanned by the image of $x^{-1}dx$.

**Example 5.2.** Let $X$ be a non-singular complex algebraic variety. Thus, $X$ may be regarded as a complex manifold if endowed with the usual *complex* topology. With respect to this topology, we have

$$H^\bullet_{\text{sing}}(X, \mathbb{C}) \cong H^\bullet(X, \underline{\mathbb{C}}) \cong H^\bullet_{\text{DR}}(X, \mathbb{C}).$$

The first isomorphism can be shown by identifying $H^\bullet_{\text{sing}}(X, \mathbb{C})$ with the simplicial cohomology of $X$ with respect to a triangulation of $X$. Any triangulation of $X$ gives rise to an open covering of $X$ in which for every vertex $v$, there is a corresponding open $St(v)$, the star of $v$, which is the interior of the union of all simplices which have $v$ as a vertex. It is a rather formal consequence of the definition of Čech cohomology that the simplicial cohomology of $X$ with respect to the given traingulation (and with $\mathbb{C}$-coefficients) is the same as the Čech cohomology of $\underline{\mathbb{C}}$ with respect to the corresponding open covering. (We have an isomorphism even on the level of cochains, see [GH]). By refining the triangulation if necessary, and incorporating Proposition 4.4, we then see that these Čech cohomology groups calculate $H^\bullet(X, \underline{\mathbb{C}})$.

For the second isomorphism, let $\mathscr{A}^i$ be the sheaf of smooth complex-valued $i$-forms on $X$. Then the complex

$$0 \to \underline{\mathbb{C}} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \mathscr{A}^2 \to \dots$$

can be shown to be an acyclic resolution of $\underline{\mathbb{C}}$. Assuming this, we may compute $H^\bullet(X, \underline{\mathbb{C}})$ by finding the homology of the cochain complex formed by taking the global sections of $\mathscr{A}^\bullet$, which is nothing but the usual de Rham complex on $X$. This proves the first isomorphism.

To show that the above complex is a resolution we use the Poincaré Lemma which states that the higher de Rham cohomology groups of the open unit disc in $\mathbb{C}^n$ all vanish. This will show that the above complex is exact on the level of stalks and hence is exact.

To prove the acyclicity of the sheaves $\mathscr{A}^i$, we have to use the notion of *soft* sheaves. These are the exact analogues of flasque sheaves, except that we replace open subsets by closed subsets in the definition. In the same exact way as in Proposition 2.15, we can show that soft sheaves are acyclic for the functor of global sections. Now it is enough to show that each $\mathscr{A}^i$ is soft. Let $K$ be a closed subset of $X$, and $s$ a section of $\mathscr{A}^i$ on $K$. There is a locally finite cover of $X$, say $\{U_j\}$, and for each $j$, a section $s_j$ of $\mathcal{A}^i$ on $U_j$ such that $s_j$ coincides with $s$ on $K \cap U_j$. Let $\{f_j\}$ be a *partition of unity* with respect to this covering (which is a collection of global smooth functions $f_j$, such that the support of $f_j$ lies in $U_j$, and that $\sum f_j = 1$). Then for each $j$, the section $f_j s_j$ extends to $X$, and $\sum f_j s_j$ is a global section of $\mathscr{A}^i$ which extends $s$. (Sheaves for which there exists a similar notion of partition of unity are called *fine*; we've just essentially shown that fine sheaves are soft, thus acyclic).

We see that using the complex topology of $X$, the sheaf cohomology of the constant sheaf $\underline{\mathbb{C}}$ contains a lot of interesting topological information about $X$.

However, if we wish to calculate the cohomology of $\underline{\mathbb{C}}$ on $X$ with respect to the *Zariski* topology of $X$, we get

$$H^i_{\text{Zar}}(X, \underline{\mathbb{C}}) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

This is because the constant sheaf is flasque for the Zariski topology on $X$ (by the irreducibility of $X$).

We see that the Zariski topology on an algebraic variety is too coarse a topology in the above example. In the coming weeks we will discuss other topologies, still "algebraic" in nature, such as the *étale topology*, which are fine enough to produce interesting cohomology theories.

# References


[G] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1964.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.

[W] C. A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics 38, Cambridge University Press, Cambridge, 1994.