Some remarks on Frobenius and Lefschetz in étale cohomology

Gabriel Chênevert

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In this lecture I will discuss some more or less related issues revolving around the main idea relating (étale) cohomology and the zeta function of a scheme X over $\mathbb{F}_p$, which is: via the Lefschetz trace formula, studying the zeta function amounts to studying the representation of the Frobenius morphism on cohomology.

I will start to try to clarify a bit *which* Frobenius morphism we're interested in, and then we'll look explicitely at some examples of 0-dimensional schemes (for which the Lefschetz trace formula takes a particularly simple form!).

1 The absolute Frobenius morphism

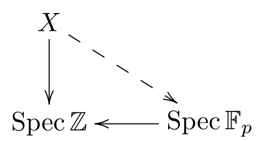
Let's start by studying the Frobenius morphism in some generality. The first thing to do is to restrict ourselves to the subcategory of schemes which *do* admit a Frobenius morphism.

Throughout, p will be understood to stand for a fixed prime number.

Definition 1.1. A scheme X is said to be of *characteristic* p if $p\,\mathcal{O}_X = 0$.

Of course, a given scheme cannot have two distinct prime characteristics unless its structure sheaf is 0, i.e. unless it is $\operatorname{Spec} 0$, the initial object in the category of schemes.

Remark that saying that X is of characteristic p amounts to saying that the (unique!) morphism $X \to \operatorname{Spec}\mathbb{Z}$ factors through $\operatorname{Spec}\mathbb{F}_p$.



Thus, saying that a scheme is of characteristic p is the same thing as saying that it may be viewed as a scheme over $\operatorname{Spec}\mathbb{F}_p$.

Definition 1.2. Let X be a scheme of characteristic p. We define the *(absolute) Frobenius endomorphism* of X

$$\mathrm{Fr}_X : X \longrightarrow X$$

(we will drop the X from the notation Fr_X when this causes no ambiguity) as the morphism which is the identity on $|X|$ and the p^{th}-power map on $\mathcal{O}_X$ (this really defines a morphism of sheaves of rings since $p\,\mathcal{O}_X = 0$).

Example 1.3. If $X = \operatorname{Spec} A$ is affine, the Frobenius endomorphism of X arises from the Frobenius endomorphism $a \mapsto a^p$ of A (check that this really induces the identity on $|\operatorname{Spec} A|$).

This endomorphism behaves functorially, in the sense that for each morphism $Y \to X$ (which automatically makes Y into a scheme over $\mathbb{F}_p$ provided X is one), the following diagram commutes.

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{Fr}} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow[\mathrm{Fr}]{} & X \end{array}$$

In particular, the Frobenius endomorphism of X is an endomorphism of X as a scheme over $\mathbb{F}_p$ (since the Frobenius morphism of $\operatorname{Spec} \mathbb{F}_p$ is the identity).

Remark 1.4. Similarly, if X is a scheme over $\mathbb{F}_q$, where $q = p^n$, then it may also be viewed as a scheme over $\mathbb{F}_p$, so it has a Frobenius morphism Fr, and Fr^n (which raises fonctions to the q^{th} power) is a morphism of X as a scheme over $\mathbb{F}_q$. When it is clear that X is to be considered as a scheme over $\mathbb{F}_q$, we may want to call Fr^n *the* Frobenius morphism of X and denote it by Fr.

2 Functoriality of cohomology

Before discussing the action of the Frobenius morphism of X on its étale cohomology, it might be useful to recall some general functoriality properties of cohomology of (étale) sheaves.

Let T_X be a Grothendieck topology with final object X and denote by $Sh(X)$ the category of sheaves on T_X with values in some fixed abelian category (i.e. of contravariant functors from T_X into this abelian category satisfying the sheaf axiom). If $Sh(X)$ has enough injectives, then for each $\mathcal{F} \in Sh(X)$, the cohomology groups

$$H^\bullet(X, \mathcal{F})$$

are defined in the usual way: first apply the global sections functor to an injective resolution of $\mathcal{F}$, then take the cohomology of the resulting complex. We may roughly think of the cohomology as a bifunctor, covariant in the second variable and contravariant in the first. This is made precise by the following propositions.

Proposition 2.1. *For $\mathcal{F}$, $\mathcal{G} \in Sh(X)$, a morphism $\mathcal{F} \to \mathcal{G}$ induces a natural map*

$$H^\bullet(X, \mathcal{F}) \longrightarrow H^\bullet(X, \mathcal{G}).$$

Proof. Choose injective resolutions of $\mathcal{F}$ and $\mathcal{G}$. By the lifting property of injective resolutions, the morphism $\mathcal{F} \to \mathcal{G}$ induces a morphism between them, which is unique up to homotopy.

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}^\bullet \\ & & \downarrow & & \vdots \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{J}^\bullet \end{array}$$

Applying the global sections functor to $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ and taking cohomology, we get the required map. $\square$

Recall that if $f : T_X \to T_Y$ is a continuous map between two Grothendieck topologies with final objects (i.e. a functor $f^{-1} : T_Y \to T_X$ preserving fibered products, coverings and final objects), then to any sheaf $\mathcal{F}$ on T_X we can associate a sheaf $f_*\mathcal{F}$ on T_Y, defined by

$$f_*\mathcal{F} := \mathcal{F} \circ f^{-1}.$$

This defines a *direct image* functor

$$f_* : Sh(X) \longrightarrow Sh(Y).$$

It is also possible to define an *inverse image* functor

$$f^* : Sh(Y) \longrightarrow Sh(X)$$

provided we have a good *sheafification* process at hand (see [M] for details). If this is the case, for $\mathcal{G} \in Sh(Y)$ we can define $f^*\mathcal{G}$ by mimicking the usual definition, i.e. by sheafifying the presheaf defined by

$$U \mapsto \varinjlim \mathcal{G}(V)$$

where the direct limit is taken over all open sets $V \in T_Y$ such that there exists an "inclusion" $U \to f^{-1}(V)$.

Remark 2.2. If $f : T_X \to T_Y$ is an homeomorphism of Grothendieck topologies with inverse g, then the functor f^* coincides with g_*. Indeed, in this case, the directed set over which we take the direct limit has a maximal element, namely $g^{-1}(U)$, and the above presheaf is just the sheaf $g_*\mathcal{F}$.

Example 2.3. Recall that any morphism of schemes $f : X \to Y$ induces a continuous function $f : X_{\text{ét}} \to Y_{\text{ét}}$, i.e. a functor $f^{-1} : Y_{\text{ét}} \to X_{\text{ét}}$, obtained by base-changing the étale open sets of Y to étale open sets of X (remember that the property of being étale is preserved under base change).

$$\begin{array}{ccc} X \times_Y U & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Thus, in this case for $\mathcal{F} \in Sh(X_{\text{ét}})$ we have

$$f_*\mathcal{F}(U) = \mathcal{F}(X \times_Y U)$$

and for $\mathcal{G} \in Sh(Y)$, $f^*\mathcal{G}$ is the sheafification (again see [M]) of the presheaf

$$U \mapsto \varinjlim \mathcal{G}(V)$$

where the direct limit is taken over all commutative diagrams of the following form.

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Example 2.4. If X is a scheme and $\iota : X_{\text{ét}} \to X_{\text{Zar}}$ the natural map between its étale and Zariski sites, then the direct image functor

$$\iota_* : Sh(X_{\text{ét}}) \longrightarrow Sh(X_{\text{Zar}})$$

is the functor which forgets the values of a sheaf on non-Zariski open sets, and the inverse image

$$\iota^* : Sh(X_{\text{Zar}}) \longrightarrow Sh(Y_{\text{ét}})$$

is just the *étalization functor* $\mathcal{F} \mapsto \mathcal{F}^{\text{ét}}$ which we already discussed.

Proposition 2.5. *If $f : X \to Y$ is a morphism of schemes, then the inverse image functors of f commute with étalization, i.e. the following diagram of functors is commutative.*

$$\begin{array}{ccc} Sh(Y_{\text{Zar}}) & \xrightarrow{\text{ét}} & Sh(Y_{\text{ét}}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ Sh(X_{\text{Zar}}) & \xrightarrow[\text{ét}]{} & Sh(X_{\text{ét}}) \end{array}$$

Proof. This diagram arises from the following commutative diagram of continuous maps.

$$\begin{array}{ccc} Y_{\text{Zar}} & \xleftarrow{\iota} & Y_{\text{ét}} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ X_{\text{Zar}} & \xleftarrow[\iota]{} & X_{\text{ét}} \end{array}$$

□

Since étale constant sheaves are just étalizations of Zariski constant sheaves and the inverse image of a Zariski constant sheaf is constant, we get the following corollary.

Corollary 2.6. *Inverse images of étale constant sheaves are étale constant sheaves.*

If $f : T_X \to T_Y$ is a continuous map of Grothendieck topologies, the functors f^* and f_* between $Sh(X)$ and $Sh(Y)$ are *adjoint*, meaning that for all $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$ we get a natural isomorphism

$$\text{Hom}_{Sh(X)}(f^*\mathcal{G}, \mathcal{F}) \cong \text{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which gives us natural *adjunction morphims*

$$f^* f_* \mathcal{F} \longrightarrow \mathcal{F} \qquad \text{and} \qquad \mathcal{G} \longrightarrow f_* f^* \mathcal{G}$$

corresponding to $1_{f_*\mathcal{F}}$ and $1_{f^*\mathcal{G}}$ respectively.

Moreover, in general f^* is exact, which by adjointness implies that f_* is left exact.

Proposition 2.7. *Suppose $Sh(X)$ and $Sh(Y)$ have enough injective and that we have a continous map $f : T_X \to T_Y$ which induces such a f^* functor. Then, for any sheaf $\mathcal{F}$ on Y, we get a natural map*

$$f^\bullet : H^\bullet(Y, \mathcal{F}) \longrightarrow H^\bullet(X, f^*\mathcal{F}).$$

Proof. Choose a an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Applying the functor f^*, which is exact, we obtain a resolution (most probably not injective anymore) of $f^*\mathcal{F}$. It has a map (unique up to homotopy) into any chosen injective resolution of $f^*\mathcal{F}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*\mathcal{F} & \longrightarrow & f^*\mathcal{I}^\bullet & & \text{(resolution)} \\ & & \| & & \downarrow & & \\ 0 & \longrightarrow & f^*\mathcal{F} & \longrightarrow & \mathcal{J}^\bullet & & \text{(injective resolution)} \end{array}$$

The composed map

$$\Gamma(Y, \mathcal{I}^\bullet) \longrightarrow \Gamma(X, f^*\mathcal{I}^\bullet) \longrightarrow \Gamma(X, \mathcal{J}^\bullet)$$

where the first map is given by the definition of f^* and the second comes from $f^*\mathcal{I}^\bullet \to \mathcal{J}^\bullet$, gives rise to the desired map in cohomology.

□

Remark 2.8. Similarly, if $\mathcal{F}$ is a sheaf on X there is a map

$$H^\bullet(Y, f_*\mathcal{F}) \longrightarrow H^\bullet(X, \mathcal{F})$$

obtained as the composition

$$H^\bullet(Y, f^*\mathcal{F}) \longrightarrow H^\bullet(X, f^*f_*\mathcal{F}) \longrightarrow H^\bullet(X, \mathcal{F})$$

where the second map comes from the adjunction map

$$f^*f_*\mathcal{F} \longrightarrow \mathcal{F}.$$

In particular, if $f : X \to Y$ is a morphism of schemes and Λ a constant (étale) sheaf, since $f^*\Lambda = \Lambda$ we get a map

$$f^\bullet : H^\bullet_{\text{ét}}(Y, \Lambda) \longrightarrow H^\bullet_{\text{ét}}(X, \Lambda)$$

in cohomology. Applying this to the constant sheaves $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, we obtain a map

$$f^\bullet : H^\bullet(Y, \mathbb{Q}_\ell) \longrightarrow H^\bullet(X, \mathbb{Q}_\ell)$$

at the level of ℓ-adic cohomology.

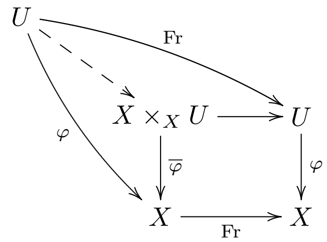
3 The action of Frobenius on cohomology

Now let's look at what the Frobenius map $\mathrm{Fr} : X \to X$ induces at the level of the ℓ-adic cohomology of a scheme of characteristic $p > 0$. To understand the continuous map of the étale site of X that it induces, we need to know how base-changing by Frobenius affects étale morphisms.

Lemma 3.1. *For any étale morphism $\varphi : U \to X$, the following diagram is Cartesian, i.e. we may identify the fibered product $X \times_X U$ with U.*

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{Fr}} & U \\ \varphi \downarrow & & \downarrow \varphi \\ X & \xrightarrow[\mathrm{Fr}]{} & X \end{array}$$

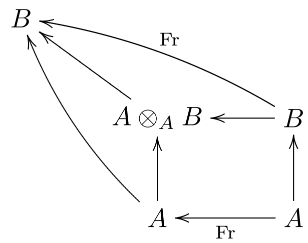
Proof. We already know that the given diagram commutes, thus, by the universal property of the fibered product, there exists a unique diagonal arrow in the following diagram (this is what is called $\mathrm{Fr}_{U/X}$, the *Frobenius morphism of* U *relative to* X in [H]).



We have to show that this is an isomorphism under the assumption that φ is étale. By the local description of étale maps, we can reduce the problem to the case where $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} B$ with

$$B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n), \quad \det\left(\frac{\partial f_i}{\partial x_j}\right) \in B^\times.$$

In the corresponding diagram of rings



the diagonal map is given by $a \otimes b \mapsto ab^p$. If it is possible (is it?) to show directly that it is an isomorphism, then this would complete the proof.

Else we have to go through an indirect argument as in [H, prop. 2]. First, we know that $\overline{\varphi}$ is étale because it is obtained by base-changing the étale map φ. But since $\varphi = \overline{\varphi} \circ \mathrm{Fr}_{U/X}$ and that both φ and $\overline{\varphi}$ are étale, we conclude that $\mathrm{Fr}_{U/X}$ is étale. Moreover, from the fact that both Frobenius morphisms Fr_X and Fr_U are universally bijective (i.e. that they remain bijective under base change), it is possible to conclude that $\mathrm{Fr}_{U/X}$ is also universally bijective (see [GD, I 3.6, 3.7], where universally injective is called *radiciel*). Since $\mathrm{Fr}_{U/X}$ is both étale and universally bijective, it is an isomorphism.

□

What this lemma asserts is that the continuous function

$$\mathrm{Fr} : X_{\text{ét}} \longrightarrow X_{\text{ét}}$$

induced by the absolute Frobenius morphism of X is (naturally equivalent to) the identity, which gives us the following corollary.

Corollary 3.2. *If* X *is a scheme of characteristic* p*, then for any étale sheaf* $\mathcal{F}$ *on* X*, both* $\mathrm{Fr}^*\mathcal{F}$ *and* $\mathrm{Fr}_*\mathcal{F}$ *are canonically isomorphic to* $\mathcal{F}$*, and the map*

$$\mathrm{Fr}^\bullet : H^\bullet_{\text{ét}}(X, \mathcal{F}) \longrightarrow H^\bullet_{\text{ét}}(X, \mathcal{F})$$

is the identity.

4 The 4 Frobeniuses on $\overline{X}$

In this section, in order to simplify the notation, let k denote the field $\mathbb{F}_p$ of p elements and fix an algebraic closure $k \hookrightarrow \overline{k}$ of k. If X is a scheme over k, we can extend the scalars to get a scheme $\overline{X}$ over $\overline{k}$.

$$\overline{X} := \operatorname{Spec} \overline{k} \times_k X$$

$$\begin{array}{ccc} \overline{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} k \end{array}$$

It appears that on $\overline{X}$, there coexists four different Frobenius morphisms:

1. the *absolute Frobenius morphism*
$$\mathrm{Fr} : \overline{X} \longrightarrow \overline{X}$$
which we discussed in the previous sections;

2. the *relative Frobenius morphism*
$$\mathrm{Fr}_r := 1_{\operatorname{Spec} \overline{k}} \times_k \mathrm{Fr}_X$$
obtained by base change of the Frobenius morphism of X (it is also sometimes called the *$\overline{k}$-linear Frobenius morphism of* X, because it is a morphism of $\overline{X}$ as a $\overline{k}$-scheme);

3. the *arithmetical Frobenius morphism*
$$\mathrm{Fr}_a := \mathrm{Fr}_{\operatorname{Spec} \overline{k}} \times_k 1_X$$
obtained by base-changing the Frobenius morphism of $\operatorname{Spec} \overline{k}$;

4. the *geometrical Frobenius morphism*
$$\mathrm{Fr}_g := \mathrm{Fr}_{\operatorname{Spec} \overline{k}}^{-1} \times_k 1_X$$
which is the inverse of the arithmetical Frobenius morphism.

Example 4.1. In the case where $X = \operatorname{Spec} A$ and $A = k[t_1, \ldots, t_n]$ is a finitely generated k-algebra, then
$$\overline{X} = \operatorname{Spec}(\overline{k} \otimes_k A) = \operatorname{Spec} \overline{k}[t_1, \ldots, t_n].$$
Then on an element of $\overline{k}[t_1, \ldots, t_n]$, which is a polynomial in the t_i's with coefficients in $\overline{k}$:

1. the relative Frobenius Fr_r corresponds to raising the t_i's to the p;

2. the arithmetical Frobenius Fr_a corresponds to raising the coefficents to the p;

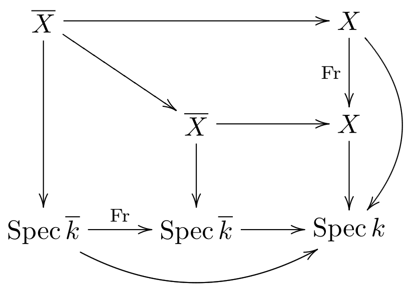
3. the geometrical Frobenius Fr_g corresponds to taking p^{th} roots of the coefficients;

4. and the absolute Frobenius Fr corresponds to raising both the t_i's and the coefficients to the p^{th} power (which is the same thing as raising our element to the p).

Let's look at how these morphisms are related. By functoriality of base change, we have

$$\begin{aligned}\mathrm{Fr}_r \circ \mathrm{Fr}_a &= (1 \times_k \mathrm{Fr}) \circ (\mathrm{Fr} \times_k 1) \\ &= \mathrm{Fr} \times_k \mathrm{Fr} \\ &= (\mathrm{Fr} \times_k 1) \circ (1 \times_k \mathrm{Fr}) \\ &= \mathrm{Fr}_a \circ \mathrm{Fr}_r.\end{aligned}$$

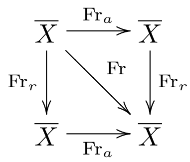
But, by definition, $\mathrm{Fr} \times_k \mathrm{Fr}$ is the unique diagonal arrow which makes the following diagram commutative.



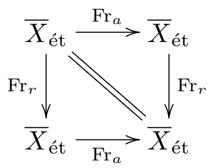
Since the absolute Frobenius Fr of X also makes it commutative, by unicity of $\mathrm{Fr} \times_k \mathrm{Fr}$ we get that

$$\mathrm{Fr}_r \circ \mathrm{Fr}_a = \mathrm{Fr} = \mathrm{Fr}_a \circ \mathrm{Fr}_r,$$

i.e. that the absolute Frobenius morphism of $\overline{X}$ is the composition of its relative and arithmetical Frobenius morphisms.



Thus the continuous maps they induce on the étale site of $\overline{X}$ are inverse one to the other (because Fr induces the identity).



This means that Fr_r is an homeomorphism of $\overline{X}_{\text{ét}}$ with inverse Fr_a, i.e. the relative Frobenius Fr_r and the geometrical Frobenius $\mathrm{Fr}_g = \mathrm{Fr}_a^{-1}$ induce the *same* continuous function

$$F : \overline{X}_{\text{ét}} \longrightarrow \overline{X}_{\text{ét}}$$

which we may call the *geometrical Frobenius correspondence* on $\overline{X}_{\text{ét}}$.

The following proposition follows.

Proposition 4.2. *For any étale sheaf $\mathcal{F}$ on $\overline{X}$, the relative and geometrical Frobenius morphisms induce the same map in cohomology, i.e.*

$$\mathrm{Fr}_r^\bullet = \mathrm{Fr}_g^\bullet : H^\bullet_{\text{ét}}(\overline{X}, F^*\mathcal{F}) \longrightarrow H^\bullet_{\text{ét}}(\overline{X}, \mathcal{F}).$$

In particular, they act the same

$$\mathrm{Fr}_r^\bullet = \mathrm{Fr}_g^\bullet : H^\bullet(\overline{X}, \mathbb{Q}_\ell) \longrightarrow H^\bullet(\overline{X}, \mathbb{Q}_\ell)$$

in ℓ-adic cohomology.

If X is of finite type, we know that the fixed points of Fr_r^n, the n^{th} iterate of the relative Frobenius morphism, is the set of points in $\overline{X}$ with coordinates x_i such that

$$x_i^{p^n} = x_i$$

i.e. the set $X(\mathbb{F}_{p^n})$ of $\mathbb{F}_{p^n}$-rational points of X. If X is smooth and proper over k, we can compute

$$N_n(X) := \#X(\mathbb{F}_{p^n})$$

either by applying the Lefschetz trace formula to Fr_r^n *or* to Fr_g^n on $H^\bullet(\overline{X}, \mathbb{Q}_\ell)$, $\ell \neq p$, since they induce the same thing on cohomology (hence the name "geometrical Frobenius").

Remark 4.3. Even if we worked with a Weil cohomology on which Fr_r and Fr_g do *not* induce the same map (any example ?), we could still equally well compute this number by applying Lefschetz to either one of the relative or geometrical Frobenius, since the number of fixed points of a function does not depend on the cohomology theory that we are using.

5 Lefschetz trace formula for 0-dimensional schemes

In this section, we will establish directly the Lefschetz trace formula in the case of a scheme of dimension 0. Let's begin by looking at how a morphism $f : X \to X$ acts on $H^0(X, \mathbb{Q}_\ell)$ for any scheme X.

Let X be a scheme and $\pi_0(X)$ the set of its connected components. For any abelian group Λ, we know that

$$H^0_{\text{ét}}(X, \Lambda) = \Gamma(X, \Lambda) = \Lambda^{\pi_0(X)}.$$

We see that following diagram commutes, where the bottom arrow is the Λ-linear map induced by $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(X)$ (which sends the connected component of a point x to the connected component of $f(x)$).

$$\begin{array}{ccc} H^0_{\text{ét}}(X, \Lambda) & \xrightarrow{f^0} & H^0_{\text{ét}}(X, \Lambda) \\ \| & & \| \\ \Lambda^{\pi_0(X)} & \longrightarrow & \Lambda^{\pi_0(X)} \end{array}$$

Writing $f^0 : \Lambda^{\pi_0(X)} \to \Lambda^{\pi_0(X)}$ in matrix notation with regards to the natural basis of $\Lambda^{\pi_0(X)}$, we see that its trace is the number of connected components stabilized by f, i.e. the number of fixed points of $\pi_0(f) : \pi_0(X) \to \pi_0(X)$, this integer being interpreted in the abelian group Λ.

In particular, the trace of

$$f^0 : H^0(X, \mathbb{Q}_\ell) \longrightarrow H^0(X, \mathbb{Q}_\ell)$$

really *is* the number of connected components of X stablized by f (since the characteristic of $\mathbb{Q}_\ell$ is 0).

In the particular case where X is a 0-dimensional scheme, the connected components of X are just its points, so that X and $\pi_0(X)$ are identified, identifying f with $\pi_0(f)$. Thus, the number number of fixed points $L(f, X)$ of f may be computed via

$$L(f, X) = \mathrm{Tr}(f^0; H^0(X, \mathbb{Q}_\ell))$$

(this is the Lefschetz trace formula for 0-dimensional schemes!).

Example 5.1. If X is a scheme of finite type over $k = \mathbb{F}_q$, then for any $n \geq 1$

$$N_n(X) = L(F^n, \overline{X}) = \mathrm{Tr}\,(F^n)^0$$

where F denotes either the relative or geometrical Frobenius morphism of $\overline{X}$. Substituting this into the formula

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(X)}{n} t^n\right)$$

for the zeta function of X, we get the expected result

$$Z(X,t) = \frac{1}{\det(1 - F^0 t)}.$$

Example 5.2. Let's look explicitely at the map induced by the relative Frobenius morphism in the case where $X = \operatorname{Spec}\big(k[x]/(f)\big)$ where f is a non-constant polynomial. Factoring

$$f = f_1^{e_1} \cdots f_m^{e_m}$$

into irreducibles in $k[x]$ and using the Chinese remainder theorem to get

$$X = \operatorname{Spec}\big(k[x]/(f)\big) \cong \operatorname{Spec}\left(\bigoplus_{i=1}^{m} k[x]/(f_i^{e_i})\right) \cong \coprod_{i=1}^{m} \operatorname{Spec}\big(k[x]/(f_i^{e_i})\big)$$

we see that if suffices to consider the case where f has a unique irreducible factor.

Moreover, since $\operatorname{Spec}\big(k[x]/(f^m)\big)$ and $\operatorname{Spec}\big(k[x]/(f)\big)$ have the same number of K-rational points for any field K, we may also restrict ourselves to the case where f is irreducible.

In this case, since $k = \mathbb{F}_q$ is a perfect field, f is separable over $\overline{k}$; if its degree is n, let $\alpha_1, \ldots \alpha_n$ stand for its n distinct roots in $\overline{k}$.

Now, the relative Frobenius morphism

$$\mathrm{Fr}_r : \overline{X} \longrightarrow \overline{X}$$

corresponds to

$$\overline{k}[x]/(f) \longleftarrow \overline{k}[x]/(f)$$

which sends (the class of) a polynomial $g(x)$ to (the class of) $g(x^q)$. Combining this with the isomorphism

$$\overline{k}[x]/(f) \cong \overline{k}^n$$

which sends $g(x)$ to the vector $(g(\alpha_1), \ldots, g(\alpha_n))$, we get a commutative diagram

$$\begin{array}{ccc} \overline{k}[x]/(f) & \longleftarrow & \overline{k}[x]/(f) \\ \cong \downarrow & & \downarrow \cong \\ \overline{k}^n & \longleftarrow & \overline{k}^n \end{array}$$

in which the bottow arrow is

$$(g(\alpha_1), \ldots, g(\alpha_n)) \mapsto (g(\alpha_1^q), \ldots, g(\alpha_n^q))$$

i.e. indeed corresponds to the permutation $\alpha_i \mapsto \alpha_i^q$ of the roots, which are the points of $\overline{X}$.

6 Examples of global zeta functions

Now let's look at some examples of 0-dimensional schemes $X \to \operatorname{Spec}\mathbb{Z}$ for which, for any prime number p, we are able to compute the zeta function $Z(X_p, t)$ of

$$X_p := X \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p = X \otimes \mathbb{F}_p$$

as a scheme over $\mathbb{F}_p$ and then multiply them all together to obtain

$$\zeta(X, s) = \prod_p Z(X_p, p^{-s})$$

the *global zeta function* of X.

Example 6.1. First of all, the trivial example of $X := \operatorname{Spec} 0 \to \operatorname{Spec} \mathbb{Z}$, i.e. $X = \varnothing$ with the 0 sheaf of functions. For any prime number p, $X_p = \operatorname{Spec}(0 \otimes \mathbb{F}_p) = \operatorname{Spec} 0$, which has no points over any extension $\mathbb{F}_{p^n}$ of $\mathbb{F}_p$, thus

$$Z(X_p, t) = \exp(0) = 1$$

(which agrees with the fact the relative Frobenius induces the 0 linear map on $H^0(\overline{X}_p, \mathbb{Q}_\ell)$), and so

$$\zeta(X, s) = \prod_p 1 = 1.$$

Example 6.2. Now, the simplest non-trivial example: let's consider $X := \operatorname{Spec} \mathbb{Z}$ as a scheme over $\operatorname{Spec} \mathbb{Z}$ (we may view X as $\operatorname{Spec} \mathbb{Z}[x]/(x)$). Then for each p and n, X_p has exactly one F_{p^n}-rational point, so that

$$Z(X_p, t) = \exp\left(\sum_{r=1}^{\infty} \frac{t^r}{r}\right) = \exp(-\log(1-t)) = \frac{1}{1-t}$$

(and indeed Frobenius acts as the identity on H^0). Thus

$$\zeta(X, s) = \prod_p Z(X_p, p^{-s}) = \prod_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which is nothing but $\zeta(s)$, the Riemann zeta function.

Example 6.3. If $X := \operatorname{Spec} \mathbb{Z}[x]/(x(x-1))$, then X has 2 points over any $\mathbb{F}_{p^n}$, so that for any p

$$Z(X_p, t) = \exp\left(\sum_{r=1}^{\infty} 2\frac{t^r}{r}\right) = \exp(-2\log(1-t)) = \frac{1}{(1-t)^2}$$

which agrees with the fact that Frobenius acts as the identity on the 2-dimensional vector space H^0. And so we get

$$\zeta(X, s) = \prod_p \frac{1}{(1-p^{-s})^2} = \left(\prod_p \frac{1}{1-p^{-s}}\right)^2 = \zeta(s)^2.$$

Since by the Chinese remainder theorem $X = \operatorname{Spec} \mathbb{Z} \sqcup \operatorname{Spec} \mathbb{Z}$, this should come as no surprise.

Example 6.4. Let $f = x^2 - x - 1$ and consider $X := \operatorname{Spec} \mathbb{Z}[x]/(f)$ (as in [S]). For any prime number $p \neq 2$, by the quadratic formula, the equation

$$x^2 - x - 1 \equiv 0 \mod (p)$$

is equivalent to

$$x^2 \equiv 5 \mod (p)$$

thus it has exactly $\left(\frac{5}{p}\right) + 1$ solutions, which is equal to $\left(\frac{p}{5}\right) + 1$ because $5 \equiv 1 \mod (4)$.

Thus the number of solutions of $x^2 - x - 1 \equiv 0 \mod (p)$, $p \neq 2$, is

$$\begin{cases} 1 & \text{if } p = 5 \\ 2 & \text{if } p \equiv \pm 1 \mod (5) \\ 0 & \text{if } p \equiv \pm 2 \mod (5) \end{cases}$$

and now we find that this is also holds for $p = 2$.

For $p = 5$, f factors as $(x-3)^2$ in $\mathbb{F}_5[x]$, so that X_5 has exactly one point over each extension $\mathbb{F}_{5^n}$ of $\mathbb{F}_5$. This means that

$$Z(X_5, t) = \frac{1}{1-t}.$$

For $p \equiv \pm 1 \mod (5)$, f factors as a product of two linear polynomials in $\mathbb{F}_p[x]$, so that X_p has two points over each $\mathbb{F}_{p^n}$, thus

$$Z(X_p, t) = \frac{1}{(1-t)^2}.$$

For $p \equiv \pm 2 \mod (5)$, f is irreducible in $\mathbb{F}_p[x]$, thus $\mathbb{F}_p[x]/(f) \cong \mathbb{F}_{p^2}$, so that

$$N_n(X_p) = \#\{\mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_{p^n}\} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

so that in this case we get

$$Z(X_p, t) = \exp\left(\sum_{n=1}^{\infty} \frac{t^{2n}}{n}\right) = \exp(-\log(1-t^2)) = \frac{1}{1-t^2}.$$

Putting all this together, we get the global zeta function of X:

$$\begin{aligned} \zeta(X, s) &= \frac{1}{1-5^{-s}} \prod_{p \equiv \pm 1} \frac{1}{(1-p^{-s})^2} \prod_{p \equiv \pm 2} \frac{1}{1-p^{-2s}} \\ &= \prod_p \frac{1}{1-p^{-s}} \prod_p \frac{1}{1-\left(\frac{p}{5}\right)p^{-s}} \\ &= \zeta(s) L(\chi, s) \end{aligned}$$

where $L(\chi, s)$ is the Dirichlet L-series for the quadratic character $\chi = \left(\frac{\cdot}{5}\right)$, defined as

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

References

[GD] A. Grothendieck and J. A. Dieudonné, *Élements de géométrie algébrique I*, Springer-Verlag, 1971.

[H] C. Houzel, *Morphisme de Frobenius et rationalité des fonctions L*, exposé XV in *Cohomologie ℓ-adique et fonctions L* (SGA 5), Lecture Notes in Math. 589, Springer-Verlag, 1977.

[K] N. M. Katz, *Review of ℓ-adic cohomology*, Proc. Symp. Pure Math. (55), 1994, 21 –30.

[M] J. S. Milne, *Lectures on étale cohomology*, `http://www.jmilne.org/math/`, 1998.

[S] J.-P. Serre, *On a theorem of Jordan*, Bulletin of the AMS (40), n° 4, oct. 2003, 429 – 440.