Winter school in pure and applied math

Mathematical general relativity

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Harmonic oscillator

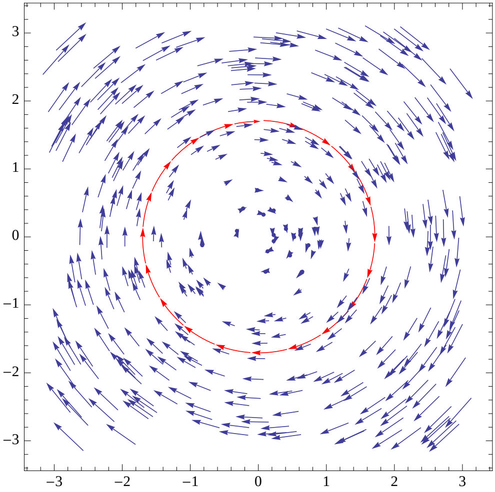
$$\ddot{x} = -x \qquad \text{or} \qquad \begin{aligned} \dot{v} &= -x \\ \dot{x} &= v \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(x^2 + v^2) &= 2x\dot{x} + 2v\dot{v} \\ &= 2xv - 2vx = 0 \end{aligned}$$

$$x(0)^2 + v(0)^2 = C$$

$$\Downarrow$$

$$x(t)^2 + v(t)^2 = C$$

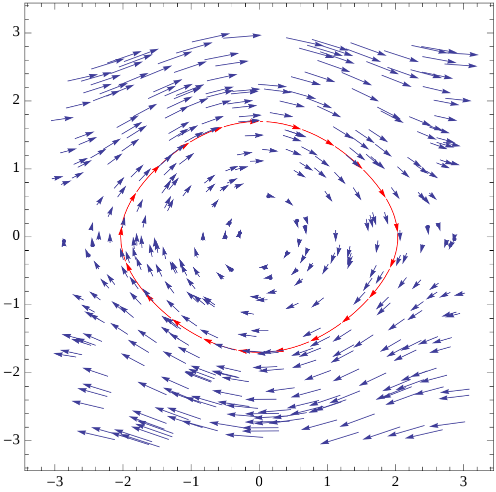


Physical pendulum

$$\ddot{x} = -\sin x \qquad \text{or} \qquad \begin{aligned} \dot{v} &= -\sin x \\ \dot{x} &= v \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(-2\cos x + v^2) &= 2(\sin x)\dot{x} + 2v\dot{v} \\ &= 2(\sin x)v - 2v\sin x \\ &= 0 \end{aligned}$$

$$-2\cos x(0) + v(0)^2 = C$$
$$\Downarrow$$
$$-2\cos x(t) + v(t)^2 = C$$



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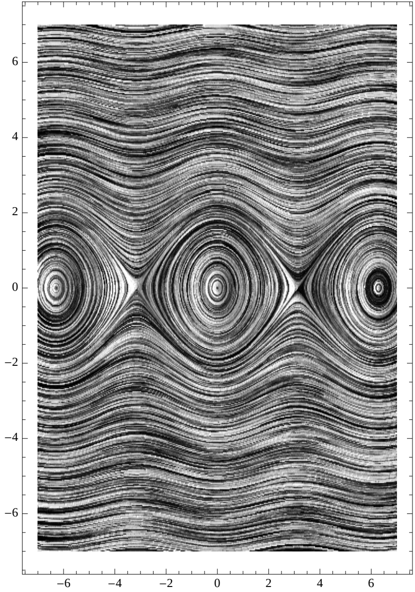

Constrained pendulum

$$\ddot{x} + x = 0$$
$$d \cdot x = 0$$

where $d \in \mathbb{R}^2$. For any $y \in \mathbb{R}^2$,

$$x = (I - dd^T)y$$

satisfies $d \cdot x = 0$. We have

$$(I - dd^T)(\ddot{y} + y) = 0$$

Let $d = e_2$. Then

$$\ddot{y}_1 + y_1 = 0$$

but no equation for y_2! x does not depend on y_2, so $y_2 = y_2(t)$ can be anything, e.g., take $y_2 = y_1$

Maxwell's equations

$$\partial_t B = -\nabla \times E, \qquad \nabla \cdot B = 0,$$
$$\partial_t E = \nabla \times B, \qquad \nabla \cdot E = 0.$$

$$\nabla \cdot B = 0 \quad \Rightarrow \quad B = \nabla \times A$$
$$\partial_t B = -\nabla \times E \quad \Rightarrow \quad \nabla \times (\partial_t A + E) = 0 \quad \Rightarrow \quad \partial_t A + E = -\nabla \varphi$$

$$\mathcal{C} := \partial_t(\nabla \cdot A) + \Delta\varphi = 0, \qquad -\partial_t(\partial_t A + \nabla\varphi) = \nabla \times \nabla \times A$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \Delta A \quad \Rightarrow \quad \mathcal{E} := \partial_t^2 A - \Delta A + \nabla(\partial_t \varphi + \nabla \cdot A) = 0$$

$$\partial_t \mathcal{C} = \nabla \cdot \partial_t^2 A + \Delta \partial_t \varphi$$
$$\nabla \cdot \mathcal{E} = \nabla \cdot \partial_t^2 A - \nabla \cdot \Delta A + \nabla \cdot \nabla \partial_t \varphi + \nabla \cdot \nabla(\nabla \cdot A) = \partial_t \mathcal{C}$$

Gauge freedom

$$[\partial_t(\nabla\cdot A)+\Delta\varphi]\big|_{t=0}=0, \qquad \partial_t^2 A-\Delta A+\nabla(\partial_t\varphi+\nabla\cdot A)=0$$
$$\Rightarrow \quad \partial_t(\nabla\cdot A)+\Delta\varphi=0$$

$$B=\nabla\times A, \qquad -E=\nabla\varphi+\partial_t A$$

$$A'=A+\nabla\lambda \qquad \Rightarrow \qquad \nabla\times A'=\nabla\times A+\nabla\times\nabla\lambda=B$$
$$\varphi'=\varphi-\partial_t\lambda \qquad \Rightarrow \qquad \partial_t A'+\nabla\varphi'=\partial_t A+\partial_t\nabla\lambda+\nabla\varphi-\nabla\partial_t\lambda=-E$$

$$\partial_t\varphi'+\nabla\cdot A'=\partial_t\varphi-\partial_t^2\lambda+\nabla\cdot A+\Delta\lambda$$

$$\partial_t^2\lambda-\Delta\lambda=\partial_t\varphi+\nabla\cdot A \qquad \Rightarrow \qquad \partial_t\varphi'+\nabla\cdot A'=0$$

Einstein's equations

The Lorentzian manifold $(M, \mathbf{g})$ satisfies

$$\mathrm{Ric}(\mathbf{g}) = 0. \tag{$\mathcal{E}$}$$

Suppose $M = \mathbb{R} \times \Sigma$, each $\Sigma_t = \{t\} \times \Sigma$ is spacelike. On each Σ_t, one has

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 0, \\ \mathrm{div}_g K - d(\mathrm{tr}_g K) &= 0. \end{aligned} \tag{$\mathcal{C}$}$$

Conversely, if $(\mathcal{C})$ holds on Σ_0, and $(\mathcal{E})$ holds in M, then $(\mathcal{C})$ holds for all Σ_t.

$$\mathrm{Ric}(\mathbf{g}) = \Box \mathbf{g} + N(\partial \mathbf{g}, \partial \mathbf{g}) + \partial \Box x^\alpha.$$

Einstein's equations

- Special solutions: Minkowski, Schwarzschild, de Sitter, Friedmann, Kerr, ...
- Local existence for smooth initial data: Choquet-Bruhat '52
- Incompleteness theorems: Penrose, Hawking $\sim$'60
- Unique maximal development: Choquet-Bruhat, Geroch '69
- Local existence for initial metric in $\mathrm{H}^{5/2+\epsilon}$: Hugh, Kato, Marsden '74
- Nonlinear stability of Minkowski space: Christodoulou, Klainerman $\sim$'90
- Local existence for initial metric in $\mathrm{H}^{2+\epsilon}$: Klainerman, Rodnianski $\sim$'00
- Black hole formation in vacuum: Christodoulou '08

Black hole stability problem

Prove that any nearby solution to a Kerr solution will stay close and asymptotically converge to a Kerr solution.

Progress:

- Linear wave equations on Kerr background: Rodnianski, Dafermos, Blue, Sterbenz, Tataru, ...
- Local uniqueness of the Kerr family: Klainerman, Alexakis, Ionescu

Einstein's constraint equations

$$
\begin{aligned}
R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 0, \\
\mathrm{div}_g K - d(\mathrm{tr}_g K) &= 0.
\end{aligned}
$$

- Positive mass theorem: Schoen, Yau, Witten ∼'80
- Conformal method: Lichnerowisz, York, Isenberg, Maxwell, ...
- Riemannian Penrose inequality: Huisken, Ilmanen, Bray '97-99
- Gluing: Corvino, Schoen, ...

Books

- SEAN CARROLL. Spacetime and Geometry: An Introduction to General Relativity
- ROBERT WALD. General Relativity
- NORBERT STRAUMANN. General Relativity: With Applications to Astrophysics
- ALAN RENDALL. Partial Differential Equations in General Relativity
- DEMETRIOS CHRISTODOULOU. Mathematical Problems of General Relativity
- YVONNE CHOQUET-BRUHAT. General Relativity and the Einstein Equations