Adaptive boundary element methods with convergence rates

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Outline

Boundary integral equations	$Au = f$
Boundary element methods	$A_n u_n = f_n$
A posteriori error estimates	$E(A, f, u_n) \sim \|u - u_n\|$
Convergence analysis	$\|u - u_n\| \to 0$?
Convergence rates	$\|u - u_n\| \lesssim n^{-\alpha}$?
Inverse-type inequalities	$\|Av_n\|_s \lesssim n^{-a}\|v_n\|$

Double layer potential

Given ρ continuous on a surface Γ, the *double layer potential*

$$u(x) = K\rho(x) := \int_\Gamma \rho(y) \frac{\partial}{\partial n_y} \left(\frac{1}{|x-y|} \right) \mathrm{d}\Gamma_y,$$

is harmonic in $\mathbb{R}^3 \setminus \Gamma$. In 1839, Gauss proposed to use the double layer potential to solve the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u = \phi \quad \text{on} \quad \partial\Omega,$$

by finding ρ on $\Gamma := \partial\Omega$, so that $(K\rho)(x) \to \phi(y)$ as $x \to y \in \partial\Omega$ from the interior. With $x^\pm \to x \in \partial\Omega$ from outside and inside, respectively, we have

$$u(x^+) - u(x^-) = 4\pi\rho(x), \qquad u(x^+) + u(x^-) = 2u(x).$$

From this we deduce

$$(K - 2\pi I)\rho = \phi \quad \text{on} \quad \partial\Omega.$$

Boundary integral equations

During 1870-1877 Carl Neumann established solvability of

$$(I - \tfrac{1}{2\pi}K)\rho = \phi \quad \text{on} \quad \Gamma,$$

for convex domains (with some exceptions). After over a century of development, we now have the same result for Lipschitz domains, which was proved by Gregory Verchota in 1984.

In general, there are many ways to convert (interior or exterior) boundary value problems for Ω into an integral equation

$$Au = f \quad \text{on} \quad \Gamma.$$

Typically, A has a singular kernel, $A : H^t(\Gamma) \to H^{-t}(\Gamma)$ is self-adjoint and bounded, and satisfies

$$\langle Au, u\rangle \geq \alpha \|u\|_{H^t}^2,$$

with $\alpha > 0$ and $t \in \{0, \pm\frac{1}{2}\}$. In particular, A is invertible.

Boundary element methods

People numerically solved boundary integral equations since mid 60's, but only after the discovery by Leslie Greengard and Vladimir Rokhlin of the fast multipole method in mid 80's, that it became competitive to direct discretizations of BVPs. BEMs are an adaptation of finite element methods to boundary integral equations.

For a triangulation T of Γ, let $S = S(T)$ be the space of piecewise constant functions on Γ subordinate to T. Then the Galerkin approximation $u_T \in S$ of u from the subspace $S \subset H^t$ $(t < \frac{1}{2})$ is the solution of

$$\langle A u_T, v \rangle = \langle f, v \rangle, \qquad \forall v \in S.$$

We have the Galerkin orthogonality

$$\| u - u_T \|^2 + \| u_T - v \|^2 = \| u - v \|^2, \qquad v \in S,$$

and the related best approximation property

$$\| u - u_T \| = \inf_{v \in S} \| u - v \|, \qquad \text{where} \qquad \| \cdot \|^2 = \langle A \cdot, \cdot \rangle.$$

Adaptive boundary element methods

The best approximation property implies the *a priori* error estimate

$$\|u - u_T\| \leq C \max_{\tau \in T} \operatorname{diam}(\tau)^s \|u\|_{H^s}, \qquad (s \leq 1).$$

If $u \notin H^1$, the convergence rate is slower than the optimal $h \sim N^{-1/2}$. Adaptive methods are observed, and in some cases proven to recover this rate.

Local *a posteriori* error indicators, $\eta(T, \tau)$, are supposed to measure how much error the triangle τ contains, e.g., $\|u - u_T\|_{H^t(\tau)}$. We need a parameter $0 < \theta < 1$, and an initial triangulation T_0. Then we repeat the following for $k = 0, 1, \ldots$.

- Compute $u_k = u_{T_k}$, and the error indicators $\eta(T_k, \tau)$, $\tau \in T_k$.
- Choose a minimal subset $R \subset T_k$, such that

$$\sum_{\tau \in R} \eta(T_k, \tau) \geq \theta \sum_{\tau \in T_k} \eta(T_k, \tau).$$

- Refine (at least) all triangles in R, to get T_{k+1}.

Some prior work on *a posteriori* error indicators

Residual is equivalent to error: $\|r_T\|_{H^{-t}} \equiv \|f - Au_T\|_{H^{-t}} \sim \|u - u_T\|_{H^t}$.
There is a localization issue for $t \notin \mathbb{N}_0$. Recall the Slobodeckij norm

$$|v|^2_{s,\omega} = \int_{\omega\times\omega} \frac{|v(x) - v(y)|^2}{|x-y|^{2+2s}} \mathrm{d}x\mathrm{d}y.$$

- Faermann '00-'02: for $-1 < t \leq 0$, global equivalence

$$\|r_T\|^2_{H^{-t}} \sim \sum_{z\in N_T} |r_T|^2_{-t,\omega(z)}.$$

- Carstensen, Maischak, Stephan '01: for $-1 < t \leq 0$, global upper bound

$$\|r_T\|^2_{H^{-t}} \lesssim \sum_{\tau\in T} h^{2(1-t)} |r_T|^2_{1,\tau}.$$

- Carstensen, Maischak, Praetorius, Stephan '04, Nochetto, von Petersdorff, Zhang '10: for $t > 0$, global upper bound

$$\|r_T\|^2_{H^{-t}} \lesssim \sum_{\tau\in T} h^{2t} |r_T|^2_{0,\tau}.$$

Gantumur '11: Lower bounds and local results. Example of a local result for $t = 0$:

Lemma

Let T' be a refinement of T, and let $\gamma = \bigcup_{\tau \in T \setminus T'} \tau$. Then we have

$$\alpha \| u_T - u_{T'} \| \leq \| r_T \|_{L^2(\gamma)} \leq \beta \| u_T - u_{T'} \| + 2 \| r_T - v \|_{L^2(\gamma)}$$

for any function $v \in S_{T'}$.

Proof of the second inequality.

Let $v \in S_{T'}$ be supported in γ. Then we have

$$\| v \|_\gamma^2 = \langle v, v \rangle = \langle v - r_T, v \rangle + \langle A(u_{T'} - u_T), v \rangle \leq \left(\| v - r_T \|_\gamma + \| A(u_{T'} - u_T) \|_\gamma \right) \| v \|_\gamma$$

implying that $\| r_T \|_\gamma \leq \| r_T - v \|_\gamma + \| v \|_\gamma \leq 2 \| r_T - v \|_\gamma + \| A(u_{T'} - u_T) \|$. □

Oscillation

Suppose r_T is piecewise H^r. Then

$$\inf_{v\in S_{T'}} \|r_T - v\|_\gamma^2 \leq C_J^2 \sum_{\tau\in T\setminus T'} h_\tau^{2r} |r_T|_{r,\tau}^2.$$

Define

$$\operatorname{osc}(T,\omega) := \left(\sum_{\tau\in T, \tau\subset\omega} h_\tau^{2r} |f - Au_T|_{r,\tau}^2 \right)^{\frac{1}{2}},$$

for $\omega \subseteq \Gamma$ and $v \in S_T$, so that we have

$$\alpha \| u_T - u_{T'} \| \leq \| r_T \|_\gamma \leq \beta \| u_T - u_{T'} \| + 2C_J \operatorname{osc}(T,\gamma).$$

Other works on convergence analysis

Symm's integral equation ($t = -\frac{1}{2}$).

- Ferraz-Leite, Ortner, Praetorius '10: With $\tilde{T}$ the uniform refinement of T, use error estimators of the type

$$\eta(T,\tau) = h_\tau^{1/2} \| u_T - u_{\tilde{T}} \|_{L^2(\tau)}.$$

  Assume saturation (1985-):

$$\| u - u_{\tilde{T}} \| \leq \alpha \| u - u_T \|, \qquad (\alpha < 1).$$

  Then $\| u - u_k \| \leq C\rho^k$ with $\rho < 1$.

- Aurada, Ferraz-Leite, Praetorius '11: Estimator convergence $\sum_\tau \eta(T_k,\tau) \to 0$ without saturation.
- Feischl, Karkulik, Melenk, Praetorius '11: Weighted residual estimator from [CMS01], geometric error reduction and convergence rate, without saturation.

Geometric error reduction

Assume

$$\sum_{\tau\in T} h_\tau^{2r}|Av|_{r,\tau}^2 \le C_A\|v\|^2, \qquad v\in S_T,$$

for all admissible T. Let T, T' be admissible partitions with T' being a refinement of T, and let $\gamma = \bigcup_{\tau\in T\setminus T'}\tau$. Suppose, for some $\theta\in(0,1]$ that

$$\|r_T\|_\gamma^2 + \operatorname{osc}(T,\gamma)^2 \ge \theta\left(\|r_T\|_\Gamma^2 + \operatorname{osc}(T,\Gamma)^2\right).$$

Then there exist constants $\delta\ge 0$ and $\rho\in(0,1)$ such that

$$\|u-u_{T'}\|^2 + \delta\operatorname{osc}(T',\Gamma)^2 \le \rho\left(\|u-u_T\|^2 + \delta\operatorname{osc}(T,\Gamma)^2\right).$$

Proof sketch:

$$\|u-u_T\| \lesssim \|r\|_\Gamma \lesssim \|r\|_\gamma \lesssim \|u_T - u_{T'}\|.$$

$$\|u-u_T\|^2 = \|u_T-u_{T'}\|^2 + \|u-u_{T'}\|^2.$$

We know $\|u-u_k\| \leq C\rho^k$ with $\rho<1$. How fast does $\#T_k$ grow?
Define approximation classes

$$\mathscr{A}_s = \{u \in L^2 : \inf_{\#T\leq N} \inf_{v\in S_T} \|u-v\| \leq CN^{-s}\}.$$

It is known that $W^{2s,p} \subset \mathscr{A}_s$ with $\frac{1}{p}=s+\frac{1}{2}$, and that $W^{2s,p}$ is much larger than H^{2s}, and friendlier to solutions of BVP and BIE.

Define $\mathscr{A}_{r,s}$ by replacing $\|u-v\|$ with $\|u-v\|+\text{osc}$. We expect $\mathscr{A}_{r,s}$ to be close to $\mathscr{A}_s$.

Assume

$$\sum_{\tau\in T} h_\tau^{2r} |Av|_{r,\tau}^2 \leq C_A \|v\|^2, \qquad v \in S_T,$$

for all admissible T. Let $\theta \in (0,\theta^*)$. Let f be piecewise H^r in the initial triangulation, and $u \in \mathscr{A}_{r,s}$ for some $s>0$. Then

$$\|u-u_k\| \leq C|u|_{A_{r,s}} (\#T_k)^{-s}.$$

Inverse-type inequalities

$$\sum_{\tau\in T} h_\tau^{2r}|Av|_{r,\tau}^2 \le C_A\|v\|^2, \qquad v\in S_T.$$

If $A=I$ or multiplication by a smooth function, then it is the standard inverse inequality. Validity of this inequality depends on how A shifts low frequencies to high frequencies locally, and how it moves frequencies around in space. We decompose $L^2 = S_T\oplus H_T$ and correspondingly, $Av=(Av)_S+(Av)_H$. The low frequency component poses no problem:

$$\sum_{\tau\in T} h_\tau^{2r}|(Av)_S|_{r,\tau}^2 \lesssim \|(Av)_S\|^2 \le \|Av\|^2 \lesssim \|v\|^2.$$

For each triangle $\tau\in T$, we decompose v as $v=v_\tau+(v-v_\tau)$, where v_τ is the part of v near τ. Then the high frequency component of Av locally decomposes into near-field interactions and far-field interactions:

$$(Av)_H|_\tau = (Av_\tau)_H|_\tau + (A(v-v_\tau))_H|_\tau.$$

For boundary integral operators, the far-field part is harmless, and the near-field part is ok if the underlying surface is regular (e.g., $C^{1,1}$).

Open problems

- to prove the inverse-type inequality for polyhedral surfaces
- to characterize the approximation classes associated to the proposed adaptive BEMs
- to extend the analysis to transmission problems, and adaptive FEM-BEM coupling
- complexity analysis, i.e., the problem of quadrature and linear algebra solvers
- convergence rate for adaptive BEMs based on non-residual type error estimators