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Solutions of the Einstein constraint equations with freely specified mean curvature

Gantumur Tsogtgerel

University of California, San Diego

Joint with M. Holst, J. Isenberg and G. Nagy

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Initial value formulation of the Einstein equations

The Lorentzian manifold $(M, \mathbf{g})$ satisfies

$$G(\mathbf{g}) := \mathrm{Ric}(\mathbf{g}) - \tfrac{1}{2}R(\mathbf{g})\mathbf{g} = 0.$$

Suppose $M = \mathbb{R} \times \Sigma$, each $\Sigma_t = \{t\} \times \Sigma$ is spacelike. On each Σ_t, one has

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 0, \\ \mathrm{div}_g K - d(\mathrm{tr}_g K) &= 0. \end{aligned} \tag{C}$$

Conversely, if (C) holds on some Riemannian manifold (Σ, g), then there are

- a Lorentzian manifold $(M, \mathbf{g})$
- and an embedding $\theta : \Sigma \to M$

such that $G(\mathbf{g}) = 0$ and that $\theta_* g$ and $\theta_* K$ are the first and second fundamental forms of $\theta\Sigma \subset M$ [Choquet-Bruhat '52].

The conformal method

Let $(\Sigma, \hat{g})$ be a Riemannian manifold, σ be a symmetric tensor with $\mathrm{div}_{\hat{g}}\, \sigma = 0$, $\mathrm{tr}_{\hat{g}}\, \sigma = 0$, and let $\tau \in C^\infty(\Sigma)$. With ϕ a positive scalar, and w a vector field, put

$$g = \phi^4 \hat{g}, \quad K = \phi^{-2}(\sigma + L_{\hat{g}} w) + \tfrac{1}{3} \tau \phi^4 \hat{g},$$

where $L_{\hat{g}} w = \mathcal{L}_w \hat{g} - \frac{2}{3} \hat{g}\, \mathrm{div}_{\hat{g}} w$. Then (C) is equivalent to

$$\begin{aligned} -8\Delta_{\hat{g}} \phi + R(\hat{g})\phi + \tfrac{2}{3}\tau\phi^5 - \left|\sigma + L_{\hat{g}} w\right|_{\hat{g}}^2 \phi^{-7} &= 0, \\ -\mathrm{div}_{\hat{g}} L_{\hat{g}} w + \tfrac{3}{2}\phi^6 d\tau &= 0. \end{aligned}$$

Let us rewrite the above as

$$\begin{aligned} A\phi + R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} =: A\phi + f(w, \phi) &= 0, \\ Bw + \phi^6 d\tau &= 0. \end{aligned}$$

Note that $\mathrm{tr}_g K = \tau$ and that if $\tau = \mathrm{const}$ the system decouples.

Fixed point approach

[Holst, Nagy, GT '07, '08]

$$A\phi + f(w,\phi) = 0, \qquad Bw + \phi^6 d\tau = 0.$$

With $S : \phi \mapsto -B^{-1}(\phi^6 d\tau)$ this can be written as

$$A\phi + f(S(\phi),\phi) = 0.$$

Let $0 < \phi_- \leqslant \phi_+ < \infty$ be global barriers, i.e.,

$$A\phi_- + f(w,\phi_-) \leqslant 0, \qquad A\phi_+ + f(w,\phi_+) \geqslant 0,$$

for all $w \in S([\phi_-,\phi_+])$. Then for $s > 0$ large, and any $w \in S([\phi_-,\phi_+])$

$$T_w : \phi \mapsto (A + sI)^{-1}(s\phi - f(w,\phi))$$

is monotone increasing on $U = [\phi_-,\phi_+]$, and for $\phi \in U$

$$T(\phi) \equiv T_{S(\phi)}(\phi) \leqslant T_{S(\phi)}(\phi_+) \leqslant \phi_+, \qquad T(\phi) \geqslant \phi_-,$$

so $T : U \to U$. Since T is compact, there is a fixed point in U.

Global super-solution

[Holst, Nagy, GT '07, '08]

We want to find $\phi > 0$ such that

$$A\phi + f(w, \phi) = A\phi + R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} \geqslant 0.$$

for all $w \in S([0, \phi])$. Recall that $a(w) = \left|\sigma + L_{\hat{g}} w\right|_{\hat{g}}^2$. Elliptic estimates give

$$a(w) \leqslant p + q\|\phi\|_{C^0}^{12}, \qquad \text{with } q \sim |d\tau|^2$$

Assume that $R = \text{const} > 0$, $\tau = \text{const}$, and let $\phi = \text{const} > 0$.

$$\begin{aligned} R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} &\geqslant \tfrac{2}{3}\tau\phi^5 + R\phi - p\phi^{-7} - q\phi^{-7}\phi^{12} \\ &\geqslant \phi^{-7}\left(R\phi^8 - (q - \tfrac{2}{3}\tau)\phi^{12} - p\right) \end{aligned}$$

If p is small enough (depending on how large q is), choosing $\phi > 0$ sufficiently small one can ensure that the above is nonnegative.

Asymptotically Euclidean manifolds

[Choquet-Bruhat, Isenberg, York '07], [Holst, Isenberg, Nagy, GT '09]

Let $\varphi > 0$ be the solution to

$$-\Delta\varphi = \eta^{-\beta},$$

for a suitable $\beta > 0$. Then $\phi = \lambda\varphi$ is a global supersolution if

$$-\varphi\Delta\lambda - 2(d\varphi)(d\lambda) + \varepsilon\eta^{-\beta} \geqslant 0,$$

for certain $\varepsilon > 0$.

Manuscripts, Collaborators, Acknowledgments

HNT2 M. HOLST, G. NAGY, AND GT, Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics. *Phys. Rev. Let.*, 100:161101, 2008. Also available as arXiv:0802.1031 [gr-qc]

HNT1 M. HOLST, , G. NAGY, AND GT, Rough solutions of the Einstein constraint equations on closed manifolds without near-CMC conditions. arXiv:0712.0798 [gr-qc]. To appear in *Comm. Math. Phys.*.

HINT M. HOLST, , J. ISENBERG, G. NAGY, AND GT, A class of solutions of the Einstein constraint equations on asymptotically Euclidean manifolds with freely specified mean curvature. In preparation.

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