An optimal adaptive wavelet method

for strongly elliptic operator equations

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Workshop on Fast Numerical Solution of PDE's
20-22 December 2005 Utrecht

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Motivation and overview

[Cohen, Dahmen, DeVore '01, '02]

- Using a wavalet basis Ψ, transform $Au = g$ to a matrix-vector system $\mathbf{A}\mathbf{u} = \mathbf{g}$
- Solve it by an iterative method
- They apply to $\mathbf{A}$ symmetric positive definite
- If A is perturbed by a compact operator?
- Normal equation: $\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{g}$
- Condition number squared, application of $\mathbf{A}^T\mathbf{A}$ expensive
- We modified [Gantumur, Harbrecht, Stevenson '05]

Sobolev spaces

- Let Ω be an n-dimensional domain
- $H^s := (L_2(\Omega), H_0^1(\Omega))_{s,2}$ for $0 \leq s \leq 1$
- $H^s := H^s(\Omega) \cap H_0^1(\Omega)$ for $s > 1$
- $H^s := (H^{-s})'$ for $s < 0$

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Model problem

- $A : H^1 \to H^{-1}$ linear, self-adjoint, H^1-elliptic:

$$\langle Au, u\rangle \geq c\|u\|_1^2 \qquad u \in H^1 \quad (= H_0^1(\Omega))$$

- $B : H^{1-\sigma} \to H^{-1}$ linear $\quad (\sigma > 0)$
- $L := A + B : H^1 \to H^{-1}$

Find $u \in H^1$ s.t. $\quad Lu = g \qquad (g \in H^{-1})$

- Regularity: $g \in H^{-1+\sigma} \quad \Rightarrow \quad \|u\|_{1+\sigma} \lesssim \|g\|_{-1+\sigma}$
- Example: Helmholtz equation

$$\langle Lu, v\rangle = \int_\Omega \nabla u \cdot \nabla v - \kappa^2 uv$$

Riesz bases

$\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for H^1
– each $v \in H^1$ has a unique expansion

$$v = \sum_{\lambda \in \nabla} \langle \tilde{\psi}_\lambda, v \rangle \psi_\lambda \quad \text{s.t.} \quad c\|v\|_1^2 \le \sum_{\lambda \in \nabla} |\langle \tilde{\psi}_\lambda, v \rangle|^2 \le C\|v\|_1^2$$

- $\tilde{\Psi} = \{\tilde{\psi}_\lambda\} \subset H^{-1}$ is the dual basis: $\langle \tilde{\psi}_\lambda, \psi_\mu \rangle = \delta_{\lambda\mu}$
- For $v \in H^1$, we have $\mathbf{v} = \{\mathbf{v}_\lambda\} := \{\langle \tilde{\psi}_\lambda, v \rangle\} \in \ell_2(\nabla)$

Wavelet basis

- $\Psi = \{\psi_\lambda\}$ Riesz basis for H^1
- Nested index sets $\nabla_0 \subset \nabla_1 \subset \ldots \subset \nabla_j \subset \ldots \subset \nabla$,
- $\mathcal{S}_j = \operatorname{span}\{\psi_\lambda : \lambda \in \nabla_j\} \subset H^1$
- $\operatorname{diam}(\operatorname{supp}\psi_\lambda) = \mathcal{O}(2^{-j})$ if $\lambda \in \nabla_j \setminus \nabla_{j-1}$
- All polynomials of degree $d-1$, $P_{d-1} \subset \mathcal{S}_0$

$$\inf_{v_j \in \mathcal{S}_j} \|v - v_j\|_1 \leq C \cdot 2^{-j(s-1)/n} \|v\|_s \qquad v \in H^s \ (1 \leq s \leq d)$$

- If $\lambda \in \nabla \setminus \nabla_0$, we have $\langle P_{\tilde{d}-1}, \psi_\lambda \rangle_{L_2} = 0$

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Equivalent discrete problem

[CDD01, CDD02]

- Wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$
- Stiffness $\mathbf{L} = \langle L\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu}$ and load $\mathbf{g} = \langle g, \psi_\lambda \rangle_\lambda$

Linear equation in $\ell_2(\nabla)$

$$\mathbf{L}\mathbf{u} = \mathbf{g}, \qquad \mathbf{L} : \ell_2(\nabla) \to \ell_2(\nabla) \text{ invertible and } \mathbf{g} \in \ell_2(\nabla)$$

- $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$ is the solution of $Lu = g$
- $\|\mathbf{u} - \mathbf{v}\|_{\ell_2} \eqsim \|u - v\|_1$ with $v = \sum_\lambda \mathbf{v}_\lambda \psi_\lambda$
- A good approx. of $\mathbf{u}$ induces a good approx. of u

Galerkin solutions

- $\mathbf{A} := \langle A\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu}$ SPD, recall $L = A + B$
- $\|\!| \cdot |\!\| := \langle \mathbf{A}\cdot, \cdot \rangle^{\frac{1}{2}}$ is a norm on ℓ_2
- $\Lambda \subset \nabla$
- $\mathbf{L}_\Lambda := \mathbf{P}_\Lambda \mathbf{L}|_{\ell_2(\Lambda)} : \ell_2(\Lambda) \to \ell_2(\Lambda)$, and $\mathbf{g}_\Lambda := \mathbf{P}_\Lambda \mathbf{g} \in \ell_2(\Lambda)$

Lemma

$\exists j_0$: If $\Lambda \supset \nabla_j$ with $j \geq j_0$, a unique solution $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ to $\mathbf{L}_\Lambda \mathbf{u}_\Lambda = \mathbf{g}_\Lambda$ exists, and

$$\|\!|\mathbf{u} - \mathbf{u}_\Lambda|\!\| \leq [1 + \mathcal{O}(2^{-j\sigma/n})] \inf_{\mathbf{v} \in \ell_2(\Lambda)} \|\!|\mathbf{u} - \mathbf{v}|\!\|$$

Ref: [Schatz '74]

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Quasi-orthogonality

- $j \geq j_0$
- $\nabla_j \subset \Lambda_0 \subset \Lambda_1$
- $\mathbf{L}_{\Lambda_i}\mathbf{u}_i = \mathbf{g}_{\Lambda_i}$, $i = 0, 1$

$$\left| |\!|\!|\mathbf{u} - \mathbf{u}_0|\!|\!|^2 - |\!|\!|\mathbf{u} - \mathbf{u}_1|\!|\!|^2 - |\!|\!|\mathbf{u}_1 - \mathbf{u}_0|\!|\!|^2 \right| \leq \mathcal{O}(2^{-j\sigma/n}) \left(|\!|\!|\mathbf{u} - \mathbf{u}_0|\!|\!|^2 + |\!|\!|\mathbf{u} - \mathbf{u}_1|\!|\!|^2 \right)$$

Ref: [Mekchay, Nochetto '04]

A sketch of a proof

$$\|\mathbf{u}-\mathbf{u}_0\|^2 = \|\mathbf{u}-\mathbf{u}_1\|^2 + \|\mathbf{u}_1-\mathbf{u}_0\|^2 + 2\langle \mathbf{A}(\mathbf{u}-\mathbf{u}_1), \mathbf{u}_1-\mathbf{u}_0\rangle$$

$$\langle \mathbf{L}(\mathbf{u}-\mathbf{u}_1), \mathbf{u}_1-\mathbf{u}_0\rangle = 0$$

$$\begin{aligned}
\langle \mathbf{A}(\mathbf{u}-\mathbf{u}_1), \mathbf{u}_1-\mathbf{u}_0\rangle &= -\langle \mathbf{B}(\mathbf{u}-\mathbf{u}_1), \mathbf{u}_1-\mathbf{u}_0\rangle \\
&= -\langle B(u-u_1), u_1-u_0\rangle \\
&\leq \|B\|_{1-\sigma\to -1}\|u-u_1\|_{1-\sigma}\|u_1-u_0\|_1
\end{aligned}$$

$\|u-u_1\|_{1-\sigma} \leq \mathcal{O}(2^{-j\sigma/n})\|u-u_1\|_1$ (Aubin-Nitsche)

Error reduction

$$\|\mathbf{u} - \mathbf{u}_1\|^2 \leq [1 + \mathcal{O}(2^{-j\sigma/n})] \left(\|\mathbf{u} - \mathbf{u}_0\|^2 - \|\mathbf{u}_1 - \mathbf{u}_0\|^2\right)$$

Lemma

Let $\mu \in (0, 1)$, and Λ_1 be s.t.

$$\|\mathbf{P}_{\Lambda_1}(\mathbf{g} - \mathbf{L}\mathbf{u}_0)\| \geq \mu\|\mathbf{g} - \mathbf{L}\mathbf{u}_0\|$$

Then we have

$$\|\mathbf{u} - \mathbf{u}_1\| \leq [1 - \kappa(\mathbf{A})^{-1}\mu^2 + \mathcal{O}(2^{-j\sigma/n})]^{\frac{1}{2}} \|\mathbf{u} - \mathbf{u}_0\|$$

Ref: [CDD01]

Exact algorithm

SOLVE$[\varepsilon] \rightarrow \mathbf{u}_k$
$k := 0;\ \Lambda_0 := \nabla_j$
`do`
  `Solve` $\mathbf{L}_{\Lambda_k}\mathbf{u}_k = \mathbf{g}_{\Lambda_k}$
  $\mathbf{r}_k := \mathbf{g} - \mathbf{L}\mathbf{u}_k$
  `determine a set` $\Lambda_{k+1} \supset \Lambda_k$`, with minimal cardinality, such that` $\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k\| \geq \mu\|\mathbf{r}_k\|$
  $k := k + 1$
`while` $\|\mathbf{r}_k\| > \varepsilon$

Approximate Iterations

Approximate right-hand side

$$\mathbf{RHS}[\varepsilon] \longrightarrow \mathbf{g}_\varepsilon \text{ with } \|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2} \leq \varepsilon$$

Approximate application of the matrix

$$\mathbf{APPLY}[\mathbf{v}, \varepsilon] \longrightarrow \mathbf{w}_\varepsilon \text{ with } \|\mathbf{L}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$$

Approximate residual

$$\mathbf{RES}[\mathbf{v}, \varepsilon] := \mathbf{RHS}[\varepsilon/2] - \mathbf{APPLY}[\mathbf{v}, \varepsilon/2]$$

Best N-term approximation

Given $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda \in \ell_2$, approximate $\mathbf{u}$ using N nonzero coeffs

$$\aleph_N := \bigcup_{\Lambda \subset \nabla : \#\Lambda = N} \ell_2(\Lambda)$$

- $\aleph_N$ is a nonlinear manifold
- Let $\mathbf{u}_N$ be a best approximation of $\mathbf{u}$ with $\#\operatorname{supp} \mathbf{u}_N \le N$
- $\mathbf{u}_N$ can be constructed by picking N largest in modulus coeffs from $\mathbf{u}$

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Nonlinear vs. linear approximation

Nonlinear approximation

If $u \in B_\tau^{1+ns}(L_\tau)$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-1}{n})$

$$\varepsilon_N = \|\mathbf{u}_N - \mathbf{u}\| \leq \mathcal{O}(N^{-s})$$

Linear approximation

If $u \in H^{1+ns}$ for some $s \in (0, \frac{d-1}{n}]$, uniform refinement

$$\varepsilon_j = \|\mathbf{u}_j - \mathbf{u}\| \leq \mathcal{O}(N_j^{-s})$$

- H^{1+ns} is a proper subset of $B_\tau^{1+ns}(L_\tau)$
- [Dahlke, DeVore]: $u \in B_\tau^{1+ns}(L_\tau) \backslash H^{1+ns}$ "often"

Approximation spaces

- Approximation space
  $\mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \leq \mathcal{O}(N^{-s})\}$
- Quasi-norm $|\mathbf{v}|_{\mathcal{A}^s} := \sup_{N \in \mathbb{N}} N^s \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2}$
- $u \in B_\tau^{1+ns}(L_\tau)$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-1}{n})$
  $\Rightarrow \mathbf{u} \in \mathcal{A}^s$

Assumption

$$\mathbf{u} \in \mathcal{A}^s \text{ for some } s \in (0, \tfrac{d-1}{n})$$

Best approximation

$$\|\mathbf{u} - \mathbf{v}\| \leq \varepsilon \text{ satisfies } \#\mathrm{supp}\,\mathbf{v} \leq \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

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Requirements on the subroutines

Complexity of **RHS**

$\mathbf{RHS}[\varepsilon] \to \mathbf{g}_\varepsilon$ terminates with $\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{g}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s} + 1$

Complexity of **APPLY**

For $\#\mathrm{supp}\, \mathbf{v} < \infty$

$\mathbf{APPLY}[\mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon$ terminates with $\|\mathbf{L}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{w}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{v}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s} |\mathbf{v}|_{\mathcal{A}^s}^{1/s} + \#\mathrm{supp}\, \mathbf{v} + 1$

The subroutine **APPLY**

- $\{\psi_\lambda\}$ are piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments
- L is either differential or singular integral operator

Then we can construct **APPLY** satisfying the requirements.
Ref: [CDD01], [Stevenson ’04], [Gantumur, Stevenson ’05,’06], [Dahmen, Harbrecht, Schneider ’05]

Optimal expansion

- $\mu \in (0, \kappa(\mathbf{A})^{-\frac{1}{2}})$.
- $\nabla_j \subset \Lambda_0$ with a sufficiently large j
- $\mathbf{L}_{\Lambda_0}\mathbf{u}_0 = \mathbf{g}_{\Lambda_0}$

Then the smallest set $\Lambda_1 \supset \Lambda_0$ with

$$\|\mathbf{P}_{\Lambda_1}(\mathbf{g} - \mathbf{L}\mathbf{u}_0)\| \geq \mu\|\mathbf{g} - \mathbf{L}\mathbf{u}_0\|$$

satisfies

$$\#(\Lambda_1 \setminus \Lambda_0) \lesssim \|\mathbf{u} - \mathbf{u}_0\|^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

Ref: [GHS05]

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Adaptive Galerkin method

SOLVE$[\varepsilon] \to \mathbf{w}_k$
$k := 0;\ \Lambda_0 := \nabla_j$
do
  Compute an appr.solution $\mathbf{w}_k$ of $\mathbf{L}_{\Lambda_k}\mathbf{u}_k = \mathbf{g}_{\Lambda_k}$
  Compute an appr.residual $\mathbf{r}_k$ for $\mathbf{w}_k$
  Determine a set $\Lambda_{k+1} \supset \Lambda_k$, with
  modulo constant factor minimal
cardinality,
  such that $\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k\| \geq \mu\|\mathbf{r}_k\|$
  $k := k+1$
while $\|\mathbf{r}_k\| > \varepsilon$

Conclusion

SOLVE$[\varepsilon] \to \mathbf{w}$ terminates with $\|\mathbf{g} - \mathbf{L}\mathbf{w}\|_{\ell_2} \leq \varepsilon$.

- $\#\operatorname{supp} \mathbf{w} \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim$ the same expression

Ref: [CDD01, GHS05]

Open:

- Singularly perturbed problems
- Adaptive initial index set

References

[CDD01] A. Cohen, W. Dahmen, R. DeVore. Adaptive wavelet methods for elliptic operator equations — Convergence rates. *Math. Comp.*, 70:27–75, 2001.

[GHS05] Ts. Gantumur, H. Harbrecht, R.P. Stevenson. An optimal adaptive wavelet method without coarsening of the iterands. Technical Report 1325, Utrecht University, March 2005. To appear in *Math. Comp.*.

[Gan05] Ts. Gantumur. An optimal adaptive wavelet method for nonsymmetric and indefinite elliptic problems. To appear.

This work was supported by the Netherlands Organization for Scientific Research and by the EC-IHP project "Breaking Complexity".