Lecture 26: Review

1. Eigenvalues, eigenvectors, eigenspaces, diagonalization
2. Inner product, length, orthogonality, orthogonal complement
3. Orthogonal projections, the Gram-Schmidt process
4. QR factorization, least-squares problems, orthogonal diagonalization of symmetric matrices

Eigenvalues, -vectors, -spaces, diagonalization

Let A be $n \times n$ matrix

- The eigenvalues of A : solutions of the characteristic equation $\det(A - \lambda I) = 0$
- The eigenspace corresponding to the eigenvalue λ_k : the space $\mathrm{Nul}\,(A - \lambda_k I)$
- An eigenvector corresponding to the eigenvalue λ_k : an element of $\mathrm{Nul}\,(A - \lambda_k I)$

Diagonalization: $A = PDP^{-1}$, where D is a diagonal matrix

- A is diagonalizable iff it has n linearly independent eigenvectors
- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the eigenvectors, and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues

$$A = PDP^{-1} \qquad \text{with} \qquad P = [\mathbf{v}_1 \ldots \mathbf{v}_n], \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

- distinct eigenvalues, or symmetric matrix $\Rightarrow$ diagonalizable
- dimension of $\mathrm{Nul}\,(A - \lambda_k I)$ is less than the multiplicity of $\lambda_k \Rightarrow$ not diagonalizable

Inner product, length, orthogonality

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v},\qquad \|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}},\qquad \text{dist}\,(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|,\qquad \mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

- Normalization: $\mathbf{w}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}$
- Orthogonality: $\mathbf{u}\cdot\mathbf{v}=0 \qquad (\mathbf{u}\perp\mathbf{v})$
- Orthogonal complement: $H^{\perp}=\{\mathbf{z}\in\mathbb{R}^n : \mathbf{z}\perp H\}$
- For any matrix A,

$$(\text{Row}\,A)^{\perp}=\text{Nul}\,A \qquad \text{and} \qquad (\text{Col}\,A)^{\perp}=\text{Nul}\,A^T$$

- Let $H=\text{Span}\,\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$, and $P=[\mathbf{v}_1,\ldots,\mathbf{v}_p]$. We have $H=\text{Col}\,P$.

$$H^{\perp}=(\text{Col}\,P)^{\perp}=\text{Nul}\,P^T$$

Orthogonal projections, Gram-Schmidt process

Let $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for H.

- For $\mathbf{x} \in \mathbb{R}^n$, let

$$\hat{\mathbf{x}} = \alpha_1 \mathbf{u}_1 + \ldots + \alpha_p \mathbf{u}_p \qquad \text{with} \quad \alpha_k = \frac{\mathbf{u}_k \cdot \mathbf{x}}{\mathbf{u}_k \cdot \mathbf{u}_k} \qquad (k = 1, \ldots, p)$$

- $\text{Proj}_H \mathbf{x} = \hat{\mathbf{x}}$ is called the orthogonal projection of $\mathbf{x}$ onto H
- $\mathbf{x} - \hat{\mathbf{x}} \perp H$, in other words, $\mathbf{x} - \hat{\mathbf{x}} \in H^{\perp}$

Gram-Schmidt: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \mapsto \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)} \mathbf{u}_1$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)} \mathbf{u}_1 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_2)}{(\mathbf{u}_2 \cdot \mathbf{u}_2)} \mathbf{u}_2$$

QR, least-squares, orthogonal diagonalization

If A has linearly independent columns

- Use Gram-Schmidt and normalization to orthonormalize the columns of $A \longrightarrow Q$
- Calculate $R = Q^T A$, then $A = QR$

Least-squares problems

- $A\hat{\mathbf{x}} = \text{Proj}_{\text{Col}\, A}\mathbf{b} \quad \Leftrightarrow \quad A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$
- The columns of A are linearly independent $\quad \Leftrightarrow \quad A^T A$ is invertible
- $A^T A$ is invertible and $A = QR$, then $A^T A\hat{\mathbf{x}} = A^T\mathbf{b} \quad \Leftrightarrow \quad \hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$

Orthogonal diagonalization of symmetric matrices: Let $A = A^T$

- Find the eigenvalues and bases for the eigenspaces
- Orthonormalize the bases
- Collecting all the vectors from these bases into the columns of U, we have $A = UDU^{-1} = UDU^T$, with the diagonal matrix D consisting of the eigenvalues