Lecture 25: Section 6.4

The Gram-Schmidt orthogonalization

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis for a subspace H of $\mathbb{R}^n$. Define

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{(\mathbf{v}_2 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1 \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_2)}{(\mathbf{u}_2 \cdot \mathbf{u}_2)}\mathbf{u}_2 \\
&\dots \\
\mathbf{u}_p &= \mathbf{v}_p - \frac{(\mathbf{v}_p \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1 - \frac{(\mathbf{v}_p \cdot \mathbf{u}_2)}{(\mathbf{u}_2 \cdot \mathbf{u}_2)}\mathbf{u}_2 - \dots - \frac{(\mathbf{v}_p \cdot \mathbf{u}_{p-1})}{(\mathbf{u}_{p-1} \cdot \mathbf{u}_{p-1})}\mathbf{u}_{p-1}
\end{aligned}$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for H. In addition

$$\text{Span}\,\{\mathbf{u}_1, \dots, \mathbf{u}_k\} = \text{Span}\,\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \qquad \text{for } 1 \le k \le p$$

An alternative description of the Gram-Schmidt process:

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{v}_1, \qquad \mathbf{u}_2 = \mathbf{v}_2 - \text{Proj}_{\,\text{Span}\,\{\mathbf{u}_1\}}\mathbf{v}_2, \qquad \mathbf{u}_3 = \mathbf{v}_3 - \text{Proj}_{\,\text{Span}\,\{\mathbf{u}_1,\mathbf{u}_1\}}\mathbf{v}_3 \\
&\dots, \qquad \mathbf{u}_p = \mathbf{v}_p - \text{Proj}_{\,\text{Span}\,\{\mathbf{u}_1,\dots,\mathbf{u}_{p-1}\}}\mathbf{v}_p
\end{aligned}$$

Applications of the Gram-Schmidt process

QR factorization

Let A be a $k \times n$ matrix with linearly independent columns. Then $A = QR$, where

- the columns of Q $(k \times n)$ form an orthonormal basis for $\operatorname{Col} A$
- R is an upper triangular matrix with positive entries on its diagonal (so invertible)

Q can be obtained by applying the Gram-Schmidt on the columns of A, followed by normalization. We have $R = Q^T A$.

Least-squares problems

If the columns of A are linearly independent and $A = QR$, then

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b} \quad \Leftrightarrow \quad \hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$$

Orthogonal diagonalization of symmetric matrices

Let $A = A^T$, let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A, and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be bases for the corresponding eigenspaces $\operatorname{Nul}(A - \lambda_i I)$. To orthonormalize $\mathcal{V}_i$ we can use the Gram-Schmidt, and get the orthonormal basis $\mathcal{U}_i$ for $\operatorname{Nul}(A - \lambda_i I)$.