Review

1. Inverse, LU factorization, determinant
2. Subspace, basis, dimension
3. Null space, column space
4. Coordinate systems, change of coordinates

Inverse, LU factorization, determinant

Inverse of A

$$[A\ I] \sim [I\ A^{-1}] \qquad \text{(zero row means no inverse)}$$

LU factorizition

Let U be an echelon form of A, obtained by only row replacement operations. Then $A = LU$, with L unit lower diagonal.

Determinant

Let B be an echelon form of A. Then

$$\det A = b_{11}b_{22}\dots b_{nn} \qquad \text{(diagonal entries of } B\text{)}$$

$$\det A^T = \det A, \qquad \det(AD) = \det A \det D, \qquad \det(A^{-1}) = (\det A)^{-1}$$

Subspace, basis, dimension

Let V be a vector space, H a subset of V. H is a vector space (and so subspace) if

- $\mathbf{0} \in H$
- $\mathbf{u}, \mathbf{v} \in H \quad \Rightarrow \quad \mathbf{u} + \mathbf{v} \in H$
- $\alpha \in \mathbb{R}, \mathbf{u} \in H \quad \Rightarrow \quad \alpha\mathbf{u} \in H$

or

- $H = \mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$

or

- in case $V = \mathbb{R}^n$, $H = \mathrm{Nul}\,A$ for some $k \times n$ matrix A

Basis, dimension

$\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_q\}$ is a basis for H (and $\dim H = q$) iff

- $\mathcal{U}$ spans H
- $\mathcal{U}$ is linearly independent

$\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$ iff $A = [\mathbf{u}_1 \ldots \mathbf{u}_n]$ is invertible

Null space, column space

- $\mathbf{u} \in \operatorname{Nul} A \quad \Leftrightarrow \quad A\mathbf{u} = \mathbf{0}$
- $\mathbf{u} \in \operatorname{Col} A \quad \Leftrightarrow \quad A\mathbf{x} = \mathbf{u}$ has a solution
- The pivot columns of A form a basis for $\operatorname{Col} A$
- Write the solution of $A\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = x_{i_1}\mathbf{v}_1 + \ldots x_{i_q}\mathbf{v}_q$$

  with $x_{i_1}, \ldots, x_{i_q}$ free varaibles. Then $\mathbf{v}_1, \ldots, \mathbf{v}_q$ form a basis for $\operatorname{Nul} A$
- Let B be an echelon form of A. Then the nonzero rows of B form a basis for $\operatorname{Row} A = \operatorname{Col} A^T$
- Number of columns of $A = \operatorname{rank} A + \dim \operatorname{Nul} A, \qquad (\operatorname{rank} A = \dim \operatorname{Col} A)$
- $\operatorname{rank} A^T = \dim \operatorname{Row} A = \operatorname{rank} A$

Coordinate systems, change of coordinates

Coordinate system

Let $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for H. Let $\mathbf{x} \in H$.

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n \quad \Leftrightarrow \quad [\mathbf{x}]_{\mathcal{V}} = \begin{bmatrix} \alpha_1 \\ \ldots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$

$$[\mathbf{u}_1]_{\mathcal{V}} = \begin{bmatrix} \alpha_{11} \\ \ldots \\ \alpha_{n1} \end{bmatrix}, \ldots, [\mathbf{u}_n]_{\mathcal{V}} = \begin{bmatrix} \alpha_{1n} \\ \ldots \\ \alpha_{nn} \end{bmatrix}, \qquad P = \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \ldots & \ldots & \ldots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{bmatrix}$$

$\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for H iff P is invertible. With $Q = P^{-1}$, we have

$$[\mathbf{x}]_{\mathcal{V}} = P[\mathbf{x}]_{\mathcal{U}} \qquad \text{and} \qquad [\mathbf{x}]_{\mathcal{U}} = Q[\mathbf{x}]_{\mathcal{V}} \qquad \text{for all } \mathbf{x} \in H.$$

Let $H = \mathbb{R}^n$, $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$, $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Then $U = VP$, so $P = V^{-1}U$ and $Q = U^{-1}V$. In other words, $[V \ U] \sim [I \ P]$, and $[U \ V] \sim [I \ Q]$.