Lecture 18 (Sections 3.2, 3.3)

1. Determinant of product, linearity
2. Cramer's rule
3. Determinants as area or volume

Determinant of product, linearity

Determinant of product

If A and B are $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$

For any square matrix $A = [\mathbf{a}_1 \ldots \mathbf{a}_n]$, and any $\mathbf{u} \in \mathbb{R}^n$, let $A_k(\mathbf{u})$ be the matrix obtained from A by replacing column k by $\mathbf{u}$.

$$A_k(\mathbf{u}) = [\mathbf{a}_1 \ldots \mathbf{a}_{k-1} \;\; \mathbf{u} \;\; \mathbf{a}_{k+1} \ldots \mathbf{a}_n]$$

Linearity of determinant

Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(\mathbf{u}) = \det A_k(\mathbf{u})$. Then T is linear.

- $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$
- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

Cramer's rule

Cramer's rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_k = \frac{\det A_k(\mathbf{b})}{\det A}, \qquad k = 1, 2, \ldots, n$$

Inverse matrix formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$

where $C_{ik} = (-1)^{i+k} A_{ik}$ are the cofactors.

Determinants as area or volume

Area of a parallelogram

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, and $A = [\mathbf{u}\ \mathbf{v}]$. Then the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is equal to $|\det A|$.

Volume of a parallelepiped

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, and $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$. Then the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to $|\det A|$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $T(\mathbf{x}) = A\mathbf{x}$. If S is a region in $\mathbb{R}^2$, then

$$\text{Area } T(S) = |\det A| \cdot \text{Area } S$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $T(\mathbf{x}) = A\mathbf{x}$. If S is a region in $\mathbb{R}^3$, then

$$\text{Volume } T(S) = |\det A| \cdot \text{Volume } S$$