Lecture 15 (Sections 4.5, 4.6)

Dimension

Theorem 9

Let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and let $q > p$. Then any collection of q vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_q\}$ (in H) is linearly dependent.

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_q\}$ are bases for H then $p = q$.

Definition

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for H, then n is called the dimension of H, written $\dim H$.

Let $n = \dim V \geq 1$ and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then
S is lin. indep. $\Leftrightarrow$ S spans V $\Leftrightarrow$ S is a basis for V.

- $\dim \mathbb{R}^n = n, \qquad \dim P_k = k + 1$
- $\dim \text{Nul}\, A$ is the number of free variables in $A\mathbf{x} = \mathbf{0}$
- $\dim \text{Col}\, A$ is the number of pivot columns in A

Rank

Theorem 13

If $A \sim B$, then $\text{Row}\, A = \text{Row}\, B$. If B is in echelon form, the nonzero rows of B form a basis for $\text{Row}\, A = \text{Row}\, B$.

Definition

$$\text{rank}\, A = \dim \text{Col}\, A$$

Theorem 14

Let A be $k \times n$ matrix. Then $\text{rank}\, A = \dim \text{Row}\, A$ and $\text{rank}\, A + \dim \text{Nul}\, A = n$.

Theorem

Let A be $n \times n$ matrix. Then

$$A \text{ is invertible} \Leftrightarrow \text{Col}\, A = \mathbb{R}^n \Leftrightarrow \text{rank}\, A = n \Leftrightarrow \text{Nul}\, A = \{\mathbf{0}\} \Leftrightarrow \dim \text{Nul}\, A = 0$$