Lecture 13 (Section 4.2)

Null space and column space

Let A be $k \times n$ matrix

The null space of A is the set of all solutions to $A\mathbf{x} = \mathbf{0}$.

$$\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

- The null space of a $k \times n$ matrix is a subspace of $\mathbb{R}^n$
- $\operatorname{Nul} A$ is the set of all $\mathbf{x}$ that are mapped into $\mathbf{0}$ via the mapping $\mathbf{x} \mapsto A\mathbf{x}$

The column space of A is the set of all linear combinations of the columns of A.

$$\operatorname{Col} A = \{\mathbf{b} \in \mathbb{R}^k : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

- The column space of a $k \times n$ matrix is a subspace of $\mathbb{R}^k$
- $\operatorname{Col} A$ is the range of the mapping $\mathbf{x} \mapsto A\mathbf{x}$
- $\operatorname{Col} A = \mathbb{R}^k \Leftrightarrow$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^k$

Kernel and range

Let V and W be vector spaces. The mapping $T : V \to W$ is linear if

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and for all $\alpha, \beta \in \mathbb{R}$

The kernel of T is the set of all $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{0}$

$$\operatorname{Ker} T = \{\mathbf{u} \in V : T(\mathbf{u}) = \mathbf{0}\}$$

The range of T is the set of all $\mathbf{z} \in W$ such that $\mathbf{z} = T(\mathbf{x})$ for some $\mathbf{x} \in V$

$$\operatorname{Ran} T = \{T(\mathbf{x}) : \mathbf{x} \in V\}$$

- $\operatorname{Ker} T$ is a subspace of V, and $\operatorname{Ran} T$ is a subspace of W
- If $T : \mathbf{x} \mapsto A\mathbf{x}$, then $\operatorname{Ker} T = \operatorname{Nul} A$, and $\operatorname{Ran} T = \operatorname{Col} A$