Lecture 12 (Section 4.1)

Vector space

A nonempty set V is called a vector space if

- $\mathbf{u} + \mathbf{v} \in V$, for any $\mathbf{u}, \mathbf{v} \in V$
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$, for any $\mathbf{u}, \mathbf{v} \in V$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$
- there is $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$, for any $\mathbf{u} \in V$
- for each $\mathbf{u} \in V$, there is some $\mathbf{z} \in V$ such that $\mathbf{u} + \mathbf{z} = \mathbf{0}$ (define $-\mathbf{u} = \mathbf{z}$)

- $\alpha\mathbf{u} \in V$, for any $\alpha \in \mathbb{R}$ and $\mathbf{u} \in V$
- $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$, for any $\alpha \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$
- $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$, for any $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in V$
- $(\alpha\beta)\mathbf{u} = \alpha\beta\mathbf{u}$, for any $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in V$
- $1\mathbf{u} = \mathbf{u}$, for any $\mathbf{u} \in V$

Subspace

H is a subspace of a vector space V if

- H is a subset of V, i.e., if $\mathbf{u} \in H$ then $\mathbf{u} \in V$
- the zero vector $\mathbf{0} \in V$ is in H, i.e., $\mathbf{0} \in H$
- if $\mathbf{u}, \mathbf{v} \in H$ then $\mathbf{u} + \mathbf{v} \in H$
- if $\alpha \in \mathbb{R}$ and $\mathbf{u} \in H$ then $\alpha\mathbf{u} \in H$

Span

Theorem 4.1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V

Definition of Span

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} := \{\mathbf{u} = c_1\mathbf{v}_1 + \ldots c_p\mathbf{v}_p : c_1, \ldots, c_p \in \mathbb{R}\}$$

- if $\mathbf{u} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ then $\mathbf{u} = c_1\mathbf{v}_1 + \ldots c_p\mathbf{v}_p$ with some $c_1, \ldots, c_p \in \mathbb{R}$
- if $\mathbf{u} = c_1\mathbf{v}_1 + \ldots c_p\mathbf{v}_p$ with some $c_1, \ldots, c_p \in \mathbb{R}$, then $\mathbf{u} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$