Lecture 8 (Section 1.9)

The matrix of a linear transformation

Linear transformations of $\mathbb{R}^2$

Onto and one-to-one mappings

The matrix of a linear transformation

Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a linear mapping. Then, for all $\mathbf{x} \in \mathbb{R}^n$, we have

$$T(\mathbf{x}) = A\mathbf{x}$$

where

- $A = [T(\mathbf{e}_1) \dots T(\mathbf{e}_n)]$
- $I = [\mathbf{e}_1 \dots \mathbf{e}_n]$ the identity matrix in $\mathbb{R}^n$

Linear transformations of $\mathbb{R}^2$

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ scaling by factor a in x_1-direction, by b in x_2-direction

$\begin{bmatrix} 1 & d \\ c & 1 \end{bmatrix}$ if $c = 0$: shear in x_1-direction, $d = 0$: shear in x_2-direction

$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ rotation by angle ϕ

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ reflections through $x_2 = x_1$, and $x_2 = -x_1$, respectively

Onto and one-to-one mappings

Let $T : \mathbb{R}^n \to \mathbb{R}^k$ and if T is linear, let $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. (If not, A is not defined)

Definition

T is onto (or surjective) if each $\mathbf{b} \in \mathbb{R}^k$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$. Equivalent are

- $\mathrm{Range}(T) = \mathbb{R}^k$
- Columns of A span $\mathbb{R}^n$

Definition

T is one-to-one (or injective) if each $\mathbf{b} \in \mathrm{Range}(T)$ is the image of exactly one $\mathbf{x} \in \mathbb{R}^n$. Equivalent are

- Each $\mathbf{b} \in \mathbb{R}^k$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$
- Columns of A are linearly independent
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution