Lecture 4 (Sections 1.4, 1.5)

Surjectiveness

Computation of $A\mathbf{x}$

Matrix-vector product

Homogeneous systems

Nonhomogeneous systems

Surjectiveness

For a given A, is $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Theorem 4

Let A be a $k \times n$ matrix. Then the following statements are equivalent:

- For each $\mathbf{b} \in \mathbb{R}^k$, $A\mathbf{x} = \mathbf{b}$ has a solution
- Each $\mathbf{b} \in \mathbb{R}^k$ is a linear combination of the columns of A ($\Leftrightarrow$ The columns of A span $\mathbb{R}^k$)
- A has a pivot position in every row ($\Leftrightarrow$ No echelon form of A has zero row)

Computation of $A\mathbf{x}$

$$A = [\mathbf{a}_1 \dots \mathbf{a}_n], \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad A\mathbf{x} = x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n$$

$$A = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + (-6) \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 7 \cdot 1 + (-6) \cdot (-4) \\ 7 \cdot 3 + (-6) \cdot 2 \\ 7 \cdot 0 + (-6) \cdot 5 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

Matrix-vector product

Theorem 5

Let A be a $k \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$. Then

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $A(c\mathbf{u}) = c(A\mathbf{u})$
- $A\mathbf{0} = \mathbf{0}$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Homogeneous systems

Homogeneous system: $A\mathbf{x} = \mathbf{0}$

- $\mathbf{x} = \mathbf{0}$ is a solution, since $A\mathbf{0} = \mathbf{0}$ (so the system is consistent)
- It is called the trivial solution

Since no free variable means only one (unique) solution, $A\mathbf{x} = \mathbf{0}$ has nontrivial solution if and only if there is at least one free variable.

- Parametric vector equation of a plane

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \qquad (s, t \in \mathbb{R})$$

- “Hyperplane” (the subset spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_m$)

$$\mathbf{x} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \ldots + s_m\mathbf{u}_m \qquad (s_1, \ldots, s_m \in \mathbb{R})$$

Nonhomogeneous systems

Theorem 6

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent, and let $A\mathbf{p} = \mathbf{b}$.

- Any solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ can be written as

  $$\mathbf{x} = \mathbf{p} + \mathbf{v}_h$$

  where $\mathbf{v}_h$ is a solution of the homogeneous system $A\mathbf{v}_h = \mathbf{0}$
- If $\mathbf{v}_h$ is a solution of the homogeneous system $A\mathbf{v}_h = \mathbf{0}$, then

  $$\mathbf{x} = \mathbf{v}_h + \mathbf{p}$$

  is a solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$

Writing a solution set in parametric vector form

- Row reduce the augmented matrix to reduced echelon form
- Express each basic variable in terms of free variables
- Write $\mathbf{x}$
- Decompose $\mathbf{x}$ into linear combinations of vectors