MATH 20F WINTER 2007 MIDTERM EXAM I

JANUARY 31

GUIDELINES:

- *Please put your* **name**, **ID number**, **TA's name**, *and* **section time** *on your exam sheet.*
- *No books, notes, or calculators are allowed.*
- *Write your solutions clearly and give explanations for your work. Answers without justifications will not be given credit.*
- *If any question is not clear, ask for clarification.*

PROBLEMS:

1. [*10pts*] Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 &= 12 \\ x_2 + x_3 &= 5 \\ 3x_1 - 2x_2 + x_3 &= 11. \end{cases}$$

a). [*5pts*] Write down the augmented matrix of the system, and use the row reduction algorithm to find a row echelon form of the matrix. Circle the pivot positions in the final matrix and list the pivot columns. Determine whether the system is consistent. If the system is consistent, find all solutions of the system. Write it in parametric vector form.

b). [*5pts*] Write the system in the form $A\mathbf{x} = \mathbf{b}$, with A being the coefficient matrix of the system. Determine if there exist $\mathbf{c}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{c}$ has *no* solution, and if there is such $\mathbf{c}$, give an example. Describe the set of all $\mathbf{c}$ in $\mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{c}$ *does* have a solution.

2. [*10pts*] Consider the matrix A and vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 562.23 \\ 362.82 \end{bmatrix}.$$

a). [*5pts*] Are the columns of A linearly independent or linearly dependent? Justify your answer. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

b). [*5pts*] Is $\mathbf{b}$ in the set spanned by the columns of A? Do the columns of A span $\mathbb{R}^2$?

(SEE OTHER SIDE)

3. [*10pts*] Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear mappings given by

$$S(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 \\ -x_2 \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and

$$T(\mathbf{x}) = \begin{bmatrix} 7x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a). [*5pts*] Is S onto? Is it one-to-one? Justify your answer.
b). [*5pts*] Find the standard matrix of the composition $L = S \circ T$, that is, the mapping $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $L(\mathbf{x}) = S(T(\mathbf{x}))$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

4. [*10pts*] Mark each statement TRUE or FALSE. Justify each answer.
a). When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, Span $\{\mathbf{u}, \mathbf{v}\}$ is a plane.
b). The columns of a $k \times n$ matrix are linearly independent if and only if it has n pivot columns.
c). If A is a $k \times n$ matrix, then the range of $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^k$.
d). The columns of the standard matrix for a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ are the images of the columns of the $n \times n$ identity matrix.
e). The linear mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is onto if and only if its standard matrix has n pivot columns.