Contemporary Mathematics

On the Engquist Majda Absorbing Boundary Conditions for Hyperbolic Systems

Adi Ditkowski and David Gottlieb

This paper is dedicated to Stan Osher on the occasion of his 60th birthday

Abstract. In their classical paper [**2**], the authors presented a methodology for the derivation of far field boundary conditions for the absorption of waves that are almost perpendicular to the boundary. In this paper we derive a general order absorbing boundary conditions of the type suggested by Engquist and Majda. The derivation utilizes a different methodology which is more general and simpler. This methodology is applied to the two and three dimensional wave equation, to the three dimensional Maxwell's equations and to the equations of advective acoustics in two dimensions.

I. Introduction

A long standing problem in simulating wave phenomena has been the issue of finding infinite space solutions on a finite numerical domain. Reflections from the boundaries of the numerical domain may distort the solution and even lead to instabilities. A pioneering contribution to this field has been the work of Engquist and Majda [**2**]. They presented a methodology to construct boundary conditions that minimize reflections of waves traveling in directions close to perpendicular to the boundary. In their paper they show how to construct absorbing boundary conditions with reflection coefficients that are $O(\theta^{2p})$ where θ is the angle between the incident wave and the normal to the boundary. Higdon [**8**] derived their boundary conditions in a simpler and more general form - for the classical wave equation.

In this paper we present a different methodology for deriving p^{th} order boundary conditions for three different cases. We start in Section II by discussing the wave equation in two space dimensions and construct the most general p^{th} order absorbing boundary condition. We also prove a uniqueness theorem for this case, showing that there is basically only one scheme that yields a p^{th} order method. We use this relatively simple equation to demonstrate, in detail, our methodology. The

---

2000 *Mathematics Subject Classification.* Primary 35LXX, 78M99.

This research was supported in part by DARPA contract HPC/F33615-01-C-1866 and by AFOSR grant F49620-02-1-0113. The authors would like to acknowledge Bertil Gustafsson and W. Hall for fruitful discussions.

boundary conditions derived are easily extended to general boundaries and three dimensions.

In Section III we discuss the three dimensional Maxwell's equations in half space. In this case there are two incident waves and two reflected waves and, thus, we have four reflection coefficients. By using the methodology presented in Section II we construct a simple p^{th} order absorbing boundary conditions that is easily extendable to complicated smooth boundaries.

In Section IV we discuss the advective acoustics case in two space dimensions. In this case there are three families of waves, two acoustics waves and an entropy wave, and therefore there is a difference between the inflow and outflow boundaries. We use the methodology of Section II to derive boundary conditions of arbitrary order in both cases.

An important issue is whether this type of boundary conditions are well posed. Gustafsson observed that the Engquist Majda boundary conditions admit a generalized eigenvalue in the classical wave equation case. This eigenvalue exists also in the case of the Maxwell's equations (for $p > 1$) and in the case of the Euler equations for advective acoustics. In the latter case there are more generalized eigenvalues. These results are shown in Section V, and will be discussed in future work.

II. The Wave Equation

Consider the wave equation in two space dimensions:

$$U_{tt} = U_{xx} + U_{yy}, \tag{2.1}$$

In the interval $0 \leq x < \infty$.

The solution U can be represented as a sum of waves of the form

$$U = e^{i\omega(t+\beta y)}\left(Ae^{i\omega\sqrt{1-\beta^2}x} + Be^{-i\omega\sqrt{1-\beta^2}x}\right), \tag{2.2}$$

where

$$\beta = \sin(\theta).$$

The first term on the right hand side of (2.2) describes a wave moving to the left whereas the second term describes a right moving wave.

We consider a local boundary condition at $x = 0$ of the form

$$\mathcal{L}U = 0 \qquad \text{at } x = 0. \tag{2.3}$$

Upon substituting the plane wave (2.2) into the boundary condition (2.3) we get

$$AF_1(\beta) + BF_2(\beta) = 0, \tag{2.4}$$

where F_1 is the result of applying the boundary operator $\mathcal{L}$ on the left moving wave (moving towards the boundary $x = 0$) and F_2 is the results of applying the boundary operator to the right moving wave (which is reflected from the boundary).

The reflection coefficient is defined as

$$R = \left|\frac{B}{A}\right| = \left|\frac{F_1}{F_2}\right|. \tag{2.5}$$

The Engquist-Majda (E-M) boundary conditions at $x = 0$ are designed to absorb the left moving wave by minimizing the reflection coefficient for small incident angles θ. In fact the p^{th} order E-M boundary condition yields

$$R = \left|\frac{B}{A}\right| = \left(\frac{1-\alpha}{1+\alpha}\right)^p,$$

where $\alpha = \sqrt{1-\beta^2} = \cos(\theta)$.

The methodology proposed by Engquist and Majda involves the approximation of the pseudo differential operator that annihilates the right moving wave (at the boundary $x = 0$)

$$\frac{d}{dx} - i\omega\sqrt{1-\beta^2}$$

in powers of β^2, and translating the polynomial in β into derivatives. The first order approximation $\sqrt{1-\beta^2} \sim 1$ yields the condition

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) U|_{x=0} \quad = \quad 0, \tag{2.6}$$

yielding the reflection coefficient $\frac{1-\alpha}{1+\alpha}$. Higher expansions (including Pade expansions) lead to higher order BC. Note, though, that the fourth approximation presented in their paper is not really fourth order but only a third order one with reflection coefficient of the form $R = \left(\frac{1-\alpha}{1+\alpha}\right)^3 \frac{3+\alpha}{3-\alpha}$, which might explain why it is not well posed. In fact the fourth order method base on fourth order derivatives is well posed.

Higdon [**8**] showed that the p^{th} order Engquist Majda absorbing boundary condition can be written in the following simple form (this is the same in three dimensions)

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^p U|_{x=0} \quad = \quad 0. \tag{2.7}$$

This result can be shown by an extremely simple argument. The result of applying the boundary operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial t}$ to the left moving wave yields $F_1 = 1 - \alpha$, and to the right moving wave yields $F_2 = 1 + \alpha$. Thus we get the reflection coefficient

$$R = \left(\frac{1-\alpha}{1+\alpha}\right)^p. \tag{2.8}$$

We will show that the most general p^{th} order absorbing boundary condition is equivalent to (2.7), and in particular those obtained by Engquist and Majda. This proves the Higdon result in a more general framework.

Although the argument presented above proves our result, we still want to present the methodology that brought about this condition for future reference.

II.1. The Derivation of the First and Second Order EM condition. In order to illustrate our methodology, we will discuss the first and second order EM condition.

The most general first order BC involves linear combinations of the first order derivatives:

$$\mathcal{L}U \quad = \quad a_0\frac{\partial U}{\partial t} + a_1\frac{\partial U}{\partial x} + a_2\frac{\partial U}{\partial y} = 0 \qquad \text{at} \quad x = 0. \tag{2.9}$$

Upon substituting the plane wave (2.2) into the boundary condition (2.9) we get

$$\begin{aligned} F_1(\beta) &= a_0 + a_1\sqrt{1-\beta^2} + a_2\beta, \\ F_2(\beta) &= a_0 - a_1\sqrt{1-\beta^2} + a_2\beta. \end{aligned}$$

In order for the reflection coefficient R, to be $O(\beta^2)$ we need to take

$$\begin{aligned} a_0 &= -a_1 \\ a_2 &= 0, \end{aligned}$$

Leading to

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) U|_{x=0} = 0.$$

For the second order boundary conditions we try the most general expression involving second derivatives:

$$a_0U_{tt} + a_1U_{tx} + a_2U_{ty} + a_3U_{xx} + a_4U_{xy} + a_5U_{yy} = 0. \tag{2.10}$$

We get

$$\begin{aligned} F_1 &= a_0 + a_1\sqrt{1-\beta^2} + a_2\beta + a_3(1-\beta^2) + a_4\beta\sqrt{1-\beta^2} + a_5\beta^2 \\ F_2 &= a_0 - a_1\sqrt{1-\beta^2} + a_2\beta + a_3(1-\beta^2) - a_4\beta\sqrt{1-\beta^2} + a_5\beta^2 \end{aligned}$$

Expanding the square root

$$\sqrt{1-\beta^2} \sim 1 - \frac{\beta^2}{2}$$

we get

$$F_1 \sim a_0 + a_1\left(1-\frac{\beta^2}{2}\right) + a_2\beta + a_3(1-\beta^2) - a_4\beta\left(1-\frac{\beta^2}{2}\right) + a_5\beta^2.$$

To get the correct order we must require:

$$\begin{aligned} a_0 + a_1 + a_3 &= 0, \\ a_2 + a_4 &= 0, \\ \frac{a_1}{2} + a_3 - a_5 &= 0, \\ a_4 &= 0. \end{aligned}$$

This leads to the following solution depending on two free parameters a and b

$$\begin{aligned} a_0 &= a \\ a_1 &= -2(a+b) \\ a_3 &= 2b + a \\ a_4 &= 0 \\ a_5 &= b \end{aligned}$$

and we get the most general second order B.C. at $x = 0$ of the form:

$$\mathcal{L}U = aU_{tt} - 2(a+b)U_{tx} + (2b+a)U_{xx} + bU_{yy} = 0.$$

Note that

$$\mathcal{L}U = (a+b)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^2 U - b\left(U_{tt} - U_{xx} - U_{yy}\right).$$

Since U is a solution to the wave equation the general BC is of the form (2.7) with $p = 2$.

II.2. The General Order E-M Method - Uniqueness. In this section we will prove the uniqueness of the boundary condition (2.7). We claim that the general p^{th} order boundary condition is of the form (2.7) with a possible addition of derivatives of the wave equation (2.1) itself.

Theorem 2.1 :

The most general absorbing boundary condition of order p at $x = 0$, is of the form

$$\left((\frac{\partial}{\partial t} - \frac{\partial}{\partial x})^p - (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y^2}) \sum_{i+j+k=p-2} a_{ijk} \frac{\partial}{\partial t}^i \frac{\partial}{\partial x}^j \frac{\partial}{\partial y}^k \right) U \quad = \quad 0 \tag{2.11}$$

Proof

We start by showing that (2.11) is of the correct order. Note that (2.11) yields:

$$\begin{aligned} F_1(\beta) &= (1-\alpha)^p = O(\theta^{2p}) \\ F_2(\beta) &= (1+\alpha)^p = O(1), \end{aligned}$$

and therefore the reflection coefficient R is given by

$$R = \left(\frac{1-\alpha}{1+\alpha} \right)^p = O(\theta^{2p})$$

i.e the correct order.

We turn now to the issue of uniqueness. The most general p^{th} order BC can be written as

$$\left(\sum_{i+j+k=p} b_{ijk} \frac{\partial}{\partial t}^i \frac{\partial}{\partial x}^j \frac{\partial}{\partial y}^k \right) U \quad = \quad 0 \tag{2.12}$$

This leads to the following expression for $F_1(\theta)$

$$F_1(\theta) = \sum_{i+j+k=p} b_{ijk} \cos^j(\theta) \sin^k(\theta),$$

which can be rewritten as

$$F_1(\theta) = \sum_{j+k \leq p} c_{jk} \cos^j(\theta) \sin^k(\theta).$$

Denote now

$$g(\theta) \quad = \quad F_1(\theta) - c_{p0}(1-\cos(\theta))^p \tag{2.13}$$

$$\quad = \quad \sum_{j=0}^{p-1} \sum_{k=0}^{p-j} c_{jk} \cos^j(\theta) \sin^k(\theta) \tag{2.14}$$

We want to verify that $g(\theta)$ is identically zero. In fact we can rewrite it as

$$g(\theta) = g_1(\theta) + \sin(\theta) g_2(\theta),$$

where g_1, g_2 are polynomials in $\cos(\theta)$ (and thus even functions of θ). Now the highest power of $\cos(\theta)$ in g_1 is $p-1$ so that the only way for it to be $O(\theta^{2p})$ is to vanish for any θ. We apply the same consideration to g_2. Thus

$$F_1(\theta) = (1-\cos(\theta))^p. \tag{2.15}$$

By taking $-\cos(\theta)$ instead of $\cos(\theta)$ in (2.15) we get

$$F_2(\theta) = (1+\cos(\theta))^p, \tag{2.16}$$

and indeed the reflection coefficient (2.8). The only boundary conditions that produce these F_1 and F_2 are (2.11).

II.3. Summary. We have shown that the general p^{th} order absorbing boundary conditions for the wave equation (2.2) is given by (2.11) and that it is unique up to addition of derivatives of the original wave equation. The analysis has been carried out explicitly for the two dimensional case, however the extension to an arbitrary number of dimensions is straightforward.

III. The 3-D Maxwell's Equations

We discuss, in this Section, the application of the methodology, presented in the last Section to the construction p^{th} order absorbing boundary conditions for the Maxwell's equations in three space dimensions.

The 3-D non dimensional Maxwell's equations in vacuum is of the form:

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} \tag{3.1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} \tag{3.2}$$

Where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively.

We consider the equations in Cartesian coordinates in the domain $0 \le x < \infty$, $-\infty < y, z < \infty$.

$$\begin{aligned} \frac{\partial E_1}{\partial t} &= \frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3}, \\ \frac{\partial E_2}{\partial t} &= \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1}, \\ \frac{\partial E_3}{\partial t} &= \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial H_1}{\partial t} &= -\left(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}\right) \\ \frac{\partial H_2}{\partial t} &= -\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right), \\ \frac{\partial H_3}{\partial t} &= -\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}\right). \end{aligned} \tag{3.4}$$

This can be written as

$$\frac{\partial \mathbf{W}}{\partial t} = A_1 \frac{\partial \mathbf{W}}{\partial x} + A_2 \frac{\partial \mathbf{W}}{\partial y} + A_3 \frac{\partial \mathbf{W}}{\partial z}$$

The number of boundary conditions to be specified at $x = 0$ corresponds to the number of negative eigenvalues of the coefficient matrix A_1. Those are $\{1, 1, 0, 0, -1, -1\}$ thus two conditions are needed at $x = 0$. In this section we derive the most general absorbing boundary conditions of the Engquist Majda type.

The general plane wave solution of the Maxwell's equations is given by

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = (A\mathbf{q_1} + B\mathbf{q_2})\, e^{i\omega(t+\alpha x+\beta y+\gamma z)} + (C\mathbf{q_3} + B\mathbf{q_4})\, e^{i\omega(t-\alpha x+\beta y+\gamma z)} \tag{3.5}$$

$$= A\mathbf{Q}_1 + B\mathbf{Q}_2 + C\mathbf{Q}_3 + D\mathbf{Q}_4 \tag{3.6}$$

Where

$$\alpha = \sqrt{1-\beta^2-\gamma^2}.$$

The vectors $\mathbf{q_i}$ are not unique, but a convenient choice is

$$\mathbf{q_1} = \begin{pmatrix} 0 \\ \gamma \\ -\beta \\ \beta^2+\gamma^2 \\ -\alpha\beta \\ -\alpha\gamma \end{pmatrix}, \quad \mathbf{q_2} = \begin{pmatrix} \beta^2+\gamma^2 \\ -\alpha\beta \\ -\alpha\gamma \\ 0 \\ -\gamma \\ -\beta\gamma \end{pmatrix}, \tag{3.7}$$

and

$$\mathbf{q_3}(\alpha,\beta,\gamma) = \mathbf{q_1}(-\alpha,\beta,\gamma),$$
$$\mathbf{q_4}(\alpha,\beta,\gamma) = \mathbf{q_2}(-\alpha,\beta,\gamma).$$

We note that the first two terms in (3.7), $\mathbf{Q}_1, \mathbf{Q}_2$, correspond to left moving waves. We would like to design two boundary conditions of the form

$$\begin{aligned} \mathcal{L}_1\mathbf{W} &= 0, \\ \mathcal{L}_2\mathbf{W} &= ,0 \end{aligned} \tag{3.8}$$

to minimize reflections of those waves from the boundary $x = 0$.

Upon substituting the plane wave (3.5) into the boundary conditions (3.8) we get a system of two equations in the unknowns A, B, C, D

$$AF_1 + BF_2 + CF_3 + DF_4 = 0, \tag{3.9}$$

$$AG_1 + BG_2 + CG_3 + DG_4 = 0. \tag{3.10}$$

Here $F_j = F_j(\alpha,\beta,\gamma)$ is the action of the first boundary operator on the wave $\mathbf{Q_j}$, and G_j is obtained as a result of the action of the second boundary condition i.e.

$$\begin{aligned} F_j &= e^{-i\omega(t-\beta y-\gamma z)}\mathcal{L}_1\mathbf{Q}_j, \\ G_j &= e^{-i\omega(t-\beta y-\gamma z)}\mathcal{L}_2\mathbf{Q}_j. \end{aligned} \tag{3.11}$$

The coefficients of the reflected waves, C, D are thus given by

$$C = A\frac{F_1G_4 - G_1F_4}{F_3G_4 - G_3F_4} + B\frac{F_2G_4 - G_2F_4}{F_3G_4 - G_3F_4} \tag{3.12}$$

$$= AR_{CA} + BR_{CB} \tag{3.13}$$

$$D = -A\frac{F_1G_3 - G_1F_3}{F_3G_4 - G_3F_4} - B\frac{F_2G_3 - G_2F_3}{F_3G_4 - G_3F_4} \tag{3.14}$$

$$= AR_{DA} + BR_{DB} \tag{3.15}$$

We have thus four reflection coefficients - two of them, R_{CA}, R_{DA}, measure the reflections of the first family of left moving waves, where as R_{CB}, R_{DB} measure the reflections of the second family. The p^{th} order Engquist-Majda boundary conditions are designed such that all reflection coefficients are $O((\beta^2+\gamma^2)^p)$. This can be done easily:

Theorem 3.1:
The Engquist-Majda boundary condition of order p is given by:

$$\mathcal{L}_1 \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} (E_3 - H_2) = 0 \tag{3.16}$$

$$\mathcal{L}_2 \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} (E_2 + H_3) = 0 \tag{3.17}$$

Proof:
We construct the F_j and the G_j as defined in (3.11) using the boundary operators in (3.16, 3.17), to obtain

$$\begin{aligned} F_1 = -\beta(1-\alpha)^p \quad &; \quad G_1 = \gamma(1-\alpha)^p, \\ F_2 = \gamma(1-\alpha)^p \quad &; \quad G_1 = \beta(1-\alpha)^p, \\ F_3 = -\beta(1+\alpha)^p \quad &; \quad G_3 = \gamma(1+\alpha)^p, \\ F_4 = \gamma(1+\alpha)^p \quad &; \quad G_1 = \beta(1+\alpha)^p. \end{aligned}$$

This leads to the following reflection coefficients

$$R_{CA} = R_{DB} = \left(\frac{1-\alpha}{1+\alpha}\right)^p \quad ; \quad R_{CB} = R_{DA} = 0. \tag{3.18}$$

Since $\alpha = \sqrt{1-\beta^2-\gamma^2}$ the reflection coefficients are all $O\left((\beta^2+\gamma^2)^p\right)$ Which proves the theorem.

□

Note that the functions $E_3 - H_2$ and $E_2 + H_3$ are the one dimensional characteristic variables in the x-direction. They emerge naturally when one looks for a first order boundary operators of the form

$$\begin{aligned} \sum_{i=1,3} a_{1i}E_i + \sum_{i=1,3} b_{1i}H_i &= 0, \\ \sum_{i=1,3} a_{2i}E_i + \sum_{i=1,3} b_{2i}H_i &= 0. \end{aligned}$$

Note also that the boundary conditions are not unique, in fact one can add combinations of the derivatives of each of the equations in (3.3).

We conclude this section by showing that the boundary conditions (3.16, 3.17) can be trivially generalized to a general smooth boundary. Denote by $\mathbf{n}$ the unit normal at the outer boundary of the computational domain. The characteristic vector in this direction $\mathbf{v}$ is given by [**7**] $\mathbf{v} = \mathbf{n} \times [\mathbf{E} + \mathbf{n} \times \mathbf{H}]$. The Engquist Majda boundary condition will then be

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \mathbf{n}}\right)^p \mathbf{v} = \mathbf{0}.$$

It should be mentioned that because $\mathbf{v} \cdot \mathbf{n} = 0$ we get only two independent conditions.

IV. Advective Acoustics

We consider the propagation of waves induced in a uniform two dimensional flow of a compressible fluid, by small perturbations. This phenomenon is described by the linearized Euler equations for the density perturbation, ρ, and the perturbation velocities, u and v which becomes after a standard nondimensionalization:

$$\begin{aligned} p_t + M\rho_x + u_x + v_y &= 0, \\ u_t + M u_x + \rho_x &= 0, \\ v_t + M v_x + \rho_y &= 0, \end{aligned}$$

where M is the Mach number.

This set of equations was transformed in [**1**] to

$$\begin{aligned} v_\tau + M v_\xi + \gamma \rho_\eta &= 0, \\ u_\tau + \rho_\xi - \frac{M}{\gamma} v_\eta &= 0, \\ \rho_\tau + u_\xi + \frac{v_\eta}{\gamma} &= 0. \end{aligned} \tag{4.1}$$

Here $\gamma = \sqrt{1 - M^2}$.

Note that for $M = 0, (\gamma = 1)$, the system (4.1) corresponds to the two dimensional electro-magnetic case. In the discussion below we will use x, y, t for ξ, η, τ.

The solution of the system (4.1) can be represented as a sum of three families of waves:

$$\mathbf{q} = e^{i\omega(t+\beta y)} \left[A\mathbf{q}_1 e^{-i\frac{\omega}{M}x} + B\mathbf{q}_2 e^{-i\omega\alpha x} + C\mathbf{q}_3 e^{i\omega\alpha x} \right] \tag{4.2}$$

$$= \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 \tag{4.3}$$

With

$$\mathbf{q}_1 = \begin{pmatrix} M \\ \frac{M^2\beta}{\gamma} \\ 0 \end{pmatrix} \quad \mathbf{q}_2 = \begin{pmatrix} -\beta\gamma \\ \alpha - M \\ 1 - M\alpha \end{pmatrix} \quad \mathbf{q}_3 = \begin{pmatrix} -\beta\gamma \\ -\alpha - M \\ 1 + M\alpha \end{pmatrix}, \tag{4.4}$$

and $\alpha = \sqrt{1 - \beta^2}$.

The entropy wave $\mathbf{Q}_1$ is a right moving wave, (since we assume a positive mean velocity U_0), $\mathbf{Q}_2$ is a right moving acoustic wave whereas $\mathbf{Q}_3$ is a left moving acoustic wave. When considering bounded domains, therefore, we have two different situations. In the first case. where the boundary is to the left of the domain, only the left moving acoustic wave ($\mathbf{Q}_3$) reaches the boundary from the domain and is reflected in the form of the two right moving waves, the entropy wave $\mathbf{Q}_1$, and the acoustic wave $\mathbf{Q}_2$. The situation is different in the case where the boundary is to the right of the domain: here two waves (the right moving ones, $\mathbf{Q}_1, \mathbf{Q}_2$) reach the boundary and are reflected in the form of the left moving acoustic wave $\mathbf{Q}_3$. We will discuss these two situations separately.

IV.1. Left Boundary. We consider the domain $0 \leq x < \infty$. In this case two boundary conditions have to be prescribed. We seek boundary conditions, at $x = 0$, that will minimize reflections of the left moving acoustic wave for small incident angles (small β). We denote the boundary conditions by

$$\mathcal{L}_1 \left(\begin{array}{c} v \\ u \\ \rho \end{array} \right) \Bigg|_{x=0} = 0 \tag{4.5}$$

$$\mathcal{L}_2 \left(\begin{array}{c} v \\ u \\ \rho \end{array} \right) \Bigg|_{x=0} = 0. \tag{4.6}$$

Upon substituting the solution (4.2)- (4.4) to the boundary condition (4.5, 4.6) one gets

$$AF_1(\beta) + BF_2(\beta) + CF_3(\beta) = 0, \tag{4.7}$$

$$AG_1(\beta) + BG_2(\beta) + CG_3(\beta) = 0. \tag{4.8}$$

Here $F_j = e^{-i\omega(t+\beta y)}\mathcal{L}_1 Q_j$ and $G_j = e^{-i\omega(t+\beta y)}\mathcal{L}_2 Q_j$, evaluated at $x = 0$.

We get the two reflection coefficients:

$$R_{AC} = \left| \frac{A}{C} \right| = \left| \frac{F_3 G_2 - G_3 F_2}{F_1 G_2 - G_1 F_2} \right| \tag{4.9}$$

$$R_{BC} = \left| \frac{B}{C} \right| = \left| \frac{F_3 G_1 - G_3 F_1}{F_2 G_1 - G_2 F_1} \right| . \tag{4.10}$$

The first reflection coefficient R_{AC} measures the component of the entropy wave $\mathbf{Q}_1$, the reflection of the left moving acoustic wave $\mathbf{Q}_3$ $(\mathbf{Q}_3 \to \mathbf{Q}_1)$, the second one R_{BC} measures the component of the right moving acoustic wave $\mathbf{Q}_2$ reflected from the boundary as the result of the left moving acoustic wave $\mathbf{Q}_3$ $(\mathbf{Q}_3 \to \mathbf{Q}_2)$.

We observe that, contrary to the elecromagnetics case, we can not get a first order condition using the variables themselves, we need to use the derivarives. To see that, consider the most general linear combination

$$\mathcal{L}_1 \left(\begin{array}{c} v \\ u \\ \rho \end{array} \right) \Bigg|_{x=0} = a_0 v + a_1 u + a_2 \rho = 0 \tag{4.11}$$

$$\mathcal{L}_2 \left(\begin{array}{c} v \\ u \\ \rho \end{array} \right) \Bigg|_{x=0} = b_0 v + b_1 u + b_2 \rho = 0. \tag{4.12}$$

This yield

$$F_3 = (1 + M)(a_2 - a_1) - a_0 \gamma \beta + (a_1 - M a_2)(1 - \alpha), \tag{4.13}$$

$$G_3 = (1 + M)(b_2 - b_1) - b_0 \gamma \beta + (b_1 - M b_2)(1 - \alpha). \tag{4.14}$$

Clearly the lowest order term must vanish and thus

$$\begin{aligned} a_2 &= a_1, \\ b_2 &= b_1. \end{aligned}$$

To eliminate the next order in β we have to take $a_0 = b_0 = 0$, but this implies that $\mathcal{L}_2$ is identical to $\mathcal{L}_1$, so that one of the parameters a_0,b_0 must be different from zero. Hence we get that the boundary condition yield an error $O(\theta)$, that is it is of order $\frac{1}{2}$. Note that by choosing $a_1 = a_2 = 0, a_0 = 1$ and $b_0 = 0, b_1 = b_2 = 1$ we get the characteristic boundary conditions

$$v = 0 \tag{4.15}$$

$$u + \rho = 0 \tag{4.16}$$

We conclude that the characteristic boundary conditions are of order $\frac{1}{2}$ only.

The generalization of this type of boundary conditions is trivial when observing that the result of applying the operator $\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)$ to $\mathbf{Q}_3$ is multiplying the vector $\mathbf{q}_3$ by $1 - \alpha$. We can state:

Theorem 4.1:

Consider the acoustics equations (4.1) in the domain $0 \leq x < \infty$, then the boundary conditions

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} v(x, y, t)|_{x=0} = 0 \tag{4.17}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} (u(x, y, t) + \rho(x, y, t))|_{x=0} = 0 \tag{4.18}$$

are of order $p - \frac{1}{2}$ in the incident angle $\beta = \sin\theta$.

Proof:

Upon evaluating the F_j and G_j, $j = 1, ..., 3$ we get

$$\begin{aligned} R_{AC} &= \beta(1-\alpha)^{p-1} \frac{2\alpha\gamma(1-M)}{(1-\frac{1}{M})^{p-1}\left[M(1-M)(1+\alpha) + M^2\beta^2\right]} \\ R_{BC} &= (1-\alpha)^p \frac{M^2(1+\alpha) + M(1-M)}{(1+\alpha)^{p-1}\left[M(1-M)(1+\alpha) + M^2\beta^2\right]} \end{aligned}$$

Which proves the theorem. Note that the reflected right moving acoustic wave $\mathbf{Q}_2$ is of order p. However the reflected entropy wave, $\mathbf{Q}_1$ is of order $p - \frac{1}{2}$ only.

□

A better set of boundary conditions, one that yields p^{th} order by using derivatives of order $p - 1$ can be obtained for $p > 1$. We note that (4.18) yields $G_3 = (1 - \alpha)^p(1 - M)$, which is the correct order, so one has to find a better condition then (4.17). We start with the second order case, seeking a linear combination of the first derivative that will yield a second order reflection coefficient (error of β^4). To avoid nonuniqueness by the possibility of adding the acoustic system of equation (4.1), we do not include the temporal derivatives. Thus we seek

$$\mathcal{L}_1 \begin{pmatrix} v \\ u \\ \rho \end{pmatrix} = a_{11}\frac{\partial v}{\partial x} + a_{12}\frac{\partial v}{\partial y} a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} a_{31}\frac{\partial \rho}{\partial x} + a_{32}\frac{\partial \rho}{\partial y} = 0$$

Demanding second accuracy we get the condition:

$$\sqrt{1 - M^2}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - M\frac{\partial \rho}{\partial y} = 0. \tag{4.19}$$

The boundary conditions (4.19),(4.18) yield

$$F_1 = M - \frac{1}{M} - M\beta^2, \qquad F_2 = 0, \qquad F_3 = 0 \tag{4.20}$$

$$G_1 = \frac{M(M+1)}{\gamma}\beta, \quad G_2 = (1-M)(1+\alpha^2), \quad G_3 = (1-M)(1-\alpha)^2 \tag{4.21}$$

and therefore

$$R_{AC} = 0 \qquad R_{BC} = \left(\frac{1-\alpha}{1+\alpha}\right)^2. \tag{4.22}$$

The surprising fact is that only the acoustic wave $\mathbf{Q}_2$ is reflected into the domain and not the density wave.

The generalization to an arbitrary order is now straightforward:

Theorem 4.2:
Consider the acoustics equations (4.1) in the domain $0 \leq x < \infty$. Then the following set of boundary conditions ($p > 1$),

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-2} \left(\sqrt{1-M^2}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - M\frac{\partial \rho}{\partial y}\right)\Bigg|_{x=0} = 0 \tag{4.23}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} (u(x,y,t) + \rho(x,y,t))\Bigg|_{x=0} = 0 \tag{4.24}$$

is of order p in the incident angle $\beta = \sin\theta$. In fact, the only reflection is the right acoustic wave moving into the domain. There is no reflection in the form of the density wave.

□

We end this section by noting that, as before, the boundary conditions (4.23) and (4.24) can be modified by adding derivatives of the equations (4.1).

IV.2. Right boundary. We consider now the case $-\infty < x \leq 0$. Now two waves reach the boundary $x = 0$, one is the density wave $\mathbf{Q}_1$ and the other is the right moving acoustic wave $\mathbf{Q}_2$, and one wave is reflected from the boundary. We thus need to provide one boundary condition

$$\mathcal{L}\begin{pmatrix} v \\ u \\ \rho \end{pmatrix}\Bigg|_{x=0} = 0 \tag{4.25}$$

Upon substituting the solution (4.2)- (4.4) into the boundary condition (4.25) one gets

$$AF_1(\beta) + BF_2(\beta) + CF_3(\beta) = 0, \tag{4.26}$$

were, as before, we define $F_j = e^{-i\omega(t+\beta y)}\mathcal{L}_1 Q_j$.

We have now two reflection coefficients $R_{CA} = |\frac{F_1}{F_3}|$ and $R_{CB} = |\frac{F_2}{F_3}|$. The first, R_{CA}, measures the intensity of the reflected acoustic wave $\mathbf{Q}_3$, due to the incident density wave $\mathbf{Q}_1$. The second reflection coefficient R_{CB} measure the intensity of the reflected acoustic wave $\mathbf{Q}_3$ due to the incident acoustic wave $\mathbf{Q}_2$. For these coefficients to be small in the incident angle θ we need that both F_1 and F_3 will be small.

We start by seeking a boundary condition based on the variable themselves

$$(a_0 v + a_1 u + a_2 \rho)|_{x=0} = 0.$$

This yields

$$F_1 = a_0 M + a_1 \frac{M^2 \beta}{\gamma}, \tag{4.27}$$

$$F_2 = -\beta\gamma a_0 + (\alpha - M) a_1 + (1 - M\alpha) a_2, \tag{4.28}$$

$$F_3 = -\beta\gamma a_0 - (\alpha + M) a_1 + (1 + M\alpha) a_2, \tag{4.29}$$

For F_1 to tend to zero with β we need $a_0 = 0$, and for F_2 to be the same $a_1 + a_2 = 0$, we thus get the characteristic boundary condition

$$(u - \rho)|_{x=0} = 0, \tag{4.30}$$

with

$$R_{CA} = \frac{M^2}{\gamma(1+\alpha)(1+M)}\beta \quad R_{CB} = \frac{1-\alpha}{1+\alpha}. \tag{4.31}$$

Thus we get first order for the acoustic wave reflection but only half order for the density wave reflection. Moreover, the trick used before to increase the order of the boundary condition, namely applying the operator $\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$ to the boundary condition (4.30) is not successful here, since it leads to

$$R_{CA} = \left(1 - \frac{1}{M}\right) \frac{M^2}{\gamma(1+\alpha)(1+M)},$$

We look, therefore, for combinations of the derivatives of the primitive variables

in order to gain higher order. This task turns out to be impossible - we can not get higher than first order in the reflection coefficient R_{CB} of the acoustic wave, however we can eliminate completely the reflection in the form of density wave. The boundary condition is

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \rho = 0, \tag{4.32}$$

yielding

$$F_{CA} = 0 \quad F_{CB} = \frac{1-\alpha}{1+\alpha}.$$

It is not surprising that there is no density wave reflection : the boundary condition (4.32) uses only the variable ρ which does not appear in $\mathbf{Q}_1$. This boundary condition can be generalized:

Theorem 4.3

Consider the system (4.1) in the domain $-\infty < x \leq 0$. Then the boundary condition

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^p \rho = 0, \tag{4.33}$$

is of order p. Moreover there is no density wave reflection:

$$R_{CA} = 0, \tag{4.34}$$

$$R_{CB} = \left(\frac{1-\alpha}{1+\alpha}\right)^p \tag{4.35}$$

□

As before any combination of the derivative of (4.1) can be added to (4.33). We also note that the pth order derivative is needed to get a p^{th} order BC, in contrast to the case of left boundary, in which one could get p^{th} order with $p-1$ derivatives, see (4.23), (4.24).

V. Well Posedness

We will discuss here the wellposedness of the Engquist Majda boundary conditions for the three systems of equations discussed in the former sections. To do that we utilize the Kreiss theory (see [**9**], [**2**], [**4**]). We find out that, as pointed out by Gustafsson (see [**3**]) all the three cases discussed above allow one generalized eigenvalue. This is something inherent in the E-M conditions. The meaning of this generalized eigenvalue is not clear. Gustafsson argues that it does not cause illposedness, but rather growth in time and degradation of accuracy when the equations are being solved numerically. However many other researchers (see the excellent review paper [**5**]) report using successfully E-M like boundary conditions up to very high order, so it seems that the particular numerical application of the boundary condition is important. We plan to discuss this matter in detail in future work, hence the following Section is concern with the analysis of the methods modulo this eigenvalue.

V.1. The Wave Equation. To apply the Kreiss theory to (2.7) we seek a solution of the form

$$u(x,y,t) = e^{St}e^{Rx}e^{i\nu y} \tag{5.1}$$

$$S^2 = R^2 - \nu^2 \tag{5.2}$$

Where ν is real.

If a solution, satisfying the boundary conditions, with Real$S > 0$ and iReal$R < 0$ exists, than S is called an eigenvalue and the problem is not well posed. The case Real$S = 0$ and Real$R = 0$ is called generalized eigenvalue if a perturbation of S, $\delta S > 0$ corresponds to $\delta R < 0$ in (5.2).

Substituting the solution (5.1) in the boundary condition (2.7) we get

$$(S-R)^p = 0,$$

So $S = R$, and from (5.2) $S = R = \nu = 0$. Perturbing $S = 0$ by $\delta S > 0$ can yield $\delta R = -\delta S$ and hence $S = 0$ is a generalized eigenvalue. However there is no other generalized eigenvalue or eigenvalue. We speculate that this generalized eigenvalue is a manifestation of the fact that the solution to the problem (2.1) with the absorbing boundary condition(2.7) is not unique. In fact consider

$$\begin{aligned} U_{tt} &= U_{xx} \qquad 0 \leq x \leq \infty \\ U(x,0) = 0 & \qquad U_t(x,0) = 0 \\ \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^p U|_{x=0} &= 0. \end{aligned}$$

This equation has in addition to the trivial solution also the solution

$$U = \begin{cases} 0 & t \leq x \\ (t-x)^{p-1} & t > x \end{cases} \tag{5.3}$$

V.2. Maxwell's Equations. Consider now the Maxwell's equations (3.3) with the Engquist Majda absorbing boundary conditions (3.16,3.17). For illposedness we look for solutions of the form

$$\mathbf{U} = e^{St} e^{i\mu y} e^{i\nu z} \mathbf{U}_0(x), \tag{5.4}$$

with Real$S > 0$,μ and ν real, and $\mathbf{U}_(x) \in L^2[0,\infty]$.

Substituting (5.4) into (3.3) we get

$$\mathbf{U} = e^{St} e^{i\mu y} e^{i\nu z} e^{Rx} \left[A \begin{pmatrix} 0 \\ i\nu S \\ -i\mu S \\ -(\mu^2+\nu^2) \\ -i\mu R \\ -i\nu R \end{pmatrix} + B \begin{pmatrix} -(\mu^2+\nu^2) \\ -i\mu R \\ -i\nu R \\ 0 \\ -i\nu S \\ i\mu S \end{pmatrix} \right] \tag{5.5}$$

with $R = -\sqrt{S^2+\mu^2+\nu^2}$.

We substitute (5.5) in (3.16, 3.17) to get

$$\begin{aligned} (R-S)^p \left[Ai\mu + B(-i\nu)\right] &= 0 \\ (R-S)^p \left[A(-i\nu) + B(-i\mu)\right] &= 0. \end{aligned}$$

Thus for a nontrivial solution we get

$$(R-S)^2 p(\mu^2+\nu^2) = 0 \tag{5.6}$$

Again we have a generalized eigenvalue $S = 0$.

V.3. Acoustics.

V.3.1. *Left Boundary.* We consider the Euler equations (4.1) in the domain $0 \leq x < \infty$, and discuss the well posedness of the absorbing boundary conditions (4.17,4.18). Again we look for a solution of the form

$$e^{St} e^{i\mu y} \mathbf{U}_0(x)$$

Where Real$S > 0$, μ is real and $\mathbf{U}_0$ is in $L^2[0,\infty]$.

Such a solution for the differential equations (4.1) is given by

$$\begin{pmatrix} v \\ u \\ \rho \end{pmatrix} = e^{St} e^{i\mu y} \left[A \begin{pmatrix} S \\ \frac{iM\mu}{\gamma} \\ 0 \end{pmatrix} e^{-\frac{S}{M}x} + B \begin{pmatrix} -i\gamma\mu \\ -R-MS \\ S+MR \end{pmatrix} e^{Rx} \right], \tag{5.7}$$

where R is a solution of $R^2 = S^2 + \mu^2$ and Real$R < 0$. Note that in (5.7) it is implicitly assumed that $R \neq -\frac{S}{M}$.

Substituting the solution (5.7) into the boundary conditions (4.17,4.18) we get the following algebraic system:

$$\begin{aligned} AS\left(S+\frac{S}{M}\right)^{p-1}+B(S-R)^{p-1}(-i\gamma\mu) &= 0, \\ A\left(S+\frac{S}{M}\right)^{p-1}+B(S-R)^{p}(1-M) &= 0. \end{aligned}$$

Thus, for a nontrivial solution, the Kreiss determinant

$$\left(S+\frac{S}{M}\right)^{p-1}(S-R)^{p}(S+MR)=0.$$

has to vanish.

Since $S+MR\neq 0$ we have to consider two pairs $S=R=0$ which is as before a generalized eigenvalue. Another generalized eigenvalues is given (for $p>1$), $S=0$, $R=-\mu$.

We turn now to the boundary conditions (4.23, 4.24). The Kreiss determinant condition is now

$$\left(S+\frac{S}{M}\right)^{p-2}(R-S)^{p}(S^2-M^2R^2)=0$$

For $p=2$ the only generalized eigenvalue comes from $S=0=R$, for $p>2$ we have the pair $S=0$,$R=-|\mu|$. The meaning of those generalized eigenvalues is not clear and further investigation has to be carried out.

V.3.2. *Right Boundary.* Consider now the domain $-\infty<x\leq 0$, an L^2 solution for the equation (4.1) is

$$e^{St}e^{i\mu y}e^{Rx}\begin{pmatrix} i\gamma\mu \\ MS+R \\ -S-MR \end{pmatrix} \tag{5.8}$$

With

$$R^2=S^2+\mu^2$$

To check the wellposedness we substitute (5.8) in the boundary condition (4.33) to get:

$$(S+R)^p(S+MR)=0.$$

It is clear that the only generalized eigenvalue is $S=0=R$.

References

[1] S. Abarbanel, D. Gottlieb and J.S. Hesthaven, *Well-Posed Perfectly Matched Layers for Avective Acoustics*. Journal of Computational Physics, 154, 266-283, (1999).

[2] Bjorn Engquist and Andrew Majda, *Absorbing Boundary Conditions for the Numerical Evaluation of Waves*, Math. of Comp., Volume 31, No. 139 (1977) pp. 629-651.

[3] B. Gustafsson, *Boundary Conditions for Hyperbolic Systems* SIAM J. Sci Stat. Comput., Vol. 9, No. 5 (1988).

[4] B. Gustafsson, H.O. Kreiss and J. Oliger, *Time Dependent Problems and Difference Methods*, Wiley Interscience (1995).

[5] T. Hagstrom, *Radiation boundary conditions for the numerical simulations of waves*, Acta Numerica, 8:47-106,1999.

[6] T. hagstrom, *New results on absorbing layers and radiation boubdary conditions* to appear.

[7] W. Hall, to appear.

[8] Robert L. Higdon *Absorbing Boundary Conditions for Difference Approximations to the Muti-Dimensional Wave Equation*, Math of Comp, Volume 47, No.176 (1986) pp. 437-459.

[9] H.-O. Kreiss, *Initial Boundary Value Problems for Hyperbolic Systems*, Comm. Pure Appl. Math. 23, pp.277-298, (1970).

Appendix A

The boundary conditions derived in the Section IV are given in terms of the transform variables ξ, η and τ, see equation (4.1). In this Appendix we would like to express those boundary conditions in terms of the physical variables x, y, t. The transformation used in [**1**] is:

$$\begin{aligned} \xi &= x \\ \eta &= \sqrt{1-M^2}y = \gamma y \\ \tau &= Mx + \gamma^2 t. \end{aligned} \tag{0.1}$$

Note the the boundary $\xi = 0$ is at $x = 0$.

We first deal with the operator:

$$\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi}$$

In the physical variables this operator becomes

$$(1-M^2)\frac{\partial}{\partial t} + (M-1)\frac{\partial}{\partial x}$$

We start with the boundary conditions (4.17,4.18) applied at $x = 0$, they become

$$\left((M+1)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} v(x,y,t) = 0 \tag{0.2}$$

$$\left((M+1)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} (u(x,y,t) + \rho(x,y,t)) = 0. \tag{0.3}$$

In Section IV we recommended the boundary conditions (at $x = 0$) (4.23,4.24), which become in the physical space

$$\left((M+1)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-2} \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - M\frac{\partial \rho}{\partial y}\right] = 0 \tag{0.4}$$

$$\left((M+1)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^{p-1} (u(x,y,t) + \rho(x,y,t)) = 0. \tag{0.5}$$

We consider now the domain $-\infty < x \leq 0$. The boundary condition (4.33) becomes:

$$\left((1-M)\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^{p} \rho = 0. \tag{0.6}$$

Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912