MATH 556 - ASSIGNMENT 1 SOLUTIONS

1. For the discrete variables concerned

(a) As

$$\begin{aligned}
\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{(x+y)\phi^{x+y}}{x!y!} &= \sum_{x=0}^{\infty}\frac{\phi^x}{x!}\left\{\sum_{y=0}^{\infty}\frac{(x+y)\phi^y}{y!}\right\} = \sum_{x=0}^{\infty}\frac{\phi^x}{x!}\left\{x\sum_{y=0}^{\infty}\frac{\phi^y}{y!} + \sum_{y=1}^{\infty}\frac{\phi^y}{(y-1)!}\right\} \\
&= \sum_{x=0}^{\infty}\frac{\phi^x}{x!}\left\{x\sum_{y=0}^{\infty}\frac{\phi^y}{y!} + \phi\sum_{y=0}^{\infty}\frac{\phi^y}{y!}\right\} = \sum_{x=0}^{\infty}\frac{\phi^x}{x!}\left\{xe^{\phi}+\phi e^{\phi}\right\} \\
&= e^{\phi}\left\{\sum_{x=1}^{\infty}\frac{\phi^x}{(x-1)!} + \phi\sum_{x=0}^{\infty}\frac{\phi^x}{x!}\right\} = e^{\phi}\left\{\phi\sum_{x=0}^{\infty}\frac{\phi^x}{x!} + \phi\sum_{x=0}^{\infty}\frac{\phi^x}{x!}\right\} \\
&= e^{\phi}(\phi e^{\phi}+\phi e^{\phi}) = 2\phi e^{2\phi}
\end{aligned}$$

and the joint pdf must sum to 1, we have $c = e^{-2\phi}/(2\phi)$

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(b) Using similar arguments, for $x = 0, 1, 2, \ldots,$

$$f_X(x) = P[X=x] = \sum_{y=0}^{\infty} f_{X,Y}(x,y) = c\frac{\phi^x}{x!}\sum_{y=0}^{\infty}\frac{(x+y)\phi^{x+y}}{x!y!} = c\frac{\phi^x}{x!}(xe^{\phi}+\phi e^{\phi})$$

and hence

$$f_X(x) = \frac{\phi^x e^{-\phi}(x+\phi)}{2\phi x!} \qquad x = 0, 1, 2, \ldots$$

and zero otherwise. By symmetry of form, the marginal for Y is identical.

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(c) By direct calculation, for integer $r > 0$,

$$\begin{aligned}
P[X+Y=r] &= \sum_{x=0}^{\infty} P[X=x, Y=r-x] = \sum_{x=0}^{\infty} f_{X,Y}(x, r-x) \\
&= c\sum_{x=0}^{r}\frac{r\phi^r}{x!(r-x)!} = \frac{c\phi^r}{(r-1)!}\sum_{x=0}^{r}\frac{r!}{x!(r-x)!} = \frac{c\phi^r}{(r-1)!}2^r = \frac{(2\phi)^r e^{-2\phi}}{2\phi(r-1)!}
\end{aligned}$$

For $r = 0$, $P[X+Y=0] = P[X=0, Y=0] = 0$.

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(d) The expectation of X is given by

$$\begin{aligned}
E_{f_X}[X] &= \sum_{x=0}^{\infty} x f_X(x) = \sum_{x=0}^{\infty} x\frac{\phi^x e^{-\phi}(x+\phi)}{2\phi x!} = \frac{e^{-\phi}}{2\phi}\sum_{x=1}^{\infty}\frac{\phi^x(x+\phi)}{(x-1)!} \\
&= \frac{e^{-\phi}}{2\phi}\sum_{x=1}^{\infty}\frac{\phi^x((x-1)+(1+\phi))}{(x-1)!} = \frac{e^{-\phi}}{2\phi}\left\{\sum_{x=1}^{\infty}\frac{(x-1)\phi^x}{(x-1)!} + \sum_{x=1}^{\infty}\frac{(1+\phi)\phi^x)}{(x-1)!}\right\} \\
&= \frac{e^{-\phi}}{2\phi}\left\{\sum_{x=2}^{\infty}\frac{\phi^x}{(x-2)!} + \sum_{x=1}^{\infty}\frac{(1+\phi)\phi^x)}{(x-1)!}\right\} = \frac{e^{-\phi}}{2\phi}\left\{\phi^2\sum_{x=0}^{\infty}\frac{\phi^x}{x!} + (1+\phi)\phi\sum_{x=0}^{\infty}\frac{\phi^x}{x!}\right\} \\
&= \frac{e^{-\phi}}{2\phi}\left\{\phi^2 e^{\phi} + (1+\phi)\phi e^{\phi}\right\} = \frac{\phi^2+(1+\phi)\phi}{2\phi} = \phi + \frac{1}{2}
\end{aligned}$$

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2. By independence the full joint pdf for the random variables associated with A_1 and A_2 is

$$f_{R_1,T_1,R_2,T_2}(r_1,t_1,r_2,t_2) = \frac{r_1 r_2}{\pi^2} \qquad 0 \le t_1, t_2 < 2\pi, 0 < r_1, r_2 < 1.$$

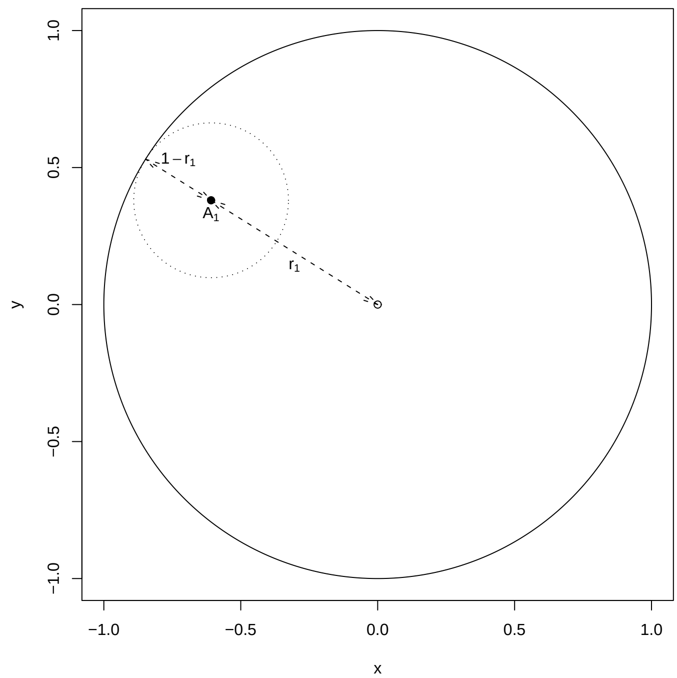
The probability of interest can be represented as an integral of this joint pdf over a region $\mathcal{C}$ defined by

$$\mathcal{C} = \{(r_1,t_1,r_2,t_2) : \text{ described circle is contained entirely within } \mathcal{D}\} \tag{1}$$

that is we wish to compute

$$\iiiint\limits_{\mathcal{C}} f_{R_1,T_1,R_2,T_2}(r_1,t_1,r_2,t_2) dr_2 dt_2 dr_1 dt_1.$$

There are many ways to formulate the solution; one simple one involves conditioning on the position of the point A_1, that is, conditioning on a specific (r_1,t_1) pair, then integrating out over these variables with respect to their joint density. Given $(R_1,T_1) = (r_1,t_1)$, we can deduce that the circle of interest lies within $\mathcal{D}$ if A_2 lies within a circle of radius $1-r_1$ centered at A_1; see diagram below. However,



(R_2,T_2) are drawn independently of (R_1,T_1), so given $(R_1,T_1) = (r_1,t_1)$, the probability that A_2 lies within a circle $\mathcal{C}_1$ of radius $1-r_1$ centered at A_1 is given by the integral

$$\iint\limits_{\mathcal{C}_1} f_{R_2,T_2}(r_2,t_2) dr_2 dt_2 = \int_0^{2\pi} \int_0^{1-r_1} \frac{r_2}{\pi} \, dr_2 dt_2 = (1-r_1)^2 \qquad 0 < r_1 < 1$$

Thus the integral in equation (1) can be computed by integrating this quantity over the distribution of (R_1,T_1); the probability of interest is thus

$$\iint\limits_{\mathcal{D}} (1-r_1)^2 f_{R_1,T_1}(r_1,t_1) dr_1 dt_1 = \int_0^{2\pi} \int_0^1 \frac{(1-r_1)^2 r_1}{\pi} \, dr_1 dt_1 = \frac{1}{6}$$

By using a change of variables from polar to Cartesian coordinates, it follows in a straightforward fashion that the distribution of the points being selected is uniform on the unit disc.

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3. (a) For $j = 0, 1, 2, \ldots$,

$$P[X = j] = \frac{P[X = j]}{P[X \geq j]} P[X \geq j] = \frac{P[(X = j) \cap (X \geq j)]}{P[X \geq j]} = P[X = j \mid X \geq j] P[X \geq j]$$

so therefore $p_j = h_j S_{j-1}$ where $S_i = P[X > i]$. Hence

$$\begin{aligned} j = 0 &: \quad p_0 = h_0 \\ j = 1 &: \quad p_1 = h_1 S_0 = h_1(1 - p_0) = h_1(1 - h_0) \\ j = 2 &: \quad p_2 = h_2 S_1 = h_2(1 - p_0 - p_1) = h_2(1 - h_0 - h_1(1 - h_0)) = h_2(1 - h_0)(1 - h_1) \end{aligned}$$

and in general

$$p_j = h_j \prod_{i=1}^{j-1} (1 - h_i)$$

(b) Directly from above

$$S_j = S_X(j) = P[X > j] = \prod_{i=1}^{j} (1 - h_i)$$

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