We now study multifactor designs, to assess the effects and interactions of several factors simultaneously.

We consider all possible combinations of

FACTOR A	with	a levels
FACTOR B	with	b levels
FACTOR C	with	c levels
	⋮	

to define the treatments in a factorial design.

Factorial Experiments

A complete factorial experiment is one in which every combination of a number of factors is utilized.

i.e. the number of treatments is equal to the total number of factor-level combinations.

We focus on two factor experiments

FACTOR A	with	a levels
FACTOR B	with	b levels

so there are ab treatments in total.

A two-way layout with $a = 3$ and $b = 5$.

	Factor B				
Factor A	1	2	3	4	5
1					
2					
3					

This design is very similar to the RBD, but now the second factor is not a blocking factor;

- that is, the *ab* treatment populations are constructed independently from the same base population, or from populations not necessarily believed to be systematically different.
- individuals from the same base population are assigned at random to one of the *ab* treatments.

In this design we can study the effect of Factor A and Factor B (**main effects**) as well as the **interaction** provided we have (balanced) replication.

We construct ANOVA F-tests based on the decomposition

$$SS = SST_A + SST_B + SSI_{AB} + SSE$$

- Sum of Squares for Treatments due to factor A ($\mathrm{SST_A}$)

$$SST_A = \sum_{i=1}^{a} br(\overline{x}_{i.} - \overline{x}_{..})^2$$

- Sum of Squares for Treatments due to factor B ($\mathrm{SST_B}$)

$$SST_B = \sum_{j=1}^{b} ar(\overline{x}_{.j} - \overline{x}_{..})^2$$

- Sum of Squares for Interaction ($\mathrm{SSI_{AB}}$)

$$SSI_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} r(\overline{x}_{ij} - \overline{x}_{i.} - \overline{x}_{.j} + \overline{x}_{..})^2$$

New notation:

- sample mean for Factor A level i

$$\overline{x}_{i.} = \frac{1}{br} \sum_{j=1}^{b} \sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, a$$

- sample mean for Factor B level j

$$\overline{x}_{.j} = \frac{1}{ar} \sum_{i=1}^{a} \sum_{t=1}^{r} x_{ijt} \qquad j = 1, \ldots, b$$

- sample mean for replicates in (i, j)th factor combination

$$\overline{x}_{ij} = \frac{1}{r} \sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, a, \; j = 1, \ldots, b$$

- overall sample mean

$$\overline{x}_{..} = \frac{1}{n} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{t=1}^{r} x_{ijt}$$

These allow computation of

$$SST_A, SST_B, SSI_{AB}, SS, SSE$$

$$MST_A, MST_B, MSI_{AB}, MSE$$

using the degrees of freedom identical to those in the RBD with replication.

Tests for

- significant effect for Factor A
- significant effect for Factor B
- significant interaction

will be carried out as before using an ANOVA table.

ANOVA Table

SOURCE	DF	SS	MS	F
FACTOR A	$a-1$	SST_A	MST_A	F_A
FACTOR B	$b-1$	SST_B	MST_B	F_B
INTERACTION	$(a-1)(b-1)$	SSI_{AB}	MSI_{AB}	F_{AB}
ERROR	$(n-ab)$	SSE	MSE	
TOTAL	$n-1$	SS		

If Factor A is not influential (H_0 specifying no difference between responses at different levels of factor A), then

$$F_A \sim \text{Fisher-F}(a-1, n-ab)$$

Similarly,

$$\begin{aligned} \text{No effect of Factor B} &: \quad F_B \sim \text{Fisher-F}(b-1, n-ab) \\ \text{No Interaction} &: \quad F_{AB} \sim \text{Fisher-F}((a-1)(b-1), n-ab) \end{aligned}$$

SEE EXAMPLES HANDOUT