NON-PARAMETRIC STATISTICS: ONE AND TWO SAMPLE TESTS EXAMPLES

EXAMPLE 1: Sign Test: Water Content Example

The following data are measurements of percentage water content of soil samples collected by two experimenters. We wish to test the hypothesis

$$H_0 \; : \; \eta = 9.0$$

for each experiment.

Experimenter 1:	$n = 10$	5.5	6.0	6.5	7.6	7.6	7.7	8.0	8.2	9.1	15.1	
Experimenter 2:	$n = 20$	5.6	6.1	6.3	6.3	6.5	6.6	7.0	7.5	7.9	8.0	8.0
		8.1	8.1	8.2	8.4	8.5	8.7	9.4	14.3	26.0		

To perform the test, we need tables of the Binomial distribution with $p = 1/2$. The individual probabilities are given by the formula

$$\Pr[X = x] = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \frac{1}{2^n} = \frac{n!}{x!(n-x)!} \frac{1}{2^n} \qquad x = 0, 1, \ldots, n$$

We test at the $\alpha = 0.05$ level. For the first experiment, with $n = 10$:

- For a test against the alternative hypothesis

  $$H_a \; : \; \eta > 9.0$$

  the test statistic is

  $$S = \text{Number of observations } \textbf{greater than } 9 \qquad \therefore \qquad S = 2$$

  and the p-value is

  $$p = \Pr[X \geq 2] = 1 - \Pr[X < 2] = 1 - \Pr[X = 0] - \Pr[X = 1] = 0.9893$$

  so we **do not** reject H_0 in favour of this H_a.

- For a test against the alternative hypothesis

  $$H_a \; : \; \eta < 9.0$$

  the test statistic is

  $$S = \text{Number of observations } \textbf{less than } 9 \qquad \therefore \qquad S = 8$$

  and the p-value is

  $$p = \Pr[X \geq 8] = \Pr[X = 8] + \Pr[X = 9] + \Pr[X = 10] = 0.0547$$

  so we **do not** reject H_0 in favour of this H_a.

- For a test against the alternative hypothesis

  $$H_a \; : \; \eta \neq 9.0$$

  the test statistic is

  $$S = \max\{S_1, S_2\} = \max\{2, 8\} = 8$$

  and the p-value is

  $$p = 2\Pr[X \geq 8] = 2(\Pr[X = 8] + \Pr[X = 9] + \Pr[X = 10]) = 0.1094$$

  so we **do not** reject H_0 in favour of this H_a.

For the second experiment, with $n = 20$:

- For a test against the alternative hypothesis $H_a : \eta > 9.0$, the test statistic is $S = 3$. The p-value is therefore

$$p = \Pr[X \geq 3] = 1 - \Pr[X < 3] = 1 - \Pr[X = 0] - \Pr[X = 1] - \Pr[X = 2] = 0.9998.$$

  so we **do not** reject H_0 in favour of this H_a.

- For a test against the alternative hypothesis $H_a : \eta < 9.0$, the test statistic $S = 17$. The p-value is therefore

$$p = \Pr[X \geq 17] = \Pr[X = 17] + \Pr[X = 18] + \Pr[X = 19] + \Pr[X = 20] = 0.0013.$$

  so we **do** reject H_0 in favour of this H_a.

- For a test against the alternative hypothesis $H_a : \eta \neq 9.0$, the test statistic is $S = \max\{S_1, S_2\} = \max\{3, 17\} = 17$. The p-value is therefore

$$p = 2\Pr[X \geq 17] = 2(\Pr[X = 17] + \Pr[X = 18] + \Pr[X = 19] + \Pr[X = 20]) = 0.0026.$$

  so we **do** reject H_0 in favour of this H_a.

This test can be implemented using SPSS, using the

Analyze → *Nonparametric Tests* → *Binomial*

pulldown menus. The test can be carried out by

(a) Selecting the *test variable* from the variables list
(b) Set the *Cut Point* equal to $\eta_0 = 9$.

A **two-sided** test is carried out at the $\alpha = 0.05$ level. The SPSS output is presented below for the two experiments in turn:

Binomial Test

		Category	N	Observed Prop.	Test Prop.	Exact Sig. (2-tailed)
% Water content	Group 1	<= 9	8	.80	.50	.109
	Group 2	> 9	2	.20		
	Total		10	1.00		

Binomial Test

		Category	N	Observed Prop.	Test Prop.	Exact Sig. (2-tailed)
% Water content	Group 1	<= 9	17	.85	.50	.003
	Group 2	> 9	3	.15		
	Total		20	1.00		

EXAMPLE 2: Mann-Whitney-Wilcoxon Test: Low Birthweight Example The birthweights (in grammes) of babies born to two groups of mothers A and B are displayed below: Thus $n_1 = 9, n_2 = 8$. From this

Group A :	$n = 9$	2164	2600	2184	2080	1820	2496	2184	2080	2184
Group B :	$n = 8$	2576	3224	2704	2912	2444	3120	2912	3848	

sample (which has ties, so we need to use average ranks), we find that

$$R_1 = 48 \qquad R_2 = 105$$

so that the two statistics are

$$\text{Wilcoxon} \quad W = R_2 = 105$$

$$\text{Mann-Whitney} \quad U = R_2 - \frac{n_2(n_2+1)}{2} = 105 - 36 = 69$$

- For the **small sample** test, from tables on p832 in McClave and Sincich, we find

  $$T_L = 51 \qquad T_U = 93$$

  Thus $W > 93$, so we

  Correction

  $$\begin{aligned} \textbf{Do not} \text{ reject } H_0 \text{ against } H_a &: \eta_1 > \eta_2 \quad \text{as} \quad W = R_2 > T_L \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 < \eta_2 \quad \text{as} \quad W = R_2 > T_U \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 \neq \eta_2 \quad \text{as} \quad W = R_2 > T_U \end{aligned}$$

  Note that the one-sided tests are carried out at $\alpha = 0.025$, the two sided test is carried out at $\alpha = 0.05$.

- For the **large sample** test, we find

  $$Z = \frac{U - \dfrac{n_1 n_2}{2}}{\sqrt{\dfrac{n_1 n_2 (n_1 + n_2 + 1)}{12}}} = 3.175$$

  Thus we

  Correction

  $$\begin{aligned} \textbf{Do not} \text{ reject } H_0 \text{ against } H_a &: \eta_1 > \eta_2 \quad \text{as} \quad Z > C_R = -1.645 \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 < \eta_2 \quad \text{as} \quad Z > C_R = 1.645 \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 \neq \eta_2 \quad \text{as} \quad Z > C_{R_2} = 1.960 \end{aligned}$$

  All tests are carried out at $\alpha = 0.05$.

This test can be implemented using SPSS, using the

Analyze → *Nonparametric Tests* → *Two Independent Samples*

pulldown menus. Note, however, that SPSS uses different rules for defining the test statistics, although it yields the same conclusions for a two-sided test.

EXAMPLE 3: Mann-Whitney-Wilcoxon Test: Treadmill Test Example
The treadmill stress test times (in seconds) of two groups of patients (disease group and healthy controls) are displayed below:

Disease :	$n = 10$	864	636	638	708	786	600	1320	750	594	750
Healthy :	$n = 8$	1014	684	810	990	840	978	1002	1110		

Thus $n_1 = 10, n_2 = 8$. From this sample (which has ties, so we need to use average ranks), we find that

$$R_1 = 70 \qquad R_2 = 101$$

so that the two statistics are

$$\begin{aligned} \text{Wilcoxon} \quad W &= R_2 = 101 \\ \text{Mann-Whitney} \quad U &= R_2 - \frac{n_2(n_2+1)}{2} = 101 - 36 = 65 \end{aligned}$$

- For the **small sample** test, from tables on p832 in McClave and Sincich, we find

$$T_L = 54 \qquad T_U = 98$$

Correction

Thus $W > 98$, so we

$$\begin{aligned} \textbf{Do not} \text{ reject } H_0 \text{ against } H_a &: \eta_1 > \eta_2 \quad \text{as} \quad W = R_2 > T_L \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 < \eta_2 \quad \text{as} \quad W = R_2 > T_U \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 \neq \eta_2 \quad \text{as} \quad W = R_2 > T_U \end{aligned}$$

Again, the one-sided tests are carried out at $\alpha = 0.025$, the two sided test is carried out at $\alpha = 0.05$.

- For the **large sample** test, we find

$$Z = \frac{U - \dfrac{n_1 n_2}{2}}{\sqrt{\dfrac{n_1 n_2 (n_1 + n_2 + 1)}{12}}} = 2.221$$

Correction

Thus we

$$\begin{aligned} \textbf{Do not} \text{ reject } H_0 \text{ against } H_a &: \eta_1 > \eta_2 \quad \text{as} \quad Z > C_R = -1.645 \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 < \eta_2 \quad \text{as} \quad Z > C_R = 1.645 \\ \textbf{Reject } H_0 \text{ against } H_a &: \eta_1 \neq \eta_2 \quad \text{as} \quad Z > C_{R_2} = 1.960 \end{aligned}$$

All tests are carried out at $\alpha = 0.05$.