556: MATHEMATICAL STATISTICS I

SOME NOTES ON CHARACTERISTIC FUNCTIONS

The characteristic function for a random variable X with pmf/pdf f_X is defined for $t \in \mathbb{R}$ as

$$\begin{aligned} C_X(t) = \mathbb{E}_{f_X}[e^{itX}] &= \mathbb{E}_{f_X}[\cos(tX) + i\sin(tX)] \\ &= \mathbb{E}_{f_X}[\cos(tX)] + i\mathbb{E}_{f_X}[\sin(tX)]. \end{aligned}$$

In general $C_X(t)$ is a complex-valued function. If X is discrete, taking values on $\mathbb{X} = \{x_1, x_2, \ldots\}$

$$\mathbb{E}_{f_X}[\cos(tX)] = \sum_{j=1}^{\infty} \cos(tx_j) f_X(x_j)$$

$$\mathbb{E}_{f_X}[\sin(tX)] = \sum_{j=1}^{\infty} \sin(tx_j) f_X(x_j)$$

Now,

$$\sum_{j=1}^{\infty} \cos(tx_j) f_X(x_j) \leq \left| \sum_{j=1}^{\infty} \cos(tx_j) f_X(x_j) \right| \leq \sum_{j=1}^{\infty} |\cos(tx_j)| f_X(x_j) \leq \sum_{j=1}^{\infty} f_X(x_j) = 1$$

with a similar result for $\sin$, so the two expectations are finite, so $C_X(t)$ exists. The same argument works for X continuous, where

$$C_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx = \int_{-\infty}^{\infty} \cos(tx) f_X(x)\, dx + i \int_{-\infty}^{\infty} \sin(tx) f_X(x)\, dx$$

EXAMPLE **Double-Exponential (or Laplace) distribution**

$$f_X(x) = \frac{1}{2} e^{-|x|} \qquad x \in \mathbb{R}$$

Then

$$C_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|}\, dx = \int_0^{\infty} \cos(tx) e^{-x}\, dx. \tag{1}$$

Integrating by parts we have

$$\begin{aligned} C_X(t) &= \left[-\cos(tx)e^{-x}\right]_0^{\infty} + \int_0^{\infty} t \sin(tx) e^{-x}\, dx \\ &= 1 + \left[-t\sin(tx)e^{-x}\right]_0^{\infty} - \int_0^{\infty} t^2 \cos(tx) e^{-x}\, dx \\ &= 1 - t^2 C_X(t) \end{aligned}$$

Therefore

$$C_X(t) = \frac{1}{1+t^2}$$

EXAMPLE **Normal distribution**

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad x \in \mathbb{R}$$

Then

$$C_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

Completing the square

$$C_X(t) \quad = \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-it)^2/2} e^{-t^2/2}\, dx.$$

Therefore

$$C_X(t) = e^{-t^2/2}.$$

The following results also hold:

- $C_X(t)$ is **continuous** for all t; this follows as $\cos$ and $\sin$ are continuous functions of x, and sums and integrals of continuous functions are also continuous. In fact, we can prove the stronger result that $C_X(t)$ is **uniformly continuous** on $\mathbb{R}$.
- $C_X(t)$ is **bounded in modulus** by 1, as

$$|C_X(t)| \le \mathbb{E}_{f_X}[|e^{itx}|] \le \mathbb{E}_{f_X}[1] \le 1$$

- The derivatives of $C_X(t)$ are not guaranteed to be finite; we can consider

$$C_X^{(r)}(t) = \frac{d^r}{dt^r} \{C_X(t)\}$$

  but this quantity may not be finite at any given t; if $r = 1$

$$C_X^{(1)}(t) = \mathbb{E}_{f_X}[-X \sin(tX)] + i\mathbb{E}_{f_X}[X \cos(tX)].$$

  but there is no guarantee that either expectation is finite. For example, for the Cauchy distribution

$$C_X(t) = e^{-|t|}$$

  which has unbounded derivative at $t = 0$.

INVERSION FORMULAE

A general inversion formulae in 1-D gives the method via which f_X or F_X can be computed from C_X.

- Let $\overline{F}_X(x)$ be defined by

$$\overline{F}_X(x) = \frac{1}{2}\left\{F_X(x) + \lim_{y \longrightarrow x^-} F_X(y)\right\}.$$

  Then for $a < b$

$$\overline{F}_X(b) - \overline{F}_X(a) = \lim_{T \longrightarrow \infty} \int_{-T}^{T} \left(\frac{e^{-iat} - e^{-ibt}}{2\pi it}\right) C_X(t)\, dt$$

- For an alternative statement, let a and $a + h$ for $h > 0$ be continuity points of F_X. Then

$$F_X(a+h) - F_X(a) = \lim_{T \longrightarrow \infty} \int_{-T}^{T} \left(\frac{1 - e^{-ith}}{2\pi it}\right) C_X(t)\, dt$$

In certain circumstances we may compute f_X from C_X more straightforwardly.

(I) If X is **discrete** taking values on the integers. Then

$$C_X(t) = \sum_{x=-\infty}^{\infty} e^{itx} f_X(x).$$

For integer j

$$\int_{-\pi}^{\pi} e^{i(j-x)t}\, dt = \begin{cases} 2\pi & \text{if } x = j \\ 0 & \text{if } x \neq j \end{cases}$$

Thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt} C_X(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt} \left\{ \sum_{j=-\infty}^{\infty} e^{itj} f_X(j) \right\} dt = \sum_{j=-\infty}^{\infty} \left\{ \int_{-\pi}^{\pi} e^{i(j-x)t}\, dt \right\} = f_X(x)$$

so we have the inversion formula: for $x \in \mathbb{Z}$

$$f_X(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt} C_X(t)\, dt.$$

(II) If X is **continuous** and **absolutely integrable**

$$\int_{-\infty}^{\infty} |C_X(t)|\, dt < \infty$$

then

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} C_X(t)\, dt$$

EXAMPLE Suppose that for $t \in \mathbb{R}$,

$$C_X(t) = e^{-|t|}.$$

Clearly this function is absolutely integrable, so we have

$$\begin{aligned} f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} e^{-|t|}\, dt &= \frac{1}{\pi}\int_{0}^{\infty} \cos(tx) e^{-t}\, dt \\ &= \frac{1}{\pi}\frac{1}{1+x^2} \end{aligned}$$

by the result in equation (1). Hence $X \sim$ Cauchy.

DIAGNOSING DISCRETE OR CONTINUOUS DISTRIBUTIONS

(I) If

$$\limsup_{|t| \longrightarrow \infty} |C_X(t)| = 1$$

then X is most likely a **discrete** random variable. Technically, X may also have a **singular** distribution: see, or example

www.math.mcgill.ca/~dstephens/556/Papers/Koopmans.pdf

but such distributions are rarely encountered in practice.

(II) If

$$\limsup_{|t|\longrightarrow\infty} |C_X(t)| = 0$$

then X is **continuous**; consequently, if

$$\lim_{|t|\longrightarrow\infty} |C_X(t)| = 0$$

then X is continuous.

INTERPRETING THE CHARACTERISTIC FUNCTION.
To get a further understanding of characteristic function, we consider the inversion formulae. For discrete random variables defined on the integers, we have

$$f_X(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt} C_X(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(xt) - i\sin(xt)]\, C_X(t)\, dt$$

One way to think about this integral is via a discrete approximation; fix

$$t_{j,N} = -\pi + \frac{2\pi j}{N} \qquad j = 0, 1, 2, \ldots, N$$

and write

$$f_X(x) \simeq \frac{1}{2\pi}\left\{\sum_{j=0}^{N} \cos(xt_{j,N}) C_X(t_{j,N}) - i\sum_{j=0}^{N} \sin(xt_{j,N}) C_X(t_{j,N})\right\}$$

(I) Suppose f_X is **degenerate** at x_0, that is,

$$f_X(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

Then by elementary calculations

$$C_X(t) = \cos(x_0 t) + i\sin(x_0 t)$$

so that

$$\mathrm{Re}(C_X(t)) = \cos(x_0 t) \qquad\qquad \mathrm{Im}(C_X(t)) = \sin(x_0 t)$$

that is, pure sinusoids with period $2\pi/x_0$.

(II) Suppose f_X is **discrete**, then as above

$$C_X(t) = \sum_{j=1}^{\infty} \cos(tx_j) f_X(x_j) + i\sum_{j=1}^{\infty} \sin(tx_j) f_X(x_j)$$

so that

$$\mathrm{Re}(C_X(t)) = \sum_{j=1}^{\infty} \cos(tx_j) f_X(x_j) \qquad\qquad \mathrm{Im}(C_X(t)) = \sum_{j=1}^{\infty} \sin(tx_j) f_X(x_j)$$

that is, a weighted sum of pure sinusoids with period $2\pi/x_1, 2\pi/x_2, \ldots$, with weights determined by f_X