Non-archimedean integration and special values of L-functions for elliptic curves over function fields

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Fratri minori Yupso,
(*etsi indefesse laboret*
ut vulgus ipsum putet
testadirapam)

Contents

Abstract

Let F be a global function field of characteristic p, ∞ a fixed place of F. The analytic uniformization of elliptic curves in characteristic p by means of Drinfeld modular curves is used to associate to an elliptic curve E over F with split multiplicative reduction at ∞ a measure μ_E on $\mathbb{P}^1(F_\infty)$. By choosing an appropriate embedding of a quadratic extension K/F into the matrix algebra $M_2(F)$, the measure μ_E is pushed forward to a measure on the p-adic group G, isomorphic to an anticyclotomic Galois group over the Hilbert class field of K. It is proven that integration of this measure yields the $\mathcal{L}$-invariant of E at ∞, when ∞ is split in K, and a Heegner point in E if ∞ is inert. An interpretation of these results in terms of special values of L-functions is suggested.

Résumé

Soit F un corps de fonctions de caractéristique p, ∞ une place de F. L'uniformisation des courbes elliptiques sur F par des courbes modulaires de Drinfeld est utilisée pour associer à une telle courbe elliptique E/F une mesure sur $\mathbb{P}^1(F_\infty)$. Après avoir choisi un plongement convenable d'une extension quadratique K/F dans $M_2(F)$, la mesure μ_E donne lieu à une mesure sur un groupe p-adique G, isomorphe au groupe de Galois d'une extension anticyclotomique du corps de classes de Hilbert de K. On démontre que certaines intégrales de cette mesure donnent l'invariant $\mathcal{L}$ de E si ∞ est déployé dans K, et donnent un point de Heegner si ∞ est inerte. Nous suggérons une interprétation de nos résultats en terme de valeurs spéciales de fonctions-L.

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Introduction

Let E be an elliptic curve over a global field F. The most prominent open problem in the arithmetic theory of elliptic curves is probably the conjecture of Birch and Swinnerton-Dyer, which, in its coarsest form, asserts the equality between the order of vanishing at $s = 1$ of the complex function $L(E/F, s)$ associated to E/F and the rank of the finitely generated abelian group $E(F)$. In a more refined version, the conjecture predicts that the leading coefficient of the Taylor expansion of $L(E/F, s)$ at 1 encodes various arithmetically meaningful quantities associated to E.

The number field case. Since the work of Iwasawa it has been a fruitful line of investigation to replace $L(E/F, s)$ by a function of a p-adic variable $L_p(E/F, s)$. Starting from the seventies, studies have been conducted on such p-adic L-functions $L_p(E, s)$ associated to modular elliptic curves $E/\mathbb{Q}$ (by work of Mazur, Manin and many others; see e.g. [**35**]). These functions have been constructed as Mellin transforms of modular forms (more precisely, as Mellin transforms of measures obtained from modular forms via modular symbols); they interpolate special values of the classical L-functions. On this basis, p-adic versions of the Birch and Swinnerton-Dyer conjecture were formulated by Mazur, Tate and Teitelbaum ([**36**]). The most noteworthy novelty they found is the fact that, when E has split multiplicative reduction at p, the p-adic L-function always vanishes at 1: in that case, as conjectured in [**36**] and proven in [**27**], the relation

$$L_p'(E, 1) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)} L_{alg}(E, 1) \tag{1}$$

holds. (Here q_E is the Tate period of E at p, $\log_p$ a p-adic logarithm and L_{alg} a suitable normalization of $L_{\mathbb{C}}$.) The coefficient on the right-hand side of (1) is called the $\mathcal{L}$-invariant $\mathcal{L}_p(E)$.

A different, purely p-adic, approach to the definition of a p-adic L-function was suggested by Schneider in [**41**]. He considers the p-adic uniformization of a modular elliptic curve $E/\mathbb{Q}$ by means of a Shimura curve: then the rigid analytic automorphic form corresponding to the canonical differential on E yields a harmonic cocycle on a certain tree T, which can be viewed as a measure on the set of ends of T; the latter is identified with $\mathbb{P}^1(\mathbb{Q}_p)$. Schneider's L-function is then the Mellin transform of the restriction of this measure to $\mathbb{Z}_p^*$. Notice that this construction requires that E has split multiplicative reduction at p and that it is a quotient of the Jacobian of a Shimura curve.

Schneider's approach has been used to attack the conjectures of [**36**] and to formulate analogous ones (see e.g. [**32**] - a work, however, flawed by some mistakes: cf. the remark in [**6**], §4.1).

Let $E/\mathbb{Q}$ be an elliptic curve and $K/\mathbb{Q}$ a quadratic imaginary field such that $(cond(E), disc(K)) = 1$. In [**3**] Bertolini and Darmon introduced a p-adic L-function $L_p(E/K, s)$ associated to the anticyclotomic $\mathbb{Z}_p$-tower above K and formulated conjectures similar to the ones of Mazur-Tate-Teitelbaum. In particular, for elliptic curves with split multiplicative reduction at p, they were later able to prove that when p is split in K an analogue of (1) holds ([**5**]), while when p is inert their L-function yields a Heegner point of $E(K)$, by evaluating the Tate map $\Phi_{\mathrm{Tate}} : \mathbb{C}_p^* \to E(\mathbb{C}_p)$ at $L_p'(E/K, 1)$ ([**4**]). In this last case, it follows by the work of Kolyvagin [**33**] that $E(K)$ has rank 1. The calculations of [**28**] and [**8**] suggest that $L_p(E/K, s)$ interpolates special values of twists of the complex L-function $L(E/K, s)$.

As first realized by Iovita (and independently by Spiess), the construction of Bertolini and Darmon is better understood as an anticyclotomic version of Schneider's L-function: this approach has been explained in [**2**] and used in [**6**] to study L-functions associated to modular forms of higher weight. The idea is to consider the Cerednik-Drinfeld p-adic uniformization of E by a Shimura curve to get a distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ and then to push forward this distribution to the anticyclotomic Galois group over K, by looking at the action of K^* on $\mathbb{P}^1(\mathbb{Q}_p)$ induced by an embedding of K into the appropriate quaternion algebra. Explicit computations of the special value $L_p(E/K, 1)$ are carried out thanks to Teitelbaum's Poisson formula ([**46**]).

The present work is an application of the methods of [**2**] to the function field setting.

The function field case. The Birch and Swinnerton-Dyer conjecture has been generalized to elliptic curves defined over a function field F of transcendence degree 1 over a finite field $\mathbb{F}_q$ ([**44**]). In this case the phenomenon is better understood: it is known that $L(E/F, s)$

is a rational function in q^{-s} (a polynomial if $j(E)$ is non-constant) and Tate has proved that the rank of $E(K)$ is at most equal to the vanishing order of the complex L-function at 1.

Besides, it has been known since the appearance of Drinfeld's great work [**11**] that in this setting all elliptic curves are modular, i.e., that they appear as factors of the Jacobian of some Drinfeld modular curve $\overline{M}_0(\mathfrak{n})$. More recently, Gekeler and Reversat ([**22**]) have provided an explicit analytic construction of the map $\overline{M}_0(\mathfrak{n}) \to E$.

The relation between measures and rigid analytic modular forms has been investigated by Teitelbaum, who, in [**47**], generalized to the function field case his own work [**46**].

A proposal for a p-adic, and characteristic p-valued, L-function in this context was made by Goss ([**26**]): he constructs Mellin transforms of measures attached to modular forms by the Schneider-Teitelbaum method (see also [**48**], p. 290-292, where a conjecture very similar to our main result, theorem 14 of the present work, is formulated).

The anticyclotomic approach of Bertolini and Darmon has inspired Pál's thesis ([**37**]), which carries over to function fields the results of [**4**], and suggests a geometric interpretation of Heegner points in terms of level structure at ∞.

This thesis. Let F be a function field of transcendence degree 1 over $\mathbb{F}_q$, with q a power of p, and fix a place ∞ of F. Fix a separable quadratic extension K of F, inert or split at ∞; denote by A (resp. B) the ring of functions in F (respectively K) regular away from ∞ (resp. the places of K above ∞). Consider an elliptic curve E/F, having conductor $\mathfrak{n}\infty$ and with split multiplicative reduction at ∞. (This is not a heavy restriction, since, unless E is defined over the constant field, $j(E)$ has a pole at some place of F and one can obtain the required properties by taking a finite extension F'/F where the conductor of E has the desired form.) In this situation, the results of [**22**] apply. For simplicity, E is understood to be the strong Weil curve in its isogeny class. We will also assume that $\mathfrak{n}$ is coprime to the discriminant of K/F.

Chapter 1 consists of preliminaries. In the first section a few basic facts about rigid analysis on the Drinfeld upper half-plane are recalled. Section 2 gives a short summary of some results from [**22**] needed in this work. Section 3 introduces the main tool for our investigations, the (multiplicative) integral over $\mathbb{P}^1(F_\infty)$. In section 4, a multiplicative version of Teitelbaum's Poisson formula is proven: this is the only part in the chapter with some claim to novelty (of course the result comes from [**47**], and the multiplicative approach was suggested in [**2**]; the idea of using van der Put's map r, which greatly simplifies the proof, seems to be original).

Chapter 2 contains the main result. In section 1, embeddings of K into $M_2(F)$ are considered: they allow to push forward to the group $G := (K \otimes_F F_\infty)^*/F_\infty^* B^*$ the measure μ_E on $\mathbb{P}^1(F_\infty)$ associated to E in the first chapter; the stress is put on the split case (i.e., when ∞ splits in K). In section 2, some class field theory is recalled and used to identify G with an "anticyclotomic" Galois group. Finally, in section 3, an explicit relation between the integral $\int_G g d\mu_E$ and the Tate period of E at ∞ is proven to hold in the split case (theorem 14); this result should be regarded as an analogue of equation (1).

In chapter 3, a brief recall of the interpretation of non-archimedean measures in terms of the Iwasawa algebra allows to define $\int_G g d\mu_E$ as special value of an L-function. A naive attempt, still very tentative, to construct such a function is explained in the second section.

Finally, chapter 4 sketches how in the inert case our techniques can be used to construct Heegner points; this was already done by Pál, in much more detail, but following the lines of [**4**], in which Schneider's construction and the Poisson formula of Teitelbaum are implicit rather than explicit. In the last section of the chapter, worked out between submission and defense, a slightly different technique of p-adic integration is used to evaluate the Gekeler-Reversat theta functions at cusps.

The assumption is made throughout the thesis that F is the rational function field $\mathbb{F}_q(T)$ and ∞ the usual place at ∞; it should be possible to generalize this to any global function field F and any place ∞.

Notations

For the convenience of the reader, we list some notations which will be used throughout this work:

- F is a function field, with constant field $\mathbb{F}_q$, q power of prime p;
- ∞ a fixed place of F;
- $A \subset F$ the ring of regular functions away from ∞;
- F_∞ the completion of F at ∞, O_∞ its ring of integers;
- K a separable quadratic extension of F, inert or split at ∞;
- B the integral closure of A in K;
- $\mathbf{C}_\infty$ the completion of an algebraic closure of F_∞;
- G the group $(K \otimes F_\infty)^*/F_\infty^* B^*$;
- $\Omega := \mathbb{P}^1(\mathbf{C}_\infty) - \mathbb{P}^1(F_\infty)$ Drinfeld's upper half-plane;
- $\mathcal{T}$ the Bruhat-Tits tree of $PGL_2(F_\infty)$, $\vec{\mathcal{E}}(\mathcal{T})$ its oriented edges;
- U_e set of ends of $\mathcal{T}$ exiting from e (also $U(e)$, according to notational convenience);
- $\Gamma := \Gamma_0(\mathfrak{n}) \subset GL_2(A)$, $\mathfrak{n}$ a (non-zero) ideal of A;
- R the A-subalgebra of $M_2(F)$ generated by Γ;
- $M_0(\mathfrak{n})$ the affine Drinfeld modular curve;
- $\overline{M}_0(\mathfrak{n})$ compactification of $M_0(\mathfrak{n})$ by cusps;
- $J_0(\mathfrak{n})$ the Jacobian of $\overline{M}_0(\mathfrak{n})$;
- ∂X the boundary of a subset X of a topological subspace Y: in particular, $\partial\Omega = \mathbb{P}^1(F_\infty)$.

In general, for v a place of F or K (the context should make clear which is the field involved), O_v will denote the respective ring of integers, with maximal ideal $\mathfrak{m}_v$, π_v a uniformizer at v in F or K and ν_v the corresponding valuation, with absolute value $|.|_v$ (normalized so that $|\pi_v|_v = \sharp(O_v/\mathfrak{m}_v)^{-1}$). In the case $v = \infty$, we will often just write $|.|$, ν, π. For $v \neq \infty$, put $\mathfrak{p}_v := A \cap \mathfrak{m}_v$.

For the time being, assume $F = \mathbb{F}_q(T)$, $A = \mathbb{F}_q[T]$.

CHAPTER 1

Prolegomena

1. Ω and $\mathcal{T}$

For a more detailed exposition of most of the contents of this section, see [**22**], section 1.

Drinfeld's upper half-plane is defined as $\Omega := \mathbb{P}^1(\mathbf{C}_\infty) - \mathbb{P}^1(F_\infty)$; it has a canonical rigid analytic structure. In particular, the algebra of globally holomorphic functions on Ω is obtained by choosing a covering system by increasing affinoids. To fix ideas, define the imaginary absolute value on Ω as the distance from F_∞:

$$|z|_i := \inf\{|z-x| \mid x \in F_\infty\}$$

and let

$$\mathcal{U}_n := \{z \in \Omega \mid q^{-n} \leq |z|_i, |z| \leq q^n\}.$$

A function $f : \mathcal{U}_n \to \mathbf{C}_\infty$ is holomorphic if it is the uniform limit of a sequence of rational functions without poles in $\mathcal{U}_n$. Then $\mathcal{O}_\Omega(\mathcal{U}_n)$ is a Banach algebra over $\mathbf{C}_\infty$, with the norm

$$\|f\|_{\mathcal{U}_n} := \sup\{|f(z)| \mid z \in \mathcal{U}_n\}$$

and $\mathcal{O}_\Omega(\Omega) := \varprojlim \mathcal{O}_\Omega(\mathcal{U}_n)$ is a Fréchet space (for more detail, cf. [**12**], [**7**]).

As usual, $GL_2(F_\infty)$ acts on Ω by Möbius transformations.

Let V be a 2-dimensional vector space over F_∞: the Bruhat-Tits tree $\mathcal{T}$ is defined having as vertices homothety classes of O_∞-lattices in V; the distance between two such vertices is given by the minimal length of a Jordan-Hölder sequence between two representatives (see [**42**], II, 1.1; another useful reference is [**34**], ch. 3, sections 4-6). Let $\mathcal{V}(\mathcal{T})$, $\mathcal{E}(\mathcal{T})$, $\vec{\mathcal{E}}(\mathcal{T})$ denote respectively the sets of vertices, edges and oriented edges of $\mathcal{T}$; given $e \in \vec{\mathcal{E}}(\mathcal{T})$, $\overline{e}$ will be its opposite, e^* the underlying non-oriented edge, $o(e)$ the origin and $t(e)$ the terminus.

Let $\mathcal{E}_\infty(\mathcal{T})$ be the set of ends of $\mathcal{T}$ (i.e., infinite half-lines, identified if equal outside some finite set of edges). There is a canonical bijection $\mathcal{E}_\infty(\mathcal{T}) \leftrightarrow \mathbb{P}(V) \simeq \mathbb{P}^1(F_\infty)$: given an end $\mathfrak{E}$, choose O_∞-lattices $\Lambda_i \subset V$

representing its vertices and contained one in each other with index q; then $\cap\Lambda_i$ is an O_∞-module of rank 1, which generates a line in $\mathbb{P}(V)$ (see e.g. [**19**], lemma 25; cf. also [**42**], p. 72). In particular, the ends exiting from an oriented edge e correspond to the points of a compact open U_e and in this way one obtains a basis $\{U_e\}_{e\in\vec{\mathcal{E}}(\mathcal{T})}$ for the topology of $\mathbb{P}^1(F_\infty)$.

The equality $\partial\Omega = \mathbb{P}^1(F_\infty)$ suggests a relation between Ω and $\mathcal{T}$. This is established by the building map $\lambda : \Omega \to \mathcal{T}$: its construction will be briefly recalled, together with some properties to be used in the following; for more detail, see [**22**] 1.5 or [**19**] §9.

Let $\mathcal{T}(\mathbb{R})$ be the realization of $\mathcal{T}$, obtained identifying edges in $\mathcal{E}(\mathcal{T})$ with copies of the segment $[0,1] \subset \mathbb{R}$. By a theorem of Goldman and Iwahori ([**24**], explained also in [**10**], ch. 3), points in $\mathcal{T}(\mathbb{R})$ classify norms on V: the basic idea is that a lattice in V can be seen as the unit ball of a certain norm.

Fix coordinates on V, so as to get an identification $V = F_\infty^2$; let $v_0 \in \mathcal{V}(\mathcal{T})$ be the vertex corresponding to the lattice $O_\infty \oplus O_\infty$. The building map $\lambda : \Omega \to \mathcal{T}(\mathbb{R})$ is defined associating to $z \in \Omega$ the norm $\nu_z(u,v) = |uz+v|$. In particular, one finds that $\lambda^{-1}(v_0) = \mathcal{U}_0$.

Following [**22**], define an action of $GL_2(F_\infty)$ on $\mathcal{T}$ so that λ is $GL_2(F_\infty)$-equivariant[1]:

$$\lambda(\gamma z) = \gamma\lambda(z)\ \forall\, \gamma \in GL_2(F_\infty).$$

Then, since $GL_2(F_\infty)$ acts transively on $\mathcal{V}(\mathcal{T})$, for any vertex v the set $\lambda^{-1}(v) = \lambda^{-1}(\gamma v_0) = \gamma\lambda^{-1}(v_0)$ is an affinoid isomorphic to $\mathcal{U}_0$.

The action of $GL_2(F_\infty)$ is well-behaved with respect to the identification $\mathcal{E}_\infty(\mathcal{T}) = \mathbb{P}^1(F_\infty)$: if the end $\mathfrak{E} = \{e_i\}$ corresponds to $a \in \mathbb{P}^1(F_\infty)$, then $\gamma\mathfrak{E} = \{\gamma e_i\}$ is the end associated to γa.[2]

Let $\mathcal{T}_n := \lambda(\mathcal{U}_n)$: it is a finite subtree.

Associate to $v \in \mathcal{V}(\mathcal{T})$ the spectral norm $\|.\|_v$ on $\mathcal{O}_\Omega(\lambda^{-1}(v))$:

$$\|f\|_v = \sup\{|f(z)| \mid z \in \lambda^{-1}(v)\}.$$

The seminorms $\{\|.\|_v\}_{v\in\mathcal{V}(\mathcal{T})}$ generate the same topology on $\mathcal{O}_\Omega(\Omega)$ as the covering $\{\mathcal{U}_n\}$: it is clear that $\max\{\|f\|_v \mid v \in \lambda(\mathcal{U}_n)\} \le \|f\|_{\mathcal{U}_n}$; vice versa, $\|f\|_{\mathcal{U}_n} = \max\{\|f\|_v \mid v \in \partial\mathcal{T}_n\}$ follows from the Mittag-Leffler decomposition on $\mathcal{O}_\Omega(\mathcal{U}_n)$ ([**12**], I.1.3). Notice also that spectral norms

[1] Notice that this is not the same action as, e.g., in [**42**], because to have a right action on the set of norms one needs a left action on V.

[2] Again, this requires taking a twist of the usual correspondence $\mathcal{E}_\infty(\mathcal{T}) \leftrightarrow \mathbb{P}(V)$ - namely, composing it with the map $z \mapsto -z^{-1}$ on $\mathbb{P}^1(F_\infty)$; see [**22**], 1.6.

are multiplicative: $\|fg\|_v = \|f\|_v \|g\|_v$.

For $\mathcal{A}$ an abelian group, let $\underline{H}(\mathcal{T}, \mathcal{A})$ denote the group of maps $\varphi : \vec{\mathcal{E}}(\mathcal{T}) \to \mathcal{A}$ satisfying $\varphi(\overline{e}) = -\varphi(e)$ and $\sum_{t(e)=v} \varphi(e) = 0$; such maps are called harmonic cochains (or cocycles).

Define a family of seminorms $\|.\|_{\mathcal{T}_n}$ on $\underline{H}(\mathcal{T}, \mathbb{Z})$ as

$$\|\varphi\|_{\mathcal{T}_n} := \max\{\varphi(e) \mid e^* \in \mathcal{E}(\mathcal{T}_n)\}.$$

The following result is due (mostly) to van der Put.

THEOREM 1. *The map*

$$r : \mathcal{O}^*_\Omega(\Omega) \to \underline{H}(\mathcal{T}, \mathbb{Z}), \quad r(f)(e) := \log_q \frac{\|f\|_{t(e)}}{\|f\|_{o(e)}}$$

is a surjective continuous group homomorphism with kernel $\mathbf{C}^*_\infty$. *(Here* $\log_q$ *is just the usual logarithm in base* q*.)*

Proofs are given in [**12**], I.8.9 and V.2, in [**38**], th. 2.1, and in [**19**], th. 32. Continuity is not mentioned in the references, but is obvious: for, if $\|f-g\|_v < \|g\|_v \ \forall\, v \in \mathcal{T}_n$, then (from multiplicativity of $\|.\|_v$) $r(f) = r(g)$ on $\mathcal{T}_n$. So r is a continuous homomorphism between "Fréchet spaces over $\mathbb{Z}$" (i.e., projective limit of topological spaces of functions, where the algebraic structure is just of abelian group).

Theorem 1 is based on a non-archimedean version of the residue theorem ([**12**] I.3.3, [**23**] II,§3 and III.1.18.4); for the relation between r and the residue map on holomorphic differentials on Ω, cf. [**22**] 1.8.

Given $a, b \in \mathbb{P}^1(F_\infty) = \mathcal{E}_\infty(\mathcal{T})$, let $A(a,b)$ denote the axis in $\mathcal{T}$ joining them (oriented from a to b). More generally, for $v, w \in \mathcal{V}(\mathcal{T})$, the notation $A(v,w)$ will be used for the geodesic (i.e., path without backtracking) from v to w.

Index the edges of $A(\infty, 0)$ as $\{e_n\}_{n\in\mathbb{Z}}$, with $t(e_n) = o(e_{n+1})$ and $t(e_0) = v_0$. Put also $v_i := t(e_i)$. Let $\mathcal{S}_i$ be the maximal subtree intersecting $A(\infty, 0)$ only in v_i: it consists of all those edges which can be joined to v_i by a path not passing through the principal axis $A(\infty, 0)$.

LEMMA 2. *Let* v *be a vertex in* $\mathcal{V}(\mathcal{S}_i)$*: then* $|z| \equiv q^{-i}$ *on* $\lambda^{-1}(v)$.

The proof is an easy consequence of the way the building map was defined; see e.g. [**16**], 1.8.

By the continuity of $|.|$, it follows that

$$\nu(t) = i \text{ for all } t \in U(e_i) - U(e_{i+1}).$$

In particular, $O^*_\infty = U(e_0) - U(e_1)$.

Given an axis $A(a,b)$, define its "characteristic function" $\chi_{A(a,b)} \in \underline{H}(\mathcal{T},\mathbb{Z})$ as

$$\chi_{A(a,b)}(e) := \begin{cases} 1 & \text{if } e \in A(a,b) \\ -1 & \text{if } \overline{e} \in A(a,b) \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma comes from [**16**], 1.14.

LEMMA 3. *Let $a, b \in \mathbb{P}^1(F_\infty)$: then* $r\left(\frac{z-a}{z-b}\right) = \chi_{A(a,b)}$.

PROOF. Since $\|f \circ \gamma\|_v = \|f\|_{\gamma v}$ for $\gamma \in GL_2(F_\infty)$ and this group acts transitively on the set of geodesics in $\mathcal{T}$ (that is, the set of ordered pairs (a,b), a and b distinct points of $\mathbb{P}^1(F_\infty)$), one is reduced to check that $r(z) = \chi_{A(0,\infty)}$; this is trivial from the fact that $\|z\|_v$ changes only along the axis $A(\infty, 0)$, as follows from lemma 2. □

The reader is reminded that $\gamma \in GL_2(F_\infty)$ is said to be hyperbolic if its eigenvalues λ_1, λ_2 are distinct in absolute value. Such an element has two fixed points a, b in $\partial\Omega$ and it acts on $\mathcal{T}$ by shifting of $\nu(\lambda_1) - \nu(\lambda_2)$ along the axis $A(a,b)$.

The prototype is $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ (which, by abuse of notation, will be called simply π). On Ω, π acts as a homothety, decreasing the absolute value of a factor q^{-1}. On $\mathcal{T}$, $\pi v_i = v_{i+1}$ for vertices in $A(\infty, 0)$; this completely determines the effect of π, since $GL_2(F_\infty)$ acts on the tree by isometries.

For more detail, see e.g. [**34**], 3, 6.1.

Define the level of a vertex v as its distance from the principal axis $A(\infty, 0)$; the level of an edge or of a path is then the minimum level of its vertices. By abuse of notation, for $z, z' \in \lambda^{-1}(\mathcal{V}(\mathcal{T}))$, $A(z, z')$ will denote the path from $\lambda(z)$ to $\lambda(z')$.

LEMMA 4. *Assume that $A(z_1, z_2)$ has level $n > 0$: then*

$$\left|\frac{z_1}{z_2} - 1\right| \leq |\pi^n|.$$

PROOF (SKETCH). Recall that $|z|$ is constant on $\lambda^{-1}(\mathcal{S}_i)$. Therefore, shifting by a power of π, one can assume that $A(z_1, z_2)$ is contained in $\mathcal{S}_0$. In these conditions, $|z_1 - z_2| \leq |\pi^n|$, as can be seen from the same reasoning as in lemma 2. □

2. Explicit analytic uniformization of E

The main reference for this section is [**22**]; see also [**18**], [**17**].

The Drinfeld modular curve $M_0(\mathfrak{n})$ is an affine algebraic curve whose set of $\mathbf{C}_\infty$-valued points can be described (in the case $A = \mathbb{F}_q[T]$) as the quotient of Ω by the action of Γ. The addition of a finite set of cusps yields a "compactified" curve $\overline{M}_0(\mathfrak{n})$.

In more general terms, $M_0(\mathfrak{n})$ is a (possibly coarse) moduli scheme representing the functor of rank 2 Drinfeld modules with $\mathfrak{n}$-level structure. These curves were introduced by Drinfeld in [**11**], where he proved (among other things) that an analogue of the Shimura-Taniyama-Weil conjecture holds for elliptic curves over global function fields.

Let E/F be an elliptic curve of conductor $\mathfrak{n}\infty$ and with split multiplicative reduction at ∞: by Drinfeld's theorems (together with results of Weil, Grothendieck, Jacquet-Langlands, Deligne and Zarhin), there is a non-constant morphism $\overline{M}_0(\mathfrak{n}) \to E$ defined over F. By means of rigid analytic geometry, Gekeler and Reversat ([**22**]) have explicitly described this map.

For simplicity, assume that E is the strong Weil curve in its isogeny class.

The presence of cusps for $M_0(\mathfrak{n})$ translates, in terms of the tree $\mathcal{T}$, into the fact that $\Gamma\backslash\mathcal{T}$ is the union of a finite graph with a finite set of half-lines.

Let $\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ be the group of cuspidal $\mathbb{Z}$-valued harmonic cocycles for Γ, i.e.,

$$\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma := \{\varphi \in \underline{H}(\mathcal{T},\mathbb{Z}) \mid \Gamma\text{-invariant finitely supported mod } \Gamma\}.$$

Elements in $\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ correspond to cusp forms for Γ ([**22**], section 4): finiteness of support means precisely that these cocycles come from forms vanishing at the cusps.

The action of Hecke operators on modular forms translates into an action on harmonic cocycles. By Drinfeld results in [**11**], the isogeny class of E corresponds to a cocycle $\varphi_E \in \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$, determined (up to a scalar) by being an eigenform for the Hecke operators, with the eigenvalues associated to E.

Let $\tilde{\Gamma} := \Gamma/\Gamma \cap Z(GL_2(F))$ ($Z(G)$ denotes the center of the group G) and $\overline{\Gamma} := \Gamma^{ab}/\mathrm{tor}(\Gamma^{ab})$; choose any vertex v of $\mathcal{T}$. One has a homomorphism $j : \overline{\Gamma} \to \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$, where $j(\alpha)(e)$ counts the number of times γe lies in $\mathrm{path}(v, \alpha v)$ as γ varies in $\tilde{\Gamma}$ ([**22**], section 3); this number is independent of v. In the case $A = \mathbb{F}_q[T]$, j is an isomorphism ([**22**], 6.4.5): put $\gamma_E := j^{-1}\varphi_E$.

Let w be any point in Ω. To $\gamma \in \Gamma$ one associates a function $u_\gamma : \Omega \to \mathbf{C}_\infty$ defined by

$$u_\gamma(z) := \prod_{\varepsilon \in \tilde{\Gamma}} \frac{z - \varepsilon w}{z - \varepsilon\gamma w}.$$

THEOREM 5. *The product u_γ converges to a holomorphic theta function on Ω; the value $u_\gamma(z)$ is independent of the choice of w. The automorphy factor $c_\gamma(\alpha) := \frac{u_\gamma(\alpha z)}{u_\gamma(z)}$ induces a group homomorphism $\overline{c} : \overline{\Gamma} \to Hom(\overline{\Gamma}, \mathbf{C}_\infty^*)$ by $\gamma \mapsto c_\gamma$. The map $\overline{\Gamma} \times \overline{\Gamma} \to \mathbf{C}_\infty^*$, defined by $(\alpha, \beta) \mapsto c_\alpha(\beta)$, is a symmetric bilinear pairing.*

All of this (and more) is proved in th. 5.4.1 of [**22**].

THEOREM 6. (**Gekeler-Reversat**) *The analytic uniformization of E is given by the commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \overline{\Gamma} \arrow[r, "\overline{c}"'] \arrow[d] & Hom(\overline{\Gamma}, \mathbf{C}_\infty^*) \arrow[r] \arrow[d, "ev_E"] & J_0(\mathfrak{n})(\mathbf{C}_\infty) \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \Lambda_E \arrow[r] & \mathbf{C}_\infty^* \arrow[r, "\Phi_{Tate}"] & E \arrow[r] & 0
\end{tikzcd}$$

Here ev_E is just the evaluation map at γ_E and $\Lambda_E := ev_E(\overline{c}(\overline{\Gamma}))$. In the case $A = \mathbb{F}_q[T]$, it is proved in [**17**] that Λ_E is equal to $Q^{\mathbb{Z}}$ for some Tate period Q.

To establish $\overline{\Gamma}\backslash\mathrm{Hom}(\overline{\Gamma}, \mathbf{C}_\infty^*) \cong J_0(\mathfrak{n})(\mathbf{C}_\infty)$ one constructs, by theta functions, an explicit Abel-Jacobi map

$$\Phi_{AJ} : \mathcal{D}iv^0(\Omega) \to J_0(\mathfrak{n})(\mathbf{C}_\infty),\ \Phi_{AJ}((a) - (b))(\gamma) = \frac{u_\gamma(a)}{u_\gamma(b)}$$

([**22**], section 7). The commutative diagram

$$\begin{tikzcd}
\mathcal{D}iv^0(\Omega) \arrow[r] \arrow[d] & \mathrm{Hom}(\overline{\Gamma}, \mathbf{C}_\infty^*) \arrow[r] \arrow[d] & \mathbf{C}_\infty^* \arrow[d, "\Phi_{\mathrm{Tate}}"] \\
\mathcal{D}iv^0(\overline{M}_0(\mathfrak{n})) \arrow[r] & J_0(\mathfrak{n})(\mathbf{C}_\infty) \arrow[r] & E
\end{tikzcd}$$

yields that the image in E of a divisor $(a) - (b)$ on Ω is

$$\Phi_{\mathrm{Tate}}\Phi_{AJ}((a) - (b))(\gamma_E) = \Phi_{\mathrm{Tate}}\left(\frac{u_{\gamma_E}(a)}{u_{\gamma_E}(b)}\right).$$

3. Integration

Let X be a topological space such that the compact open subsets form a basis for the topology. Define the group $\mathcal{M}(X, \mathbb{Z})$ of $\mathbb{Z}$-valued distributions on X as consisting of the finitely additive functions on the compact open subsets of X. In the case X is compact, distributions are measures and $\mathcal{M}_0(X, \mathbb{Z})$ is the kernel of $\mu \mapsto \mu(X)$.

Let C be a complete topological abelian group such that a basic system of neighbourhoods of the identity consists of open closed subgroups (e.g., $C = \mathbf{C}_\infty^*$). The group law on C will be written multiplicatively.

The set of continuous functions $\mathcal{C}(X, C)$ is a topological group with the compact open topology: a fundamental system of open subgroups is $\{f \mid f(\mathfrak{K}) \subset V\}$, where $\mathfrak{K}$ varies among compact subsets of X and V runs through a fundamental system of open subgroups of C.

From now on, assume that X is compact.

Definition. Given a continuous function $f : X \to C$, its integral is defined as:

$$\int_X f(t) d\mu(t) := \lim \prod_{U_n \in \mathcal{C}_\alpha} f(t_n)^{\mu(U_n)}, \tag{2}$$

where the limit is taken over the direct system of finite covers $\mathcal{C}_\alpha$ of X by compact open subsets and t_n is an arbitrary point in U_n.

PROPOSITION 7. *The limit in (2) exists and is independent of the choice of the t_n's. Besides,*

$$\int_X .d\mu : \mathcal{C}(X, C) \to C$$

is a continuous homomorphism.

PROOF (SKETCH). Let V be an open subgroup of C: the sets $f^{-1}(cV)$, $c \in C$, form an open covering of X, which can be refined to a covering by compact open subsets. Take a finite subcover $\mathcal{C}_{\alpha(V)}$: then for any $\alpha \geq \alpha(V)$, for any $U_n \in \mathcal{C}_\alpha$, having t and t' both in U_n implies that $f(t)f(t')^{-1}$ belongs to V. The existence of the limit follows.

It is immediate to see that the integral defines a group homomorphism $\mathcal{C}(X, C) \to C$. To check its continuity, just observe that if V is an open closed subgroup of C, integration over X sends $\{f \mid f(X) \subset V\}$ into V.

$\square$

Caveat. Essentially all integrals considered will be multiplicative: in the case $C = \mathbf{C}_\infty^*$, the reader is cautioned not to confuse $\int_X f d\mu$ with the usual integral taking values in the additive group $\mathbf{C}_\infty$, which will be denoted by $\int_X f d^+\mu$.

The open basis $\{U_e\}_{e\in\vec{\mathcal{E}}(\mathcal{T})}$ for the topology of $\partial\Omega = \mathbb{P}^1(F_\infty)$ is used to establish a bijection between the set of harmonic cocycles on $\mathcal{T}$, $\underline{H}(\mathcal{T},\mathbb{Z})$, and $\mathcal{M}_0(\partial\Omega,\mathbb{Z})$: to a cocycle φ one associates the measure μ_φ by $\mu_\varphi(U_e) := -\varphi(e)$. The negative sign is necessary to make theorem 8 consistent with Teitelbaum's results in [**47**], where the opposite orientation on ends of $\mathcal{T}$ seems to be used.

To compute integrals on $\partial\Omega$, it is convenient to fix a vertex v_0 in $\mathcal{T}$ and, for $e \in \vec{\mathcal{E}}(\mathcal{T})$ pointing away from v_0, let $l(e)$ denote the distance between v_0 and the origin of e. Then it is clear that

$$\int_{\partial\Omega} f(t)d\mu(t) = \lim_{n\to\infty} \prod_{l(e)=n} f(t_e)^{\mu(U_e)}$$

independently of the choice of v_0.

Let μ_E be the measure coming from φ_E.

Notice that $GL_2(F_\infty)$ acts naturally on $\mathcal{M}(\mathbb{P}^1(F_\infty),\mathbb{Z})$, as well as on $\underline{H}(\mathcal{T},\mathbb{Z})$, by $\gamma * \mu(U) := \mu(\gamma^{-1}U)$ (respectively $\gamma * \varphi(e) = \varphi(\gamma^{-1}e)$). It is obvious that $\int_{\gamma U} f d(\gamma * \mu) = \int_U (f \circ \gamma) d\mu$.

4. The Poisson formula

As in [**2**], one expects to recover u_{γ_E} as a multiplicative integral on $\partial\Omega$. Recall van der Put's map $r : \mathcal{O}_\Omega^*(\Omega) \to \underline{H}(\mathcal{T},\mathbb{Z})$. Let $\Theta_h(\Gamma)$ denote the group of holomorphic theta functions for Γ: then r induces an injection $\Theta_h(\Gamma)/\mathbf{C}_\infty^* \to \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$. (As a matter of fact, this is an isomorphism: cf. [**22**], 6.4.3.)

THEOREM 8. *Let $u \in \Theta_h(\Gamma)$ and define $\varphi = r(u)$. Choose $z_0 \in \Omega$: then for $z \in \Omega$*

$$u(z) = u(z_0) \int_{\partial\Omega} \frac{z-t}{z_0-t} d\mu_\varphi(t).$$

PROOF. The partial products $P_N(z) = \prod_{l(e)=N} (\frac{z-t_e}{z_0-t_e})^{\varphi(e)}$ converge uniformly on each $\mathcal{U}_n$: hence the integral on the right-hand side defines a nowhere vanishing rigid analytic function of z. To prove that it is u (up to a scalar $c \in \mathbf{C}_\infty^*$), it is enough to check the identity $\varphi = r\left(\int_{\partial\Omega} \frac{z-t}{z_0-t} d\mu_\varphi(t)\right)$. It is then immediate to see that $c = u(z_0)$.

By the continuity of r,

$$r\left(\int_{\partial\Omega}\frac{z-t}{z_0-t}d\mu_\varphi(t)\right)=\lim_{N\to\infty}\sum_{l(e)=N}\varphi(e)r\left(\frac{z-t_e}{z_0-t_e}\right).$$

By lemma 3, $r(z-t_e)=\chi_{A(t_e,\infty)}$. Now compute this cocycle on $e'\in\vec{\mathcal{E}}(\mathcal{T})$: without loss of generality, it can be assumed that e' points away from ∞. Then $\overline{e}'$ belongs to $A(t_e,\infty)$ if and only if t_e is in $U(e')$: thus

$$r\left(\int_{\partial\Omega}\frac{z-t}{z_0-t}d\mu_\varphi(t)\right)(e')=\lim_{N\to\infty}\sum_{l(e)=N}\varphi(e)\chi_{A(t_e,\infty)}(e')=$$

$$\lim_{N\to\infty}\sum_{\substack{t_e\in U(e')\\ l(e)=N}}-\varphi(e)=-\varphi(e').$$

The last passage is a direct consequence of the harmonicity of φ.

□

Remark. The above is a multiplicative version of Teitelbaum's result ([**47**]) that $\int_{\partial\Omega}\frac{1}{z-t}d^+\mu_\varphi(t)$ is the modular form corresponding to φ. This can be seen applying to both sides the differential logarithm operator, $d\log : f\mapsto\frac{f'}{f}$, which can be brought inside the integral by continuity. Note, however, that, as remarked in [**22**], in the characteristic p context the operator $d\log$ tends to be quite forgetful: hence theorem 8 says actually something more than in [**47**].

Besides, [**47**] takes a different approach - and proves much more - than the one here, suggested by [**2**]. It might be an interesting exercise to develop this multiplicative machinery to recover all of [**47**], thm. 11 (where the Poisson formula is proven for cusp forms of any weight $n\geq 2$) and maybe obtain a simpler proof of theorem 16 (the bijection between cusp forms and harmonic cocycles).

The following corollary is the form of theorem 8 to be used in future applications.

COROLLARY 9. *Let u,φ be as in theorem 8: then for $z,z'\in\Omega$*

$$\frac{u(z)}{u(z')}=\int_{\partial\Omega}\frac{z-t}{z'-t}d\mu_\varphi(t).$$

PROOF. Immediate, remembering that one deals with a multiplicative integral.

□

COROLLARY 10. *For $z \in \Omega$, one has*

$$u_{\gamma_E}(z) = u_{\gamma_E}(z_0) \int_{\partial\Omega} \frac{z-t}{z_0-t} d\mu_E(t).$$

PROOF. Just use $r(u_\gamma) = j(\gamma)$ ([**22**], thm. 5.6.1). $\square$

CHAPTER 2

De integralibus quibusdam

In the previous sections, a p-adic measure on $\partial\Omega$ was associated to E and used to recover geometric information about the elliptic curve (i.e., u_γ and hence the uniformizing map). In this chapter, the action of $GL_2(F_\infty)$ on $\partial\Omega$ will be exploited to transfer the measure to the profinite group G defined in the next section. It will be then proven that integration on G yields information about the arithmetic of E.

1. Embeddings

Let R be the A-algebra generated by Γ in $M_2(F)$ and K/F a separable quadratic extension, inert or split at ∞.

LEMMA 11. *Assume that* $(disc(K), \mathfrak{n}) = 1$. *Then there is an* F-*algebras homomorphism* $\Psi : K \to M_2(F)$ *such that* B *is sent into* R *if and only if all the primes dividing* $\mathfrak{n}$ *in* F *are split in* K. *The classes of these isomorphisms modulo conjugation by* Γ *are in bijection with* $\mathcal{P}ic(B)$.

PROOF. An embedding $\Psi : B \to M_2(A)$ induces an action of B on $A \oplus A$, which so becomes a B-module M, projective (since it is torsion free and finitely generated) and of rank 1. The fact that $\Psi(B)$ takes values in R means that $N = A \oplus \mathfrak{n}$ is a B-submodule of M. Thus the existence of Ψ implies the existence of a torsion B-module $M/N \simeq A/\mathfrak{n}$. Clearly M/N is cyclic, and its annihilator is contained in $\mathfrak{n}B$: that is, $M/N \simeq B/\tilde{\mathfrak{n}}$, with $\tilde{\mathfrak{n}}$ a divisor of $\mathfrak{n}B$. Let $\mathfrak{n} = \prod \mathfrak{p}^{\nu_p(\mathfrak{n})}$ be the prime decomposition: localizing, one is reduced to look for B-modules isomorphic to $A/\mathfrak{p}^{\nu_p(\mathfrak{n})}$. Let $\mathfrak{P}$ be a prime in B above $\mathfrak{p}$: if $\mathfrak{p}$ splits, then $A/\mathfrak{p}^i$ is isomorphic to $B/\mathfrak{P}^i$ for all i, while if $\mathfrak{p}$ is inert this never happens.

On the other side, projective B-modules of rank 1 are isomorphic to the fractional ideals of B. If all primes dividing $\mathfrak{n}$ split, let $\mathfrak{a}$ be any ideal of B: then (since A is PID) $\mathfrak{a} \simeq A \oplus A$, and, after choosing a basis, the natural B-action becomes an embedding $B \to M_2(A)$. Let $\mathfrak{n}_1 \subset B$ be an ideal of norm $\mathfrak{n}$, so that $A/\mathfrak{n} \simeq B/\mathfrak{n}_1$. Choosing $\mathfrak{a}$ coprime to $\mathfrak{n}_1$, one gets a surjection $\mathfrak{a} \oplus \mathfrak{n}_1 \to B$, from which $\mathfrak{a}/\mathfrak{a}\mathfrak{n}_1 \simeq A/\mathfrak{n}$: hence coordinates can be chosen so that $\mathfrak{a}\mathfrak{n}_1 = A \oplus \mathfrak{n}$, yielding an embedding of B in R. $\square$

It is still possible to find Ψ when $\mathfrak{n}$ has ramified prime factors $\mathfrak{p}$ with $\nu_{\mathfrak{p}}(\mathfrak{n}) = 1$: for also in this case (but not with higher powers) $A/\mathfrak{p} \simeq B/\mathfrak{P}$.

Choose an embedding Ψ as in lemma 11 and extend it to a map $K \otimes_F F_\infty \to M_2(F_\infty)$: this induces an action of $(K \otimes F_\infty)^*/F_\infty^*$ on $\partial\Omega$. Since K/F is separable, $\Psi(a)$ has two distinct eigenvalues $\forall\, a \in K - F$ and so there are 2 or 0 fixed points, according to whether ∞ is split or not in K.

Notice that measures on $\partial\Omega$ associated to Γ-invariant cocycles, such as μ_E, are invariant under the action of B^*.

By the S-unit theorem (which in this case follows from the finiteness of the Jacobian of the curve corresponding to K - see e.g. [**39**]), B^* is a finite group if ∞ is inert or ramified, while it is of rank one modulo torsion in the split case. If B^* is finite, then it is contained in the constant field of K; that is, $B^* = A^*$ unless K is the constant field extension.

For the rest of the chapter, assume that the infinite prime splits in K. Set

$$G := (K \otimes_F F_\infty)^*/F_\infty^* B^*.$$

Fix u a generator of the free part of B^*: then $\Psi(u)$ is hyperbolic. Let i_k be the immersions of K into F_∞, with $\nu(i_1(u)) > 0$; put

$$w := \frac{i_1(u)^2}{N_{K/F}(u)} \in F_\infty^*.$$

$K \otimes_F F_\infty$ is isomorphic to $F_\infty \times F_\infty$ by $\alpha \otimes \beta \mapsto (i_1(\alpha)\beta, i_2(\alpha)\beta)$; then $(K \otimes F_\infty)^* \simeq F_\infty^* \times F_\infty^* \to F_\infty^*$, $(x, y) \mapsto xy^{-1}$. The image of B^* is $w^{\mathbb{Z}}$. This yields an isomorphism between G and F_∞^*/B^*: let $\varrho : F_\infty^* \to G$ be the corresponding projection.

Stabilizers in $M_2(F_\infty)$ of pair of points of $\mathbb{P}^1(F_\infty)$ are 2-dimensional split F_∞-algebras: therefore Ψ is determined (up to Galois conjugation in K/F) by its fixed points; denote them as P_Ψ^-, P_Ψ^+ according to whether repulsive or attractive for $\Psi(u)$. Then $\Psi(B^*)$ acts on $\mathcal{T}$ as shift by $h := 2\nu(i_1(u))$ along $A(P_\Psi^-, P_\Psi^+)$. This allows one to define a "fundamental domain" $\mathcal{F}_\Psi$ for G: choose $e \in A(P_\Psi^-, P_\Psi^+)$ and take $\mathcal{F}_\Psi = U(e) - U(\Psi(u)e) \subset \mathbb{P}^1(F_\infty)$ (a more precise choice will be made later on).

The choice of a base point $P \in \partial\Omega - \{P_\Psi^-, P_\Psi^+\}$ gives an identification of $(K \otimes_F F_\infty)^*/F_\infty^*$ and $\partial\Omega - \{P_\Psi^-, P_\Psi^+\}$ by $a \mapsto \Psi(a)P$. This yields an isomorphism of topological spaces between $\mathcal{F}_\Psi$ and G: thus the restriction of a measure $\mu \in \mathcal{M}(\mathbb{P}^1(F_\infty), \mathbb{Z})$ to $\mathcal{F}_\Psi$ induces a measure (still denoted by μ) on G.

LEMMA 12. *For all measures* $\mu \in \mathcal{M}_0(\partial\Omega, \mathbb{Z})^\Gamma$, $\mu(\mathcal{F}_\Psi) = 0$.

PROOF. This follows immediately from $\Psi(B^*) \subset \Gamma$: for, if φ is the cocycle corresponding to μ, $\mu(\mathcal{F}_\Psi) = \varphi(e) - \varphi(\Psi(u)e)$. □

2. Galois interpretation

The goal of this section is to identify $(K \otimes_F F_\infty)^*/F_\infty^* B^*$ with a Galois group.

Let S be the set of places of K lying above ∞. Denote by $\mathbb{J}_K$ the group of idéles on K; K^*, O_v^*, K_v^* are embedded in $\mathbb{J}_K$ as usual. Put

$$O_S^* := \prod_{v \notin S} O_v^* \text{ and } K_S^* := \prod_{v \in S} K_v^*$$

and define $\mathbb{J}_S := O_S^* K_S^*$: then $\mathbb{J}_S \cap K^* = B^*$. Let K_{ab} be the maximal abelian extension of K, K'_{ab} the maximal abelian extension of K unramified at places outside S; following [**40**], define the Hilbert class field with respect to S, H, as the maximal unramified abelian extension in which all primes in S split completely. By class field theory (see e.g. [**45**]), Artin reciprocity identifies $\mathbb{J}_K/K^*$ with a dense subgroup of $\mathrm{Gal}(K_{ab}/K)$: K'_{ab} is the fixed field of O_S^*, while $\mathrm{Gal}(K_{ab}/H)$ is the completion of $K^*\mathbb{J}_S/K^* \simeq \mathbb{J}_S/B^*$. Therefore $\mathrm{Gal}(K'_{ab}/H)$ contains a dense subgroup isomorphic to $\mathbb{J}_S/O_S^* B^* \simeq K_S^*/B^*$. Besides, $\mathrm{Gal}(H/K) \simeq \mathbb{J}_K/K^*\mathbb{J}_S$ is canonically isomorphic to $\mathcal{P}ic(B)$, by sending an idéle into the corresponding divisor.

Note that K'_{ab}/F and H/F are still Galois. For, in a separable closure of F, K'_{ab} is the fixed field of (the closure of) the commutator subgroup of $\mathrm{Gal}(F_{sep}/K)$ and the group generated by inertia at places $v \not| \infty$; this group is normal, since conjugation by $\sigma \in \mathrm{Gal}(F_{sep}/F)$ just permutes the inertia groups not above ∞ and leaves invariant the commutator part. Analogously for H/F, adding to the picture the decomposition groups for primes above ∞.

Let τ be the generator of $\mathrm{Gal}(K/F)$; extend it to $\tilde{\tau} \in \mathrm{Gal}(K'_{ab}/F)$. Since $\mathrm{Gal}(K'_{ab}/K)$ is abelian, conjugation by $\tilde{\tau}$ is independent of the choice of the extension $\tilde{\tau}$: hence get an action (obviously continuous) of $\mathrm{Gal}(K/F)$ on $\mathrm{Gal}(K'_{ab}/K)$ and on $\mathrm{Gal}(K'_{ab}/H)$. On the other hand,

τ acts naturally on $\mathbb{J}_K$; the two actions are compatible with the reciprocity map: $rec(\tau a) = \tilde{\tau}\, rec(a)\, \tilde{\tau}^{-1}$. This is clear if $rec(a)$ is the Frobenius for some $\mathfrak{p}$; by Chebotarev, such are dense.

Define the "anticyclotomic" extension K_{ac}/H as the fixed field, in K'_{ab}, of the subgroup of $\mathrm{Gal}(K'_{ab}/H)$ corresponding to F_∞^*. Then $\mathrm{Gal}(K_{ac}/H)$ is isomorphic to $K_S^*/F_\infty^* B^*$ and τ acts on it as inversion: for $rec(a)rec(\tau a) = rec(N_{K/F}a) \in rec(F_\infty^*)$. It is clear that K_{ac}/H is the maximal subextension of K'_{ab}/H enjoying this property (which justifies the name anticyclotomic): for $F_\infty^* = N_{K/F}(K_S^*)$ has to act trivially on any such subfield. Note, however, that H/K is not anticyclotomic when $\mathcal{P}ic(A) \to \mathcal{P}ic(B)$ is not trivial.

It is clear from the above that $\mathrm{Gal}(K_{ac}/H) \simeq G$; if K/F splits at ∞, the two groups are isomorphic to $F_\infty^*/w^{\mathbb{Z}}$, as explained in the previous section.

3. Main results

To simplify future computations, change coordinates on $\partial\Omega = \mathbb{P}^1(F_\infty)$ by $\alpha \in PGL_2(F_\infty)$, determined by $0 = \alpha P_\Psi^+$, $\infty = \alpha P_\Psi^-$ and $\alpha P = 1$. Then one can identify $(K \otimes F_\infty)^*/F_\infty^*$ with F_∞^*, orbit of 1 under multiplication. In particular, $\gamma_w := \alpha\Psi(u)\alpha^{-1}$ acts as multiplication by w on $\mathbb{P}^1(\mathbf{C}_\infty)$ and as shift by $\nu(w)$ along the geodesic $A(\infty, 0)$ on $\mathcal{T}$. Since $\nu(w) = h > 0$, F_∞^* is a disjoint union of cosets $\pi^i w^j O_\infty^*$, $i \in \{0, ..., h-1\}$, $j \in \mathbb{Z}$: hence a "fundamental domain" mod w is

$$\mathcal{F} = \bigcup_{i=0}^{h-1} \pi^i O_\infty^* = U(e_0) - U(e_h).$$

To fix ideas, define $\mathcal{F}_\Psi := \alpha^{-1}\mathcal{F}$.

Proposition 13. *Let $\mu \in \mathcal{M}(\partial\Omega, \mathbb{Z})$ be γ_w-invariant: then, for any $z \in \Omega$,*

$$\int_{\mathbb{P}^1(F_\infty)} \frac{wz - t}{z - t} d\mu(t) = w^{m_0} \int_{\mathcal{F}} t d\mu(t),$$

where $m_0 := \mu(U(e_0))$.

Cf. prop. 3.3 of [**9**].

Proof. First of all, notice that γ_w-invariance of μ implies that $\mu(\mathcal{F}) = 0$ (same proof as for lemma 12).

One can decompose

$$\mathbb{P}^1(F_\infty) = U(\overline{e}_{-Nh}) \cup U(e_{Nh}) \cup \bigcup_{n=-N}^{N} w^n \mathcal{F}$$

and it follows, by definition of integral, that

$$\int_{\mathbb{P}^1(F_\infty)} \frac{wz-t}{z-t} d\mu(t) =$$

$$\lim_{N\to\infty} \left(\frac{wz-0}{z-0}\right)^{m_0} \left(\frac{wz-\infty}{z-\infty}\right)^{-m_0} \prod_{n=-N}^{N} \int_{w^n\mathcal{F}} \frac{wz-t}{z-t} d\mu =$$

$$\lim_{N\to\infty} w^{m_0} \prod_{n=-N}^{N} \int_{\mathcal{F}} \left(\frac{wz-w^n t}{z-w^n t}\right) d\mu(t) =$$

$$\lim_{N\to\infty} w^{m_0} \int_{\mathcal{F}} \left(\prod_{n=-N}^{N} \frac{w(z-w^{n-1}t)}{z-w^n t}\right) d\mu(t) =$$

$$\lim_{N\to\infty} w^{m_0} \int_{\mathcal{F}} \left(w^N \frac{w^{N+1}z-t}{z-w^N t}\right) d\mu(t) =$$

$$\lim_{N\to\infty} w^{m_0} \int_{\mathcal{F}} w^N d\mu(t) \int_{\mathcal{F}} \left(\frac{w^{N+1}z-t}{z-w^N t}\right) d\mu(t) =$$

$$w^{m_0} \int_{\mathcal{F}} \left(\lim_{N\to\infty} \frac{t-w^{N+1}z}{w^N t - z}\right) d\mu(t) = w^{m_0} \frac{\int_{\mathcal{F}} t d\mu(t)}{\int_{\mathcal{F}} (-z) d\mu(t)} = w^{m_0} \int_{\mathcal{F}} t d\mu(t)$$

using throughout the proof the facts that $\int$ is a continuous homomorphism and that $\int_{\mathcal{F}} c = 1$ for $c \in \mathbf{C}_\infty^*$.

□

It is immediate to check that $\alpha * \mu_E$ satisfies the conditions of proposition 13:

$$\gamma_w * \alpha * \mu_E = \alpha\Psi(u)\alpha^{-1} * \alpha * \mu_E = \alpha * \Psi(u) * \mu_E = \alpha * \mu_E.$$

Following [**9**], define the winding number

$$W_\Psi := -\sum_{i=0}^{h-1} \varphi_E(e_i(\Psi)),$$

where $e_i(\Psi) = \alpha^{-1}(e_i)$.

It follows from section 2 of chapter 1 that $c_{\Psi(u)}(\gamma_E) = Q^M$ for some $M \in \mathbb{Z}$.

THEOREM 14. *Let M be the index of γ_w in Λ_E: then $M = \frac{W_\Psi}{\nu(Q)}$ and $\int_G t d\mu_E(t) = \varrho(Q)^M$, Q the period of the Tate curve E.*

Notice the formal similarity of the term $Q^{W_\Psi/\nu(Q)}$ with the $\mathcal{L}$-invariant of equation (1).

PROOF. Exploiting the symmetry of the pairing $c_.(.)$,

$$Q^M = c_{\Psi(u)}(\gamma_E) = c_{\gamma_E}(\Psi(u)) = \frac{u_{\gamma_E}(\Psi(u)z)}{u_{\gamma_E}(z)},$$

so that, by theorem 8 and its corollaries,

$$Q^M = \int_{\mathbb{P}^1(F_\infty)} \frac{\Psi(u)z - t}{z-t} d\mu_E(t).$$

Put $a = \alpha z$ and observe that

$$\frac{\Psi(u)z - t}{z-t} = c\frac{\gamma_w a - \alpha t}{a - \alpha t}$$

for some constant $c \in \mathbf{C}_\infty^*$ (both sides have the same divisor). Then

$$\int_{\mathbb{P}^1(F_\infty)} \frac{\Psi(u)z - t}{z-t} d\mu_E(t) = \int_{\mathbb{P}^1(F_\infty)} \frac{wa - t}{a-t} d(\alpha * \mu_E)(t)$$

and, using proposition 13 and $\int_{\mathcal{F}} td(\alpha * \mu_E) = \int_{\mathcal{F}_\Psi} td\mu_E$, get

$$Q^M = w^{-\alpha * \varphi_E(e_0)} \int_{\mathcal{F}} td(\alpha * \mu_E)(t) = w^{-\varphi_E(\alpha^{-1}e_0)} \int_{\mathcal{F}_\Psi} td\mu_E,$$

so that

$$\varrho(Q^M) = \int_G td\mu_E(t)$$

(where, by abuse of notation, μ_E denotes the pushforward of the measure from $\mathcal{F}_\Psi$ to G).

It is left to compute M. In general, since $\nu(t) = i$ for t in the set $U(e_i) - U(e_{i+1})$,

$$\nu\left(\int_{\mathcal{F}} td\mu\right) = \sum_{i=0}^{h-1} i\Big(\mu U(e_i) - \mu U(e_{i+1})\Big) = -\sum_{i=1}^{h-1} \varphi(e_i) + (h-1)\varphi(e_h)$$

(where φ is the cocycle associated to the measure μ). Then, replacing the above in

$$M\nu(Q) = -(\alpha * \varphi_E)(e_0)\nu(w) + \nu\left(\int_{\mathcal{F}} td(\alpha * \mu_E)\right)$$

it follows that $M = \frac{W_\Psi}{\nu(Q)}$. □

The result is independent of the choice of base point P: changing it just shifts the fundamental domain along axis $A(P_\Psi^-, P_\Psi^+)$ and replaces α with $c\alpha$, $c \in \mathbf{C}_\infty^*$, which doesn't affect integration on sets of measure zero.

Dependence on u: the choice of a different fundamental unit might reverse the orientation on the axis of fixed points and therefore change the sign of μ and W_Ψ.

As for Ψ, notice that the right hand side of formula in theorem 14 depends on Ψ only through W_Ψ. $GL_2(F)$ acts on embeddings by conjugation, $(\gamma * \Psi)(a) := \gamma\Psi(a)\gamma^{-1}$. Assume $\Psi' = \gamma * \Psi$, $\gamma \in \Gamma$: then $P^+_{\Psi'} = \gamma P^+_\Psi$, etc., $\mu_{\Psi'} = \gamma_\Psi * \mu$ and in particular $W_{\Psi'} = \sum \varphi_E(\gamma e_i(\Psi)) = W_\Psi$.

CHAPTER 3

Hypotheses finguntur

1. Iwasawa algebras

Let R be a commutative ring, $G = \varprojlim G_a$ a profinite group, $G_a = G/H_a$. Let $\mathcal{M}(G, R) := \mathcal{M}(G, \mathbb{Z}) \otimes R$ be the set of R-valued distributions on G: it becomes an R-algebra defining the product as convolution. It is trivial to check that $\mathcal{M}(G, R)$ is the inverse limit of the algebras $\mathcal{M}(G_a, R)$: therefore if R is a topological ring $\mathcal{M}(G, R)$ is naturally endowed with a topology.

There is a canonical isomorphism of topological R-algebras between $\mathcal{M}(G, R) = \varprojlim \mathcal{M}(G_a, R)$ and the Iwasawa algebra $R[[G]] := \varprojlim R[G_a]$, given by

$$\mu \mapsto \alpha_\mu := \lim(\sum_{g \in G_a} \mu(gH_a)g)$$

(see [**35**], §7.2, for more detail).

Let I_a be the augmentation ideal of $R[G_a]$, i.e.

$$I_a := \ker(R[G_a] \to R,\, g \mapsto 1)$$

Define $I := \varprojlim I_a$ be the augmentation ideal of $R[[G]]$; then μ belongs to $\mathcal{M}_0(G, R)$ if and only if $\sum_{g \in G_a} \mu(gH_a) = 0\ \forall\, a$, that is, if and only if $\alpha_\mu \in I$.

Apply the above to $R = \mathbb{Z}$ and G the group of previous chapter. For $\mu \in \mathcal{M}_0(G, \mathbb{Z})$, define α_μ to be the associated ∞-adic L-function; its derivative at 1, "$L'(\mu, 1)$", is the image of α_μ in I/I^2. Since G is abelian, I/I^2 is canonically isomorphic to G by $[g-1] \mapsto g$; the two will be identified in the rest of this section. Let

$$L'(E, 1) := L'(\mu_E, 1).$$

PROPOSITION 15. $L'(E, 1) = \int_G g d\mu_E(g)$.

PROOF. This is just a matter of working out definitions: in the map $I \to G$,

$$\lim_a \sum_{g \in G_a} \mu(gH_a)g \mapsto \lim_a \prod_{g \in G_a} g^{\mu(gH_a)} = \int_G g d\mu(g).$$

$\square$

Together with theorem 14, this yields an analogue of formula (1).

Remark. In the number field case, the group G one deals with is (essentially) $\mathbb{Z}_p$. Fix g_0 a topological generator of such a G: then, thanks to the isomorphism $\mathbb{Z}_p[[G]] \simeq \mathbb{Z}_p[[x]]$, $g_0 \mapsto x+1$ (under which I corresponds to the ideal generated by x), α_μ can be identified with a power series F_μ and the image of α_μ in $I/I^2 \simeq \mathbb{Z}_p$ is c_1, the degree one coefficient of F_μ. Furthermore, in this case, the dual group $G^\vee := \mathrm{Hom}_{cont}(G, \mathbb{C}_p^*)$ can be identified with an open neighbourhood of 1 in $\mathbb{C}_p$ by $\chi \mapsto \chi(g_0)$ [1]. Taking as p-adic L-function the additive integral of characters, $L_p(\chi) := \int_G \chi d^+\mu$, one can take derivatives in the usual sense and the formula $\int_G \chi d^+\mu = F_\mu(\chi(g_0)-1)$ shows that $L_p'(1) = c_1$ (cf. [**35**]).

Besides, in characteristic 0 one can define a p-adic logarithm $\log : G \to \mathbb{C}_p$. Thus, in [**2**] and [**6**], $\mathbb{Z}_p$ is embedded in $G^\vee$ by $s \mapsto \chi_s$, $\chi_s(g) := \exp(s\log(g))$. Then $L_p(s) := \int_G \chi_{s-1}(t)d\mu(t)$ is analytic and its derivative is $\frac{d}{ds}F_\mu(\chi_{s-1}(g_0)-1) = \log(g_0)c_1 = \log(g_0^{c_1})$.

In the next section, an attempt (still very naive and tentative) will be made to develop a similar interpretation of L'_∞ as derivative of a function on $\mathbb{Z}_p$.

2. Towards a p-adic L-function

Let G be, as usual, the group $(K \otimes F_\infty)^*/F_\infty^*$. Define its dual group as

$$G^\vee := \mathrm{Hom}_{cont}(G, \mathbf{C}_\infty^*).$$

Let $\mathfrak{u}_1$, $\mathfrak{U}_1$ be the groups of 1-units in F_∞, $\mathbf{C}_\infty$ respectively; i.e. $\mathfrak{U}_1 := \{z \in \mathbf{C}_\infty \mid |z-1| < 1\}$, $\mathfrak{u}_1 := \mathfrak{U}_1 \cap F_\infty = 1 + \mathfrak{m}_\infty$. The following result is lemma 8.2.2 in [**25**].

LEMMA 16. *$\mathfrak{U}_1$ is a $\mathbb{Q}_p$-vector space.*

[1] A necessary and sufficient condition for $t = \chi(g_0)$ is $\lim_{n\to\infty} t^{p^n} = 1$; therefore any character is of the form $\chi = \chi_t$, $\chi_t(g_0^a) = t^a := \exp(a\log_p(t))$, t in the open ball $B(1,1)$.

PROOF. For $s = \sum_{i=-N}^{\infty} a_i p^i \in \mathbb{Z}_p$ and $t = 1 + m \in \mathfrak{U}_1$, set

$$s \cdot t := t^s = \prod_{i=-N}^{\infty} (1 + m^{p^i})^{a_i}.$$

This is well defined, since p-th roots uniquely exist in $\mathbf{C}_\infty$. The vector space axioms are trivially verified for those s expressed as a finite sum; continuity takes care of the rest. □

Let G_1 be the image of $\mathfrak{u}_1$ in G: G_1 is a pro-p-group, sitting in G with finite index $h(q-1)$. Since $\mathbf{C}_\infty^* \simeq \mathbb{Q} \times \overline{\mathbb{F}}_p^* \times \mathfrak{U}_1$ and G is compact, any character $\chi \in G^\vee$ takes values in the product of $\mathfrak{U}_1$ with some finite subgroup of $\overline{\mathbb{F}}_p^*$. Since the latter has order coprime to p, $\chi(G_1)$ is trivial in $\overline{\mathbb{F}}_p^*$: therefore $G^\vee$ decomposes as $G_t^\vee \times G_w^\vee$, where $G_t^\vee$ is isomorphic to $\mathrm{Hom}(G/G_1, C)$, C some finite cyclic group, and $G_w^\vee := \mathrm{Hom}_{cont}(G, \mathfrak{U}_1)$ (of course the subscripts t, w stay for tame, wild).

It follows from lemma 16 that $\mathfrak{U}_1$ is uniquely divisible: in particular, any homomorphism from G_1 to $\mathfrak{U}_1$ extends uniquely to all of G, yielding an isomorphism $G_w^\vee \simeq \mathrm{Hom}_{cont}(\mathfrak{u}_1, \mathfrak{U}_1)$.

Define a tentative p-adic L-function $L_{tent}(E, \chi) : G^\vee \to \mathbf{C}_\infty^*$ as

$$L_{tent}(E, \chi) := \int_G \chi d\mu_E.$$

In particular, one gets a p-adic, $\mathbf{C}_\infty^*$-valued L-function by choosing an embedding of $\mathbb{Q}_p$ in the $\mathbb{Q}_p$-vector space $G_w^\vee$. The most natural thing to do seems to be the following.

Let $\chi_1 \in G_w^\vee$ be the character determined by $\chi_1 \circ \varrho|_{\mathfrak{u}_1} = \mathrm{id}_{\mathfrak{u}_1}$. Define $g^s := \chi_s(g) := \chi_1(g)^s$. This gives a map $\mathbb{Q}_p \to G_w^\vee$ by $s \mapsto \chi_s$. Then the function

$$L_{tent}(E, s) := \int_G g^{s-1} d\mu = \chi_{s-1}\left(\int_G g d\mu\right)$$

is $\mathbb{Q}_p$-linear and has constant derivative

$$\frac{d}{ds} L_{tent}(E, s)|_{s_0} = \lim_{s \to s_0} \int_G g^{\frac{s-s_0}{s-s_0}} d\mu = \int_G g d\mu.$$

This definition was naively made by looking at the usual Mellin transform, both classical and p-adic[2]; however, in those situations, the integrals taken into consideration were additive. In our case, it would be

[2]For the latter, in a context very close to ours, see [**26**].

probably better to replace the exponent $s-1$ by some polynomial of higher degree in $\mathbb{Z}_p[s]$, in order to avoid having a constant derivative.

Observe that χ_1 seems to play role of exponential and ϱ of logarithm. There is of course some loss in taking just χ_1 instead of multiplying it with some tame character, but this seems unavoidable.

In terms of the augmentation ideal, the above can be interpreted in the following way. Integration on G gives a natural map

$$S : \mathcal{M}(G, \mathbb{Z}_p) \to \mathrm{Hom}(G_w^\vee, \mathfrak{U}_1),$$

by $\mu \mapsto L_\mu$, where

$$L_\mu(\chi) := \int_G \chi d\mu.$$

This map is a homomorphism on the additive group of measures: it is immediate that $L_{\mu_1+\mu_2} = L_{\mu_1}L_{\mu_2}$. As before, $\mathcal{M}(G,\mathbb{Z}_p)$ is identified with $\mathbb{Z}_p[[G]]$. Notice that G is contained in $\mathbb{Z}_p[[G]]$ as a set of topological generators $\{g\}$, corresponding to the Dirac deltas $\{\delta_g\}$ in $\mathcal{M}(G,\mathbb{Z}_p)$. An immediate computation yields

$$S(g)(\chi) = S(\delta_g)(\chi) = \chi(g).$$

Since $\mathbb{Z}_p[[G]] = \mathbb{Z}_p \oplus I$, the equality $S(g-1)(\chi) = \chi(g)$ gives $S(\mathbb{Z}_p[[G]]) = S(I)$. Besides S factors through I^2, since

$$S\big((g-1)(h-1)\big)(\chi) = \chi(gh)\chi(g)^{-1}\chi(h)^{-1} = 1.$$

Therefore $S(\mathbb{Z}_p[[G]]) = S(I/I^2)$ is the image of G in $\mathrm{Hom}(G_w^\vee, \mathfrak{U}_1)$.

Goss' theory of L-functions. It seems natural to try and look at our tentative L-function in the framework of Goss' L-series ([**25**]).

To define p-adic L-functions in characteristic p, Goss introduces a "complex plane" $\mathbf{S}_\infty := \mathbf{C}_\infty^* \times \mathbb{Z}_p$ ([**25**], 8.1): the idea is that $F_\infty^* \simeq \mathbb{Z} \times \mathbb{F}_q^* \times \mathfrak{u}_1$, hence its dual is $\mathbf{C}_\infty^* \times \mathrm{Hom}(\mathbb{F}_q^*, \mathbf{C}_\infty^*) \times \mathrm{Hom}_{cont}(\mathfrak{u}_1, \mathfrak{U}_1)$, which contains $\mathbf{S}_\infty$.

For $s = (x,y) \in \mathbf{S}_\infty$, $\alpha \in F_\infty^*$, he defines $\alpha^s := x^{\deg \alpha}\langle\alpha\rangle^y$. The subgroup $\mathfrak{U}_1 \times \mathbb{Z}_p$ of $G_w^\vee$ is contained in $\mathbf{S}_\infty$: one can extend L_{tent} to a function defined on all of $\mathbf{S}_\infty$ by $L_{tent}(E,s) := \int_G g^{s-(0,1)} d\mu$ (notice that for $\alpha \in \mathfrak{U}_1$, $\deg \alpha = 0$, and $\chi_0(G) \subset \mathfrak{U}_1$, ergo $g^s = \chi_0(g)^y$).

A function $f : \mathbf{S}_\infty \to \mathbf{C}_\infty$ is defined to be entire if the corresponding map $\mathbb{Z}_p \to \mathcal{O}_{\mathbf{C}_\infty^*}(\mathbf{C}_\infty^*)$ is continuous[3] (this definition is slightly stronger than 8.5.1 in [**25**]); this condition is clearly satisfied by $L_{tent}(E,s)$.

[3]The topology on $\mathcal{O}_{\mathbf{C}_\infty^*}(\mathbf{C}_\infty^*)$ is given by seminorms $\|f\|_{\mathcal{A}_n} = \sup\{|f(z)| \mid z \in \mathcal{A}_n\}$, where $\mathcal{A}_n$ is the annulus $\{z \mid q^{-n} \le |z| \ge q^n\}$.

CHAPTER 4

Paralipomena

The techniques introduced in the previous chapters can be easily extended to the case of an extension inert at ∞: this will be sketched in section 1. In section 2, a few words will be said on what can be done replacing the field K by a split quadratic algebra over F.

1. The inert case

Assume that K/F is inert at ∞: then $K \otimes_F F_\infty =: K_\infty$ is a field. The action of $\mathrm{Gal}(K_\infty/F_\infty)$ will be denoted by $x \mapsto \overline{x}$.

1.1. Heegner points. Recall that points of the Drinfeld modular curve $M_0(\mathfrak{n})$ classify rank 2 Drinfeld A-modules with level-$\mathfrak{n}$ structure.

Definition. $P \in M_0(\mathfrak{n})$ is said to be a Heegner point if the corresponding Drinfeld module $\mathcal{P}$ has complex multiplication by an order $\mathcal{R}$ in an imaginary quadratic extension of F; the action of $\mathcal{R}$ on $\mathcal{P}$ is requested to preserve the level structure.

In the conditions of the definition above, one says that $\mathcal{P}$ is associated to the order $\mathcal{R}$.

Heegner points enjoy rationality properties (as can be seen, e.g., by means of Hayes' explicit class field theory [**31**], §8; see also [**13**], §4).

The reader is reminded of the equivalence of categories between Drinfeld A-modules over $\mathbf{C}_\infty$ and homothety classes of A-lattices of the same rank in $\mathbf{C}_\infty$. In particular, an endomorphism of the module Φ corresponds to a scalar $c \in \mathbf{C}_\infty$ such that $c\Lambda \subset \Lambda$, where Λ is the lattice associated to Φ.

Lemma 17. *There is a bijection between the set of Heegner points of $M_0(\mathfrak{n})$ associated to the order B and embeddings $\Psi : B \to R$: lattices generated by fixed points of $\Psi(B)$ correspond to Drinfeld modules with complex multiplication by B.*

The proof works like in the case of elliptic curves over $\mathbb{C}$ ([**43**], 4.4).

PROOF (SKETCH). Let P be a Heegner point, $\Lambda = Aw_1 + Aw_2$ and $\Lambda_1 = Aw_1 + \mathfrak{n}w_2$ the corresponding pair of lattices. For $b \in B$, the relation $bw_i \in \Lambda$ yield an embedding $\Psi : B \to R$; it is immediate to check that $z_\Psi := \frac{w_1}{w_2}$ is a fixed point of $\Psi(B)$.

Vice versa, if z_Ψ is a fixed point of Ψ, then $(z_\Psi, 1)$ is an eigenvector for $\Psi(B)$: that is, the lattice $Az_\Psi + A$ has complex multiplication by B. □

1.2. The theorem. Let $K^*_{\infty,1} := \{x \in K^*_\infty \mid x\overline{x} = 1\}$ be the group of elements of norm 1 in K^*_∞.

If K/F is not the constant field extension $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$, the map $x \mapsto x\overline{x}^{-1}$ establishes an isomorphism between $G = K^*_\infty/F^*_\infty$ and $K^*_{\infty,1}$. Thus a convenient way to transfer measures from $\partial\Omega$ to G will be by means of the Möbius transformation $\alpha(t) = \dfrac{t - z_\Psi}{t - \overline{z}_\Psi}$, which yields a bijection $\alpha : \partial\Omega \to K^*_{\infty,1}$. Notice that in this case $K^*_{\infty,1}$ is contained in $\mathfrak{U}_1$.

In the case $K = F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$, to get G one has to factor out the image of the roots of unity: that is, $G \simeq K^*_{\infty,1} \cap \mathfrak{U}_1$. So in all cases there is an embedding $\iota : G \to \mathfrak{U}_1$.

As in the split case, define $L'(E, 1) := \iota(\int_G g d\mu)$.

THEOREM 18. *For $K \neq F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$, $\Phi_{Tate}(L'(E, 1))$ is a global point of E.*

PROOF (SKETCH). Let $z_\Psi, \overline{z}_\Psi$ be the fixed points corresponding to the embedding Ψ. Remember the Abel-Jacobi map from section 2. One has

$$\Phi_{AJ}((z_\Psi) - (\overline{z}_\Psi))(\gamma_E) = \frac{u_{\gamma_E}(z_\Psi)}{u_{\gamma_E}(\overline{z}_\Psi)} = \int_{\partial\Omega} \frac{t - z_\Psi}{t - \overline{z}_\Psi} d\mu_E(t) =$$

$$\int_{K^*_{\infty,1}} td(\alpha_*\mu_E)(t) = \iota \int_G g d\mu_E.$$

Now use the rationality properties of Heegner points and of the Abel-Jacobi map. □

In the case of the constant field extension, $K^*_{\infty,1}$ is the union of $q+1$ copies of $\iota(G)$, corresponding to the action of B^*. Since the measure μ_E is invariant with respect to this action, in this case one gets

$$\Phi_{AJ}((z_\Psi) - (\overline{z}_\Psi))(\gamma_E) = \left(\iota \int_G g d\mu_E\right)^{\frac{1}{q+1}}.$$

As in the previous chapter, also in the inert case one can use the action of $\mathbb{Z}_p$ on $\mathfrak{U}_1$ to define a p-adic L-function.

The inert case has been studied in Pál's thesis ([**37**]): his approach is somewhat different, but the special value $\mathcal{L}'(E|K)$ he gets seems to be the same as here[1]. In [**37**], as in [**4**], no mention is made of integration, which makes some computations more cumbersome. On the other side, Pál proves interpolation formulae which show links between ∞-adic and classical L-functions and could give fruitful hints for the construction of a p-adic L-function as attempted in chapter 3.

2. The cyclotomic case

In this section, the field K is replaced by the split quadratic algebra $\tilde{K} := F \times F$, into which F is embedded on the diagonal; the integral closure of A in $\tilde{K}$ is $\tilde{B} = A \times A$.

In this situation the group $\tilde{G} := (\tilde{K} \otimes_F F_\infty)^*/F_\infty^* \tilde{B}^* \simeq F_\infty^*/\mathbb{F}_q^* = \pi^{\mathbb{Z}} \times \mathfrak{u}_1$ is not compact any more and so it cannot be identified with a Galois group. Class field theory shows that $\tilde{G}$ corresponds to a dense subgroup of a "cyclotomic" Galois group (see e.g. [**30**]); in particular, the uniformizer π is sent by reciprocity into the arithmetic Frobenius, corresponding to extensions of the the constant subfield of F. the role previously played by G will be now given to the geometric cyclotomic group $G_0 \simeq \mathfrak{u}_1$.

The set of fixed points of embeddings $\tilde{K} \hookrightarrow GL_2(F)$ consists precisely of $\mathbb{P}^1(F)$.

2.1. Cusps. Let $\overline{\Omega} := \Omega \cup \mathbb{P}^1(F)$. As said in section 2 of chapter 1, the affine curve $M_0(\mathfrak{n})$ can be completed to a projective curve $\overline{M}_0(\mathfrak{n})$, whose $\mathbf{C}_\infty$-points are identified with $\Gamma \backslash \overline{\Omega}$. As in the classical situation of modular curves over $\mathbb{C}$, a rigid analytic structure at the cusps $s \in \mathbb{P}^1(F)$ is defined by means of horocyclic neighbourhoods: for $s = \infty$ one takes $\Omega_c := \{z \in \Omega \mid |z|_i \geq c\}$, quotiented by the action of Γ (for more detail, see [**22**], §2.6-2.7 and [**15**] V.2).

The theta functions studied in [**22**] can be evaluated also at cusps of Γ.

Proposition 19. *Let $s \in \mathbb{P}^1(F)$: then $u_\gamma(s) = \prod_{\varepsilon \in \tilde{\Gamma}} \frac{s - \varepsilon w}{s - \varepsilon \gamma w}$ converges to a value in $\mathbf{C}_\infty^*$; moreover, u_γ is a holomorphic function on all of $\overline{\Omega}$.*

[1] It's not completely clear to the author from the still incomplete draft of [**37**] he could see.

For the proof, see [**22**], lemmata 5.3.9 and 5.3.10.

The cusps $\Gamma\backslash\mathbb{P}^1(F)$ correspond to the set of ends of $\Gamma\backslash\mathcal{T}$: recall that this graph consists of the union of a finite graph and a finite number of half-lines (see [**42**], II.2.3). In particular, this means that, given a cusp s, one can find an end $e = e(s)$ (pointing to s) such that the conical subtree $\mathcal{C}_{s,e} \subset \mathcal{T}$ collapses to a the half-line of s in $\Gamma\backslash\mathcal{T}$. Here

$$\mathcal{C}_{s,e} := A(o(e), s) \cup \{A(v, w) \mid v \in A(t(e), s) \text{ and } d(v, w) \leq d(v, t(e)\},$$

where $d(v, w)$ is the distance between the vertices. Then, for $\varphi \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$, the cuspidality condition yields that φ is identically 0 on $\mathcal{C}_{s,e}$.

Finally, an analogue of the classical Manin-Drinfeld theorem on cusps holds.

THEOREM 20. *The group generated by cusps in $J_0(\mathfrak{n})$ is finite.*

A proof is given in [**21**] (for the case $F = \mathbb{F}_q(T)$, a different proof can be found in [**14**]).

2.2. The theorem. Consider the diagonal embedding of $\tilde{K}$ in $GL_2(F)$: the fixed points on $\partial\Omega$ are 0 and ∞. As in chapter 2, the choice of a base point allows to identify $(\tilde{K} \otimes_F F_\infty)^*/F_\infty^*$ with $\partial\Omega - \{0, \infty\}$; the measure μ_E is then pulled back to $\tilde{G}$. To get a measure on $G_0 \simeq \mathfrak{u}_1$, recall the notion of level of an edge/vertex, $l(e)/l(v)$, introduced in chapter 1, section 1, and define $\tilde{\mu}_E$ on O_∞^* by

$$\tilde{\mu}(U_e) := -\sum_{l(e')=l(e)} \varphi(e').$$

Notice that this sum is finite, because under the action of Γ the cones $\mathcal{C}_{0,e(0)}$ and $\mathcal{C}_{\infty,e(\infty)}$ collapse to half-lines in $\Gamma\backslash\mathcal{T}$. The restriction of $\tilde{\mu}_E$ to $\mathfrak{u}_1$ gives the measure on G_0.

For $r, s \in \mathbb{P}^1(F)$ and $\varphi \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$, define

$$[r, s] \cdot \varphi := \sum_{e \in A(r,s)} \varphi_E(e).$$

This is the pairing between modular symbols and cuspidal harmonic cocycles considerd by Teitelbaum in [**48**], definition 8.

Let e_v be the unique edge with origin v and pointing to 0. For $v \in \mathcal{V}(\mathcal{S}_0)$ choose a representative $s(v) = u_v \in U(\overline{e}_v)$. Extend this map to all of $\mathcal{V}(\mathcal{T})$ by letting $s(\pi^i v) = \pi^i u_v$.

THEOREM 21. *One has*

$$u_{\gamma_E}(0) = \pi^{[0,\infty]\cdot\varphi_E}\left(\int_{G_0} g d\tilde{\mu}_E\right)^{q-1}.$$

PROOF. Choose a vertex w_0 and let $d_1, ..., d_r$ be the edges of the path $A(w_0, \gamma w_0)$. Put $t(d_i) = o(d_{i+1}) = w_i$ and take $z_0, ..., z_{r-1}$ in Ω such that $\lambda(z_i) = w_i$; let $z_r := \gamma_E z_0$. Define

$$P_n := \prod_{i=0}^{r-1} \prod_{\substack{\varepsilon\in\tilde{\Gamma} \\ l(\varepsilon d_i)\leq n}} \frac{\varepsilon z_i}{\varepsilon z_{i+1}}.$$

Then

$$u_\gamma(0) = \prod_{\varepsilon\in\tilde{\Gamma}} \frac{\varepsilon z_0}{\varepsilon\gamma z_0} = \prod_{\varepsilon\in\tilde{\Gamma}}\prod_{i=0}^{r-1} \frac{\varepsilon z_i}{\varepsilon z_{i+1}} = \lim_{n\to\infty} P_n$$

by lemma 4 (the fact that one can rearrange terms in the infinite product follows from the estimates in [**22**], §5.3).

If $l(\lambda(\varepsilon z_i)) \leq n$, each time εz_i appears in P_n as terminus of an edge εd_i in the $\tilde{\Gamma}$-orbit of $A(z_0, \gamma z_0)$, it also appears as origin of the next edge.[2] Thus the only contributions to P_n which are not cancelled are those corresponding to vertices of level $n+1$. Furthermore, lemma 4 implies that $\frac{\varepsilon z_i}{s(\lambda(\varepsilon z_i))} \to 1$ uniformly as $l(\lambda(\varepsilon z_i)) \to \infty$. Hence

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \prod_{l(v)=n+1} s(v)^{X_v}.$$

To compute the exponent X_v, observe that $s(v)$ appears in the numerator exactly the number of times e_v falls in $\tilde{\Gamma}A(w_0, \gamma_E w_0)$; analogously, for the denominator one computes the occurrences of $\overline{e}_v$. But ([**22**], §3.3)

$$\varphi_E(e) = \sharp\{\varepsilon \in \tilde{\Gamma} \mid e \in \varepsilon A(w, \gamma_E w)\} - \sharp\{\varepsilon \in \tilde{\Gamma} \mid \overline{e} \in \varepsilon A(w, \gamma_E w)\}.$$

That is,

$$u_\gamma(0) = \lim_{n\to\infty} \prod_{l(v)=n} s(v)^{\varphi_E(e_v)} = \lim_{n\to\infty} \prod_{l(v)=n} \pi^{i(v)\varphi_E(e_v)} \prod_{\substack{v\in\mathcal{S}_0 \\ l(v)=n}} u_v^{-\tilde{\mu}U(e_v)}.$$

The product of units converges precisely to $\int_{O_\infty^*} u d\tilde{\mu}_E$.[3] To pass to an integral over G_0, just recall that $\tilde{\mu}_E$ is invariant under $B^* \subset \Gamma$ and that $G_0 \simeq \mathfrak{u}_1 = O_\infty^*/\mathbb{F}_q^*$.

[2]Recall that $t(d_r) = o(\gamma d_1)$, $o(d_1) = t(\gamma^{-1} d_r)$.

[3]Don't forget that e_v is oriented in the "wrong" way: O_∞^* is covered by the $U(\overline{e}_v)$'s.

As for the product of π's, recall that $i(v) = i$ for all $v \in \mathcal{V}(\mathcal{S}_i)$; besides, it follows from the harmonicity of φ_E that

$$\sum_{\substack{v \in \mathcal{S}_i \\ l(v)=n}} \varphi_E(e_v) = \varphi_E(e_{i+1}) - \varphi_E(e_i).$$

Therefore, the valuation of P_n is constant, given by

$$\nu(P_n) = \sum_{i=N_\infty}^{N_0} i(\varphi_E(e_{i+1}) - \varphi_E(e_i)) = [0, \infty] \cdot \varphi_E.$$

$\square$

According to Gekeler, [**21**], th. 1.6 (see also [**20**]), it follows from theorem 20 that $u_{\gamma_E}(0) = \frac{u_{\gamma_E}(0)}{u_{\gamma_E}(\infty)}$ is mapped by the Tate map to a torsion point on E and it is therefore a rational power of the Tate period. Theorem 21 should thus be regarded as the function field analogue of the Greenberg-Stevens theorem about the cyclotomic $\mathcal{L}$-invariant.

Conclusion

It has been proven that integration on the anticyclotomic Galois group G with respect to measures obtained from explicit uniformization of elliptic curves can yield arithmetically meaningful information also in characteristic p, thereby generalizing what was done in [**2**] in the case of number fields. More precisely, in the split case an analogue of the $\mathcal{L}$-invariant of [**36**] was obtained by integration on G; while in the inert case, we have indicated how this technique allows to recover Heegner points.

An attempt has been made to interpret the values of these integrals as special values of a still hypothetical p-adic L-function $L(E,s):\mathbb{Z}_p \to \mathbf{C}_\infty^*$. The results obtained are not yet very satisfactory (the function L_{tent} takes the right value at the special point 1, but it has constant derivative). This clearly indicates a direction for further research: to define, by some sort of Mellin transform, a p-adic L-function which encodes a large enough amount of the arithmetic of E; such a function should be related (e.g., by interpolation of special values) to the classical complex L-function $L(E,s)$. The fact that the latter is a polynomial in q^{-s} suggests that the conjectural L_p should not be very complicated.

A further goal (still far from the author's scope) would be to formulate analogues of the Birch and Swinnerton-Dyer conjectures for these L-functions.

Another possible direction of research is the generalization to the characteristic p case of the double integrals considered in [**9**].

Sed fugit interea, fugit inreparabile tempus,
singula dum capti circumvectamur amore.[4]

And so, for the moment, we stop here our investigation.

A last remark: throughout the paper it has been made use of the simplifying assumption that F and A are respectively $\mathbb{F}_q(T)$ and $\mathbb{F}_q[T]$.

[4][**49**], III, 284-285.

It should be an easy step to generalize our results to any F, A. Among the things one should take care of, there are:

- if A has class number > 1, $\Gamma\backslash\Omega$ is only one of the irreducible components of the moduli space $M_0(\mathfrak{n})$;
- for A a generic Drinfeld ring, j is injective with finite cokernel, but it is not known whether it is an isomorphism (cf. [**22**], remark 6.4.5);
- in general $Q^{\mathbb{Z}}$ is contained in Λ_E with finite index ([**22**] 9.5.1);
- the classification of embeddings $B \to R$ depended on the fact that A was a principal ideal domain.

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