RIGID MEROMORPHIC COCYCLES FOR CONGRUENCE SUBGROUPS

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Do you really need to code and prove all of this?
Can't you just solve Hilbert's 12th problem?

-F. M. Nonez

Abstract

In [16], Henri Darmon and Jan Vonk defined and laid out the groundwork for the study of rigid meromorphic cocycles, which can be seen as a p-adic counterpart to meromorphic modular functions. These cocycles seem to be a useful tool in constructing certain class fields of real quadratic fields and hence can be studied to get a more profound understanding of Hilbert's 12th problem.

In this thesis we generalize results of the aforementioned article to the congruence subgroup case, i.e. when $SL_2(\mathbb{Z}[1/p])$ is replaced by the subgroup having lower-left entry divisible by N. We give full characterizations of rigid analytic / meromorphic cocycles in the additive weight 2 case as well as the multiplicative case. The parabolic and Eisenstein subspaces are considered separately. We also construct a kernel giving a lift between rigid analytic cocycles and a certain space of cusp forms, generalizing results of [35]. Finally, we describe an efficient algorithm that computes $\Gamma_0^p(N)$-invariant modular symbol using modular symbols and M-symbols algorithms, which allow us to compute RM-values of rigid meromorphic cocycles.

Abrégé

Dans [16], Henri Darmon et Jan Vonk introduisent les fondations pour l'étude des cocycles rigides méromorphes, objets qui peuvent être vus comme une contre-partie p-adique aux fonctions modulaires méromorphes. Ces cocycles semblent utiles pour construire certains corps de classe de corps quadratiques réels et donc peuvent être étudiés pour avoir une connaissance plus profonde du 12e problème de Hilbert.

Dans cette thèse, nous généralisons les résultats de l'article mentionné ci-haut au cas de sous-groupe de congruence, c'est-à-dire lorsque $SL_2(\mathbb{Z}[1/p])$ est remplacé par le sous-groupe des matrices ayant l'élément inférieur-gauche divisible par un entier N. Nous donnons une caractérisation complète des cocycles rigides analytiques / méromorphes dans le cas additif de poids 2 ainsi que le cas multiplicatif. Les sous-espaces paraboliques et Eisenstein sont considérés séparémment. Nous construisons aussi un noyau donnant un relevé entre les cocycles rigides analytiques et un certain espace de formes modulaires, généralisant les résultats de [35]. Finalement, nous décrivons un algorithme qui calcule des symboles modulaires invariant sous $\Gamma_0^p(N)$ utilisant des idées d'algorithme reliés aux symboles-M, ce qui nous permet d'évaluer les valeurs RM de cocycles rigides méromorphes.

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Chapter 1

Introduction

Explicit class field theory is a beautiful and far-reaching subject at the center of algebraic number theory. At its core, the fundamental question is the following:

Given a number field K, explicitely construct the maximal abelian extension of K.

This is a rather vague question, but this usually means finding some "general" function whose values generate all abelian extensions of K. To this day, only an infinitesimal portion of this question has been answered. Quite famously, the case when $K = \mathbb{Q}$ or K is an imaginary quadratic field are completely solved. In the first instance, the Kronecker-Weber theorem explicitely constructs every abelian extension of $\mathbb{Q}$ using roots of unity, and in the other case, the theory of complex multiplication constructs every abelian extension of an imaginary quadratic field using torsion points of elliptic curves. Both constructions have an arithmetic and a geometric flavor to them: When $K = \mathbb{Q}$, the important arithmetic function is the exponential one, and we evaluate it at torsion points of the circle group $\mathbb{R}/\mathbb{Z}$. When K is imaginary quadratic, the arithmetic function is the j-invariant associated to an elliptic curve with complex multiplication by an order in K.

Results similar to these for other kind of number fields have been sparse. In the 70s, Harold Stark proposed a series of conjectures relating the leading term of Artin L-series with algebraic units, called Stark units, generating abelian extensions of certain number fields. Variations of this conjecture have been proven throughout the years, but little is known about the original statement. Another recent development has been made in [2] regarding complex cubic field using an elliptic gamma function.

The next natural step is to understand how to construct abelian extensions of real quadratic fields. Rigid meromorphic cocycles, introduced in the seminal paper [16], seem to be a good candidate for this matter. In essence, they are natural p-adic analogues of the j-function mentioned above, whose values at real quadratic points τ conjecturally live in an abelian extensions of $\mathbb{Q}(\tau)$. These objects naturally arise as functions on

$SL_2(\mathbb{Z}[1/p])\backslash\mathcal{H}_p^{RM}$, where $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ is the p-adic upper half plane, $\mathcal{H}_p^{RM}$ is the set of *real multiplication* points on $\mathcal{H}_p$ and $SL_2(\mathbb{Z}[1/p])$ acts as Möbius transformations on $\mathcal{H}_p$, which gives them a kind of geometric flavor.

Rigid meromorphic cocycles are elements of $H^1(SL_2(\mathbb{Z}[1/p]), \mathcal{M}^\times)$ where $\mathcal{M}^\times$ is the multiplicative group of rigid meromorphic functions on $\mathcal{H}_p$. More generally, one could replace $\mathcal{M}^\times$ by the additive group $\mathcal{M}_k$, or consider particular subgroups of $H^1(SL_2(\mathbb{Z}[1/p]), \mathcal{M}^\times)$. In their original paper [16], Darmon and Vonk give full characterizations of the spaces $H^1(SL_2(\mathbb{Z}[1/p]), \mathcal{M}_2)$ and $H^1(SL_2(\mathbb{Z}[1/p]), \mathcal{O}_2)$, where $\mathcal{O}_2$ is the additive group of analytic functions on $\mathcal{H}_p$ with action given by the usual slash operator of weight 2. They also construct *rigid meromorphic theta cocycles*, which are elements of $H^1(SL_2(\mathbb{Z}[1/p]), \mathcal{M}^\times/\mathbb{C}_p^\times)$, which can be lifted to genuine rigid meromorphic cocycles if some conditions hold, and an efficient algorithm is given to compute the values of these cocycles at real multiplication points. The real multiplication values of these cocycles also conjecturally satisfy a kind of *Gross-Zagier style factorization*, similar to the j-function in the imaginary quadratic case.

A lot of recent developments have tried to generalize these cocycles to various different setups. Examples include [23] when $SL_2(\mathbb{Z}[1/p])$ is replaced by an order in an indefinite quaternion algebra over a totally real field, [22] which considers p-arithmetic subgroups of inner forms of SL_2 for arbitrary number fields, [11] which considers general orthogonal groups (and not just $O(2,1)$), and [28] which constructs a cocycle that is in a sense analogous to rigid meromorphic cocycles, but over the classical upper-half plane.

Rigid meromorphic cocycles have a rich structure closely related to modular forms over $O(2,1)$. Evidence of this appears for example in [15] and [35] where correspondences between rigid analytic cocycles of weight $2k$ and modular forms of weight $k + 1/2$ and level $4p$ are given, in the spirit of the usual Borcherds' lift of modular forms on $O(2,1)$. A special kind of rigid analytic cocycle called the *Dedekind-Rademacher cocycle*, related to an Eisenstein series of level p, is studied in detail in [14].

1.1 Overview of this thesis

The goal of this thesis is to generalize a number of results fundamental to rigid meromorphic coycles after replacing $SL_2(\mathbb{Z}[1/p])$ by a congruence subgroup, i.e. considering the set $H^1(\Gamma_0^p(N), \mathcal{M}^\times)$, where $\Gamma_0^p(N)$ is the congruence subgroup of $SL_2(\mathbb{Z}[1/p])$ with lower-left entry divisible by N. To be able to state the results of this thesis, we introduce some useful concepts which will be studied in greater detail later.

Let's fix a prime p with $p \nmid N$. The inclusion $\Gamma_0(pN) \subset \Gamma_0(N)$ defines a projection of modular curves

$$\varphi_s \ : \ X_0(pN) \to X_0(N).$$

Let $\varphi_t := \varphi_s \circ W_p$ where W_p is the Atkin-Lehner involution at p. Then we define the p-new subspace (sometimes called the p-new kernel in the literature) of the jacobian of $X_0(pN)$ as

$$J_0(pN)^{p-\text{new}} := \ker(\varphi_s \oplus \varphi_t).$$

In general, when we have a module with $\Gamma_0(pN)$-action and $\gamma_1, ...\gamma_{p+1}$ is a set of generators for $\Gamma_0(N)/\Gamma_0(pN)$, then acting on this module by $\sum_{i=1}^{p+1} \gamma_i$ naturally gives a morphism to the same module with $\Gamma_0(N)$-action, and the p-new part is again the kernel of the map $\sum_{i=1}^{p+1} \gamma_i \oplus \sum_{i=1}^{p+1} (\gamma_i W_p)$. We will make this more precise later.

Given a right Γ-module Ω (with Γ any subgroup of $SL_2(\mathbb{Z}[1/p])$), the space of Ω-valued modular symbols, denoted $MS(\Omega)$ is defined as the set of functions

$$m \ : \ \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \Omega$$

satisfying the relations

$$m\{r,s\} = -m\{s,r\}, \quad m\{r,s\} + m\{s,t\} = m\{r,t\}.$$

It comes with a natural Γ-action given by

$$(m|\gamma)\{r,s\} := m\{\gamma r, \gamma s\}|\gamma.$$

Rigid analytic / meromorphic cocycles are closely related to Γ-invariant modular symbols. This later space being easier to understand and manipulate is the reason why theorems about rigid meromorphic cocycles will sometimes be given in terms of modular symbols.

We are now ready to summarize the contents and the results of this thesis.

Chapter 2 gives a review of the literature regarding rigid analytic / meromorphic cocycles for $SL_2(\mathbb{Z}[1/p])$. In particular, results regarding the characterization of additive rigid meromorphic cocycles and their divisors are given in Section 2.2 while the isomorphism between rigid analytic cocycles and modular forms of level p is explained in Section 2.3. Multiplicative rigid analytic cocycles and their Borcherds' lift are explained in the next two sections.

Chapter 3 generalizes results of Chapter 2 to the congruence subgroup $\Gamma_0^p(N)$. More precisely, we first characterize the divisors of rigid meromorphic cocycles, giving the following theorem:

Theorem 1.1.1. *The support of a rigid meormorphic cocycle $m \in MS^{\Gamma_0^p(N)}(\mathcal{M}_k)$ consists only of RM-points.*

In Section 3.4, we also give a condition for a divisor-valued modular symbol to come from a rigid meromorphic cocycle, culminating in this result:

Theorem 1.1.2. *There is an exact sequence*

$$0 \to MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \to MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \xrightarrow{div} MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p))$$
$$\xrightarrow{(\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}})} \frac{H^1_{par}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{par}(\Gamma_0'(N), \mathbb{C}_p)}{\mathrm{im}(\varphi_s \oplus \varphi_t)} \to 0$$

Here there is a small abuse of notation as the map $\deg$ should send divisor-valued modular symbols to a space of modular symbols. But as we will see, modular symbols can be naturally related to parabolic cohomology. We will prove this theorem at the end of Chapter 4, in Theorem 4.5.2.
The rest of Chapter 3 gives a characterization of rigid analytic cocycles in terms of p-new modular forms and studies the multiplicative lifts of these cocycles.

Chapter 4 is dedicated to generalizations of results of [35] for congruence subgroup as well as considering the weight 2 case. The fundamental result of this chapter constructs a kernel $\widehat{\Omega_k(z,\tau)}$ which is a vector-valued cusp form of level pN giving a correspondence between rigid analytic cocycles and vector-valued modular forms of weight $k + 1/2$ and level $4pN$, giving the following theorem:

Theorem 1.1.3. $\widehat{\Omega_k(z,\tau)} \in S_{k+1/2}(\Gamma_0(4pN), \rho_{L_2}) \otimes MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$*, i.e.* $\widehat{\Omega_k(z,\tau)}$ *is a* $\Gamma_0(pN)$*- vector-valued cusp form for the Weil representation associated to* L_2 *with coefficients in* $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$.

The edge case $k = 1$ is dealt with in Section 4.4 using *Hecke's convergence trick*.

Chapter 5 gives an algorithm to efficiently compute rigid meromorphic cocycles for $\Gamma_0^p(N)$, using the space of modular symbols and M-symbols as a helpful tool. The rest of this chapter gives values of these cocycles obtained by this algorithm.

1.2 Notation

We list here the important notations used throughout this thesis so that the reader can refer to this page when need be.

$\Gamma := SL_2(\mathbb{Z})$, $\Gamma^p := SL_2(\mathbb{Z}[1/p])$ and $\Gamma_0(N)$, $\Gamma_0^p(N)$ denote their respective congruence subgroup with lower-left entry divisible by N. We will commonly use $\Gamma_0'(N) := P^{-1}\Gamma_0(N)P$ where $P := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

$\alpha_1, ..., \alpha_Q$ are representatives of $\Gamma^p/\Gamma_0^p(N)$.

$\gamma_1, ..., \gamma_{p+1}$ are representatives of $\Gamma_0(N)/\Gamma_0(pN)$.

$c_1, ..., c_l$ are representatives of $\Gamma_0^p(N)\backslash\mathbb{P}^1(\mathbb{Q})$, i.e. cusps of $\Gamma_0^p(N)$.

$\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ is the p-adic upper half plane.

$\mathcal{O}$ and $\mathcal{M}$ are the rings of rigid analytic functions and rigid meromorphic functions on $\mathcal{H}_p$ respectively. We will use the usual superscript / subscript when we refer to the multiplicative group or the additive group with weight-k action. So for example, $\mathcal{M}_k$ is the additive group of rigid meromorphic functions with weight-k action given by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right), \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and $\mathcal{M}^\times$ is the multiplicative group of rigid meromorphic functions.

$\mathcal{T}$ is the Bruhat-Tits tree of $\mathcal{H}_p$ and $\mathcal{V}$, $\mathcal{V}^+$, $\mathcal{V}^-$ are its vertices, even vertices and odd vertices respectively. $\mathcal{E}$ and $\mathcal{E}^*$ are its unordered and ordered edges respectively.

$\mathcal{A}_v$ is the affinoid in $\mathcal{H}_p$ corresponding to a vertex v of $\mathcal{T}$ and $\mathcal{W}_v$ is the wide open space corresponding to a vertex v with all the edges connected to v. We will use U_e to denote the annulus associated to an edge.

$S_{2k}(N)$ (respectively $E_{2k}(N)$) is the $\mathbb{R}$-vector space of cusp forms (respectively Eisenstein series) for $\Gamma_0(N)$.

$H^1_{par}(\Gamma, \Omega)$ denotes the parabolic cohomology of a Γ-module Ω, i.e. the subset of elements of H^1 in the kernel of

$$H^1(\Gamma, \Omega) \to \oplus_{c_i, \text{ cusps of } \Gamma} H^1(\Gamma_{c_i}, \Omega),$$

where Γ_{c_i} is the stabilizer in Γ of the cusp c_i.

Chapter 2

The case $SL_2(\mathbb{Z}[1/p])$

In this chapter, we review the literature and the results regarding rigid analytic / meromorphic cocycles for $SL_2(\mathbb{Z}[1/p])$, which were introduced in [16]. We introduce some preliminaries in Section 2.1.Section 2.2 - 2.4 deals with the classification of additive / multiplicative rigid cocycles and is based on [16]. Finally,Section 2.5 is about a special kind of Borcherd's lifts associated to rigid analytic cocycles, which can be found in [15].

2.1 Preliminaries

2.1.1 The p-adic upper half plane

Let $\mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half plane and $\mathcal{T}$ be the associated Bruhat-Tits tree, i.e. the undirected graph whose vertices correspond to $\mathbb{Q}_p^\times$-homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ and where two vertices $[L_1], [L_2]$ are joined by an edge if

$$L_1' \supset L_2' \supset pL_1',$$

for some $L_1' \in [L_1]$ and $L_2' \in [L_2]$. We let $\mathcal{V}, \mathcal{E}$ and $\mathcal{E}^*$ be respectively the set of vertices, unordered edges and ordered edges. The standard vertex corresponding to $\mathbb{Z}_p^2$ will be denoted by v_0. Given an ordered edge $e \in \mathcal{E}^*$, we denote by $s(e)$ and $t(e)$ the source and the target of e respectively.
We denote the distance between two vertices by $d(u, v)$, and a vertex is said to be even if its distance with v_0 is even, and odd otherwise. We denote the set of even vertices (respectively odd vertices) by $\mathcal{V}^+$ (resp. $\mathcal{V}^-$). Similarly, an edge $e \in \mathcal{E}^*$ is even if $s(e)$ is even, and odd otherwise, and the subsets of even edges (resp. odd) is denoted by $\mathcal{E}^+$ (resp. $\mathcal{E}^-$).

We have the Γ-invariant *reduction map* (See [17])

$$\text{red} \ : \ \mathcal{H}_p \to \mathcal{T}.$$

A *vertex affinoid* is the inverse image of a vertex v under red, denoted by $\mathcal{A}_v$. The inverse image of an edge e is an annulus, denoted by U_e. Finally, for a vertex v, we will denote by $\mathcal{W}_v$ the inverse image of v and all the edges attached to v, i.e.

$$\mathcal{W}_v := \text{red}^{-1}\left(v \cup \bigcup_j e_j\right),$$

where e_j are the edges attached to v. The subset $\mathcal{W}_v$ is called the standard wide open subset attached to v. The image of $\mathcal{H}_p(\hat{\mathbb{Q}}_p^{nr})$ where $\hat{\mathbb{Q}}_p^{nr}$ is the completion of the unique maximal unramified extension of $\mathbb{Q}_p$ is $\mathcal{V}$, meaning that the reduction map sends unramified elements of $\mathcal{H}_p$ to vertices.

$\mathcal{T}^{\leqslant n}$ will denote the subgraph of $\mathcal{T}$ of vertices and edges at distance $\leqslant n$ from v_0 and $\mathcal{H}_p^{\leqslant n} := \text{red}^{-1}(\mathcal{T}^{\leqslant n})$, and similarly for $\mathcal{T}^{<n}$. These sets give a cover

$$\mathcal{H}_p = \bigcup_{n \geqslant 0} \mathcal{H}_p^{\leqslant n}.$$

Finally, we say that a function on $\mathbb{C}_p$ is rigid analytic if its restriction to any affinoid $\mathcal{A}$ of $\mathcal{H}_p$ is a uniform limit of rational functions on $\mathbb{P}_1(\mathbb{C}_p)$ having poles outside of $\mathcal{A}$, the limit being taken under the supremum norm. We denote the space of rigid anlytic functions by $\mathcal{O}$ and $\mathcal{M}$ denotes its fraction field.

The action of Γ^p (and consequently $\Gamma_0^p(N)$ also) on $\mathcal{H}_p$ and the Bruhat-Tits tree $\mathcal{T}$ by Möbius transformations are compatible under the reduction map, and thus, we have for every $\gamma \in \Gamma^p$

$$\gamma \mathcal{A}_v = \mathcal{A}_{\gamma v}, \qquad \gamma \mathcal{W}_v = \mathcal{W}_{\gamma v}.$$

A fundamental domain of $\mathcal{T}$ under Γ^p or $\Gamma_0^p(N)$ is $\{v, \overline{v}, e\}$ where v is a vertex of even degree, $\overline{v}$ is a vertex of odd degree and e is any edge. For example, one could take $v = v_0$ to be the standard vertex, $\overline{v}$ to be the vertex associated to the lattice $\frac{1}{p}\mathbb{Z}_p \oplus \mathbb{Z}_p$ and e to be the edge between these two vertices. The stabilizer in Γ^p (respectively $\Gamma_0^p(N)$) of any even vertex is conjugate to $SL_2(\mathbb{Z})$ (respectively $\Gamma_0(N)$), the stabilizer of any odd vertex is conjugate to $P \cdot SL_2(\mathbb{Z}) \cdot P^{-1}$ (respectively $P \cdot \Gamma_0(N) \cdot P^{-1}$) where

$$P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

and the stabilizer of any edge is conjugate to $\Gamma_0(p)$ (respectively $\Gamma_0(pN)$).

More general constructions of the p-adic upper half plane and its associated Bruhat-Tits tree can be found in [17], [37] and [4].

2.1.2 Modular symbols

Modular symbols are powerful tools to compute the parabolic cohomology of any Γ-module, where Γ is any congruence subgroup of $SL_2(\mathbb{Z})$ or $SL_2(\mathbb{Z}[1/p])$. Throughout this section, Ω will denote an arbitrary Γ^p-module with a right action $|\gamma$, but most of these results also hold when we replace Γ^p by $SL_2(\mathbb{Z})$.

Definition 2.1.1. *A Ω-valued modular symbol is a function*

$$\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \Omega$$

satisfying the two modular symbol relations

$$m\{r,s\} = -m\{s,r\}, \quad m\{r,s\} + m\{s,t\} = m\{r,t\} \quad \textit{for all } r,s,t \in \mathbb{P}_1(\mathbb{Q}).$$

The space of Ω-valued modular symbols is denoted by $MS(\Omega)$ and has an action of $PGL_2(\mathbb{Q})$:

$$(m|\gamma)\{r,s\} := (m\{\gamma r, \gamma s\})|\gamma,$$

where $|\gamma$ represents the action of γ on Ω. The space of Γ-invariant modular symbols is denoted by $MS^\Gamma(\Omega) := H^0(\Gamma, MS(\Omega))$. These Γ^p-invariant modular symbols satisfy

$$m\{\gamma r, \gamma s\} = m\{r,s\}|\gamma^{-1}.$$

When $(n,p) = 1$, we have the same action for the Hecke operator T_n as for modular forms, i.e. consider the double coset decomposition

$$\Gamma^p \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma^p = \bigsqcup_i \Gamma^p \gamma_i.$$

We let

$$(m|T_n) := \sum_i (m|\gamma_i),$$

which is independent of the choice of $\{\gamma_i\}$ by the Γ^p-invariance. When $n = p$, we usually denote T_p by U_p instead. The U_p-operator will be of fundamental importance in Chapter 5 when computing rigid meromorphic period functions for $\Gamma_0^p(N)$. We also define the ω_p-operator as $(m|\omega_p)\{r,s\} := (m|P)\{r,s\}$ where as above

$$P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

As mentioned at the beginning of this section, it turns out that modular symbols can be used to compute the parabolic cohomology of Γ.

Lemma 2.1.2. *There is an exact sequence*

$$0 \to \Omega^{\Gamma^p} \to \Omega^{\Gamma^p_\infty} \to MS^{\Gamma^p}(\Omega) \to H^1(\Gamma^p, \Omega) \to H^1(\Gamma^p_\infty, \Omega),$$

where Γ^p_∞ is the stabilizer of the cusp $\infty \in \mathbb{P}_1(\mathbb{Q})$ in Γ^p.
In particular, the connecting homomorphism $\delta : MS^{\Gamma^p}(\Omega) \to H^1_{par}(\Gamma^p, \Omega)$ is an Hecke-equivariant surjection, which is an isomorphism if $\Omega^\Gamma = \Omega^{\Gamma_\infty}$. Here, we define

$$H^1_{par}(\Gamma^p, \Omega) := \ker(H^1(\Gamma^p, \Omega) \to H^1(\Gamma^p_\infty, \Omega)).$$

Proof. We recall the proof given in [16]. Consider the short exact sequence of Γ^p-modules

$$0 \to \Omega \to \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega) \xrightarrow{d} MS(\Omega) \to 0,$$

where $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega)$ is the set of Ω-valued functions on $\mathbb{P}_1(\mathbb{Q})$ with Γ^p-action defined by

$$(f|\gamma)(r) := f(\gamma r)|\gamma,$$

and d is defined by $df\{r, s\} := f(s) - f(r)$. Shapiro's lemma gives us the isomorphism

$$H^1(\Gamma^p, \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega)) \cong H^1(\Gamma^p_\infty, \Omega)$$

and the result follows by taking the long exact sequence associated to the Γ^p-cohomology of the short exact sequence above. □

The case $\Omega = \mathcal{M}^\times$ and $\Omega = \mathcal{M}_k$ can be simplified using the following proposition.

Proposition 2.1.3. *The Γ_∞-invariants of $\mathcal{M}^\times$ and $\mathcal{M}_k$ are given by*

$$H^0(\Gamma_\infty, \mathcal{M}^\times) = \mathbb{C}_p^\times, \quad H^0(\Gamma_\infty, \mathcal{M}_k) = \begin{cases} \mathbb{C}_p \text{ if } k = 0, \\ 0 \text{ if } k > 0 \end{cases}$$

Proof. Since any translation-invariant rigid analytic function is constant on the standard affinoid by the Weierstrass preperation theorem, it is constant everywhere by analytic continuation. □

Using the two last results, we obtain

Corollary 2.1.4. *For all $k \geqslant 0$, the map δ of Lemma 2.1.2 induces isomorphisms*

$$MS^\Gamma(\mathcal{M}^\times) \xrightarrow{\sim} H^1_{par}(\Gamma, \mathcal{M}^\times), \quad MS^\Gamma(\mathcal{M}_k) \xrightarrow{\sim} H^1_{par}(\Gamma, \mathcal{M}_k).$$

In the hope of computing rigid meromorphic cocycles with the help of modular symbols, it will be useful to "package" the information given by modular symbols (which is an infinite number of rigid meromorphic functions indexed by $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$) into a finite set of functions. We first look at what happens when $\Gamma = SL_2(\mathbb{Z})$.

Proposition 2.1.5. *Let*

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

be the usual order 4 and order 6 matrices in $SL_2(\mathbb{Z})$, and let $\Omega^\dagger \subset \Omega$ be the subset of elements $\omega \in \Omega$ satisfying

$$\omega + S\omega = 0, \qquad \omega + U\omega + U^2\omega = 0.$$

Then the assignment $m \mapsto m\{0, \infty\}$ is a bijection from $MS^{SL_2(\mathbb{Z})}(\Omega)$ to $\Omega^\dagger$.

Proof. A more general proof for $\Gamma_0(N)$ will be given in Proposition 3.3.3. One can also see Proposition 1.4 of [16]. □

When one considers Γ^p-invariant modular symbols instead, we obtain the following.

Proposition 2.1.6. *Let $\Omega^\ddagger \subset \Omega^\dagger$ be the image of $MS^{\Gamma^p}(\Omega)$ under the assignment above. Then an element $\omega \in \Omega^\ddagger$ also satisfies*

$$D\omega = \omega$$

where

$$D = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}.$$

Proof. This is clear since m is a Γ^p-invariant modular symbol and D fixes 0 and ∞. □

More specifically, using the correspondence between $MS^{\Gamma^p}(\Omega)$ and $\Omega^\ddagger$ when $\Omega = \mathcal{M}_k$, one can see that an element $\varphi \in \mathcal{M}_k^\ddagger$ satisfies the two and three term relations

$$\varphi\left(-\frac{1}{z}\right) = -z^k\varphi(z), \qquad \varphi(z) = z^{-k}\varphi\left(\frac{z-1}{z}\right) + (z-1)^{-k}\varphi\left(\frac{-1}{z-1}\right) = 0$$

and the relation coming from the invariance under D

$$\varphi(p^2 z) = p^{-k}\varphi(z).$$

An element of $\mathcal{M}_k^\ddagger$ will be called a *rigid meromorphic period function of weight k*. Equivalently, using the isomorphism between modular symbols and rigid functions, a rigid meromorphic period function of weight

k is the value at S of the parabolic representative of a rigid meromorphic cocycle of weight k, or the value at $\{0, \infty\}$ of a a modular symbol $m \in MS^{\Gamma_p}(\mathcal{M}_k)$. In the classical setting, we use the term *rational period function* to denote an element of $(\mathcal{M}_k^{\text{rat}})^\dagger$ where $\mathcal{M}_k^{\text{rat}}$ is the set of rational functions on $\mathbb{C}$.

2.1.3 Eichler-Shimura isomorphism

We now explain briefly the Eichler-Shimura isomorphism which will be used throughout this thesis. This isomorphism will be used to relate spaces of modular forms to Γ_N-cohomology with values in $\mathbb{C}$, where Γ_N is any congruence subgroup of $SL_2(\mathbb{Z})$. As mentioned in the last section, cohomology is deeply related to modular symbols, which is the reason why the isomorphism will be stated using them.

Let $S_k(\Gamma_N)$ (respectively $E_k(\Gamma_N)$) be the space of cusp forms (respectively Eisenstein series) of weight k and level Γ_N. We also consider $P_k(\mathbb{C})$ the space of homogeneous polynomials in two variables of degree $k-2$ with the usual Γ_N-action given by

$$(P|\gamma)(x,y) := P(ax+by, cx+dy).$$

Given a cusp form $f \in S_k(\Gamma_N)$, one can define a Γ_N-invariant modular symbol with values in $P_k(\mathbb{C})^*$ (the dual of $P_k(\mathbb{C})$) by

$$I_f\{r,s\}(P) = \int_r^s f(z) \cdot P(z,1)dz.$$

Convergence of the integral comes from the fact that f is a cusp form and Γ_N-invariance is a simple computational exercise. So this gives us a map

$$\phi_k \ : \ S_k(\Gamma_N) \to MS^{\Gamma_N}(P_k(\mathbb{C})^*),$$

which gives an element of $H^1_{\text{par}}(\Gamma_N, P_k(\mathbb{C})^*)$ when composed with the map of the last section.

Proposition 2.1.7. *The composition*

$$ES_{k,par} \ : \ S_k(\Gamma_N) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{2\Re(\phi_k)\otimes 1} MS^{\Gamma_N}(P_k(\mathbb{R})^*) \otimes \mathbb{C} \to H^1_{par}(\Gamma_N, P_k(\mathbb{C})^*)$$

is an isomorphism.

The Eisenstein part can be dealt with by considering the map

$$E_k(\Gamma_N) \to H^1(\Gamma_N, P_k(\mathbb{C})^*)$$
$$f \mapsto u_f,$$

where

$$u_f(\gamma) = g|_{\gamma^{-1},-(k-2)} - g$$

and g is a $k-1$-primitive of f, i.e. g is an analytic function satisfying $g^{(k-1)} = f$. The full Eichler-Shimura isomorphism is given by the following:

Theorem 2.1.8 (Eichler-Shimura)**.** *We have a Hecke-equivariant isomorphism*

$$ES_k \ : \ (S_k(\Gamma_N) \otimes_{\mathbb{R}} \mathbb{C}) \oplus E_k(\Gamma_N) \xrightarrow{ES_{k,par}+u_f} H^1(\Gamma_N, P_k(\mathbb{C})^*).$$

2.2 Additive rigid meromorphic cocycles

We now focus on the classification of additive rigid meromorphic cocycles, i.e. elements of $MS^{\Gamma}(\mathcal{M}_2)$.

As seen in the last section, the map

$$\begin{aligned} eval\colon MS^{\Gamma}(\mathcal{M}_2) &\to \mathcal{M}_2 \\ \phi &\mapsto \phi\{0,\infty\}. \end{aligned}$$

is injective. We will use this correspondence throughout this section.

First, we will define a special class of rigid meromorphic cocycles indexed by a RM point τ. Recall that the Γ-orbit of τ is dense in $\mathcal{H}_p$ for the rigid analytic topology, and that its image under the reduction map is either a vertex or the midpoint of an edge. We define a subset

$$\Sigma_\tau := \{w \in \Gamma^p\tau \ : \ ww' < 0\}$$

where w' is the conjugate of w. It is endowed with a sign function $\delta_\infty \ : \ \Sigma_\tau \to \pm 1$ which partitions Σ_τ into positive and negative elements. More generally, let $r, s \in \mathbb{P}_1(\mathbb{Q})$ and let $\gamma(r,s)$ denote the geodesic in $\overline{\mathcal{H}}$ from r to s. Then

$$\overline{\mathcal{H}} - \gamma(r,s) = \mathcal{H}^+(r,s) \sqcup \mathcal{H}^-(r,s)$$

with the convention that $\mathcal{H}^+(r,s)$ is the region on the right of $\gamma(r,s)$. Let $\Sigma_\tau(r,s)$ be the set of RM points w in the Γ-orbit of τ such that w and w' lie in two distinct connected components of $\overline{\mathcal{H}} - \gamma(r,s)$. Then $\Sigma_\tau(r,s)$ is also endowed with a sign function

$$\delta_{r,s}(w) := \begin{cases} 1 \quad if \ w \in \mathcal{H}^+(r,s) \\ -1 \quad if \ w \in \mathcal{H}^-(r,s). \end{cases}$$

We see that in particular

$$\Sigma_\tau = \Sigma_\tau(0,\infty), \quad \delta_\infty = \delta_{0,\infty}.$$

Finally, we define the divisor

$$\Delta_\tau\{r,s\} := \sum_{w\in\Sigma_\tau(r,s)} \delta_{r,s}(w)[w] \in \mathrm{Div}(\mathcal{H}_p)$$

where $\mathrm{Div}(\mathcal{H}_p)$ is the Γ^p-module of (possibly infinite) formal sums of elements in $\mathcal{H}_p$, i.e. functions from $\mathcal{H}_p$ to $\mathbb{C}_p$. Using the fact that $\delta_{\gamma r,\gamma s}(w|\gamma) = \delta_{r,s}(w)$, one can see that Δ_τ is a Γ^p-invariant modular symbol. We say that a subset $F \subset \mathcal{H}_p$ is discrete if its intersection with any affinoid of $\mathcal{H}_p$ is finite. The subset of such divisors is denoted by $\mathrm{Div}^\dagger(\mathcal{H}_p)$

Lemma 2.2.1. *For all $r,s \in \mathbb{P}_1(\mathbb{Q})$, the sets $\Sigma_\tau(r,s)$ are discrete and the finite intersections $\Sigma_\tau(r,s)\cap\mathcal{A}_v$ and $\Sigma_\tau(r,s)\cap\mathcal{W}_v$ contain an equal number of positive and negative elements.*

Proof. First consider the case when $\{r,s\} = \{0,\infty\}$. Since $\Sigma_\tau \cap \mathcal{H}_p^{\leqslant n}$ consists of RM points with discriminant smaller than a fixed bound by [16] Proposition 1.1, this set is finite. Therefore $\Sigma_\tau\cap\mathcal{A}_v$ and $\Sigma_\tau\cap\mathcal{W}_v$ are finite sets and the divisors

$$\Delta_\tau^{\mathcal{A}_v}\{r,s\} := \sum_{w\in\Sigma_\tau(r,s)\cap\mathcal{A}_v} \delta_{r,s}(w)[w], \qquad \Delta_\tau^{\mathcal{W}_v}\{r,s\} := \sum_{w\in\Sigma_\tau(r,s)\cap\mathcal{W}_v} \delta_{r,s}(w)[w]$$

are finite divisors when $\{r,s\} = \{0,\infty\}$. Using the Γ^p-invariance property $\gamma\Delta_\tau^{\mathcal{A}_v}\{0,\infty\} = \Delta_\tau^{\mathcal{A}_{\gamma v}}\{\gamma 0,\gamma\infty\}$ and the modular symbols relation, we get that the same is true for any $\{r,s\} \in \mathbb{P}_1(\mathbb{Q})\times\mathbb{P}_1(\mathbb{Q})$. Now the degree map

$$\begin{aligned}\deg_{\mathcal{A}_v} \ :\ \mathrm{Div}^\dagger(\mathcal{H}_p) &\to \mathbb{Z}\\ \Delta &\mapsto \deg(\Delta_{\mathcal{A}_v})\end{aligned}$$

(and similarly for $\mathcal{W}_v$) induces a map $MS^{\Gamma^p}(\mathrm{Div}^\dagger(\mathcal{H}_p)) \to MS^{\Gamma_v^p}(\mathbb{Z})$. As seen in Section 2.1.1, Γ_v^p is conjugate to $SL_2(\mathbb{Z})$, and since there are no cusp forms of weight 2 for $SL_2(\mathbb{Z})$, the Eichler-Shimura isomorphism tells us that $MS^{\Gamma_v^p}(\mathbb{Z}) = 0$. Therefore $\deg(\Delta_{\mathcal{A}_v}) = \deg(\Delta_{\mathcal{W}_v}) = 0$ which concludes the proof. □

Theorem 2.2.2. *Let τ be a RM point. Then the infinite sums*

$$\Phi_\tau\{r,s\} = \sum_{w\in\Sigma_\tau(r,s)} \delta_{r,s}(w)\frac{1}{z-w}$$

converge to a rigid meromorphic function and give an element $\Phi_\tau \in MS^\Gamma(\mathcal{M}_2)$.

Proof. We direct the reader to Proposition 4.3.2 for the proof of a generalization of this result. □

A small remark has to be made when τ is ramified, i.e. when τ generates a real quadratic field where p is ramified. In that case we can define

$$\Phi_\tau^+\{r,s\} = \lim_{n \text{ odd}} \sum_{w \in \Sigma_\tau(r,s) \cap \mathcal{H}_p^{\leqslant n}} \delta_{r,s}(w) \frac{1}{z-w}, \qquad \Phi_\tau^-\{r,s\} = \lim_{n \text{ even}} \sum_{w \in \Sigma_\tau(r,s) \cap \mathcal{H}_p^{\leqslant n}} \delta_{r,s}(w) \frac{1}{z-w}.$$

Both series converge and the convergence of $\Phi_\tau\{r,s\} := \Phi_\tau^+\{r,s\} + \Phi_\tau^-\{r,s\}$ follows. Interestingly, $\Phi_\tau^+\{r,s\} - \Phi_\tau^-\{r,s\}$ can be seen as the representation of a modular symbol with values in $\mathbb{Z}$, i.e. let

$$m_\tau\{r,s\} := \deg(\Delta_\tau^{\mathcal{W}_{e_0}}\{r,s\}) = \sum_{w \in \Sigma_\tau(r,s) \cap \mathcal{W}_{e_0}} \delta_{r,s}(w)$$

where U_{e_0} is the standard annulus with stabilizer $\Gamma_0(p)$. Then

$$\Phi_\tau^+\{r,s\} - \Phi_\tau^-\{r,s\} = \mathrm{ST}(m_\tau)$$

where ST is the Schneider-Teitelbaum lift that will appear in the next section. See [15] Section 4.3 for more details.

These Φ_τ give a plethora of rigid meromorphic cocycles indexed by Γ^p-orbits of RM points. Since the divisors of Φ_{τ_1} and Φ_{τ_2} are disjoint for $\tau_1 \neq \tau_2$ in $\Gamma^p \backslash \mathcal{H}_p^{\mathrm{RM}}$, the set $\{\Phi_\tau\}_{\tau \in \Gamma^p \backslash \mathcal{H}_p^{\mathrm{RM}}}$ is linearly independent. In fact, it turns out that up to rigid analytic cocycles, there are no other rigid meromorphic cocycles.

Theorem 2.2.3. *A rigid meromorphic period function of weight two is a unique finite linear combination of the elements $\varphi_\tau := \Phi_\tau\{0,\infty\}$ and of a rigid analytic period function of weight two.*

Proof. We give a quick sketch of this proof which will also be used in the next chapter.

Let φ be a rigid meromorphic period function of weight two and let Σ_φ be its set of poles. We have seen in Section 2.2.1 that $\Sigma_\varphi^{<1} := \Sigma_\varphi \cap \mathcal{H}_p^{<1}$ is finite. Since $\Sigma_\varphi^{<1}$ is invariant under $SL_2(\mathbb{Z})$, we get that $\Sigma_\varphi^{<1}$ satisfies the two and three term relations and a classical result (See [9]) gives the decomposition

$$\Sigma_\varphi^{<1} = \sqcup_{\tau \in I_\varphi} \Sigma_\tau^\circ$$

where $I_\varphi \subset SL_2(\mathbb{Z}) \backslash (\mathcal{H}_p^{\mathrm{RM}} \cap \mathcal{H}_p^{<1})$ is a finite set and

$$\Sigma_\tau^\circ := \{w \in SL_2(\mathbb{Z})\tau \ : \ ww' < 0\}.$$

Again, a result in [9] implies that φ has simple poles on $\mathcal{H}_p^{<1}$ and that the difference

$$\varphi - \sum_{\tau \in I_\varphi} \lambda_\tau \varphi_\tau$$

where

$$\text{res}_w\varphi(z)dz := \begin{cases} \lambda_\tau & if\ w > 0 \\ -\lambda_\tau & if\ w < 0 \end{cases}$$

is a rigid meromorphic period function having no singularities on $\mathcal{H}_p^{<1}$. Restricting to the case where φ is p-even, one can then show that φ is also regular everywhere, which concludes the proof. □

The last result gives us a short exact sequence of Γ-modules

$$0 \to MS^\Gamma(\mathcal{O}_2) \to MS^\Gamma(\mathcal{M}_2) \to \bigoplus_{\tau \in \Gamma^p \backslash \mathcal{H}_p^{\text{RM}}} \mathbb{C}_p \Phi_\tau \to 0.$$

As we will see in Section 3.4, this sequence is not a priori exact when considering $\Gamma_0^p(N)$, mainly because the Φ_τ don't necessarily converge since the degree of these functions on affinoids are not 0.

We now conclude this section with some corollaries regarding Hecke operators and the relation between $\mathcal{M}_2$ and $\mathcal{M}$.

Lemma 2.2.4. *Let τ be a RM-point and let l be a prime not dividing p or the discriminant D of τ. Then $T_l(\Phi_\tau)$ is a linear combination of some $\Phi_{\tau'}$ such that τ' is of discriminant D or Dl^2, with at least one τ' of discriminant Dl^2.*

Corollary 2.2.5. *If a finite dimensional subspace V of $MS^{\Gamma^p}(\mathcal{M}_2)$ is stable under Hecker operators, then $V \subset MS^{\Gamma^p}(\mathcal{O}_2)$*

Corollary 2.2.6. *The natural inclusion $MS^{\Gamma^p}(\mathcal{O}) \to MS^{\Gamma^p}(\mathcal{M})$ is an isomorphism.*

Proof. Consider the derivative map $d : \mathcal{M} \to \mathcal{M}_2$. Its image consist of rigid meromorphic functions with vanishing residues and induces a map

$$d : MS^{\Gamma^p}(\mathcal{M}) \to MS^{\Gamma^p}(\mathcal{M}_2).$$

Using the classification of $MS^{\Gamma^p}(\mathcal{M}_2)$, we see that the image of d must be contained in $MS^{\Gamma^p}(\mathcal{O}_2)$ and thus $MS^{\Gamma^p}(\mathcal{O}) = MS^{\Gamma^p}(\mathcal{M})$. □

2.3 Additive rigid analytic cocycles

The goal of this section is to describe the isomorphism between the space of rigid analytic cocycles $MS^\Gamma(\mathcal{O}_2)$ and $MS^{\Gamma_0(p)}(\mathbb{C}_p)$, the last space being related to classical modular forms of weight 2 and level p by the Eichler-Shimura isomorphism. Again, this section will follow closely Section 2 of [16]. We shall recall the

proofs which can be translated word for word in the $\Gamma_0^p(N)$-case and we will omit the proofs that will differ in that case.

Let $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ be the dual of the space of locally analytic functions on $\mathbb{P}_1(\mathbb{Q}_p)$ with the topology of uniform convergence on compact open subsets. For $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ and h a locally analytic function, we use the notation

$$\mu(h) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t)d\mu(t)$$

and

$$\int_U h(t)d\mu(t) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} 1_U(t)h(t)d\mu(t)$$

where $U \subseteq \mathbb{P}_1(\mathbb{Q}_p)$ is any compact open subset and 1_U is the characteristic function of U.

The subset of distributions of total volume 0, i.e. such that $\mu(1) = 0$, where 1 is the constant function equal to 1 on $\mathbb{P}_1(\mathbb{Q})$, is denoted by $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$. The set $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts naturally on $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ and $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ by

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t)d(\mu|\gamma)(t) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(\gamma t)d\mu(t).$$

Given $f \in \mathcal{O}_2$ and $U_e \subseteq \mathbb{P}_1(\mathbb{Q}_p)$ a compact open subset, we define a distribution by

$$\int_{U_e} h(t)d\mu_f(t) := \mathrm{res}_e(f(z)h(z)dz)$$

where res_e is the p-adic annular residue along U_e. μ_f is called the *boundary distribution* attached to f, and using the residue theorem, one concludes that $\mu_f \in \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$.

Proposition 2.3.1. *The map $f \mapsto \mu_f$ induces an $PGL_2(\mathbb{Q}_p)$-equivariant isomorphism*

$$BD \ : \ \mathcal{O}_2 \xrightarrow{\sim} \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$$

Proof. See Theorem 2.2.1 of [17]. □

Therefore, using the last proposition, we also have an isomorphism between modular symbols

$$BD \ : \ MS^{\Gamma^p}(\mathcal{O}_2) \xrightarrow{\sim} MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))).$$

Using various results and maps related to $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$, it turns out that $MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))$ is easier to characterize.

Using the inclusion $\mathbb{Z}_p \subset \mathbb{Q}_p$, and seeing that $\Gamma_0(p)$ is the stabilizer of $\mathbb{Z}_p$ in $\mathbb{Q}_p$, we get a natural map

$$\mathrm{res}_{\mathbb{Z}_p} \ : \ MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \longrightarrow MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)),$$

where $\mathcal{D}(\mathbb{Z}_p)$ is the space of distributions on $\mathbb{Z}_p$. The image of $\text{res}_{\mathbb{Z}_p}$ is called the space of overconvergent modular symbols of weight two and level p. We direct the reader to [36] for a thourough investigation of this space.

Lemma 2.3.2. *The map $res_{\mathbb{Z}_p}$ is injective.*

Proof. We recall the proof which can be found in [16] and [15].

Let ι_p be the matrix

$$\iota_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.$$

It sends $\mathbb{Z}_p$ to its complement $\mathbb{Z}_p' := \mathbb{P}_1(\mathbb{Q}_p)\backslash\mathbb{Z}_p$ and vice-versa, and normalises $\Gamma_0(p)$. It thus induces an isomorphism

$$\iota_p \ : \ MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \to MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p'))$$

which makes the diagram

$$\begin{array}{ccc} MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & \xrightarrow{\omega_p} & MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \\ \downarrow{\scriptstyle \text{res}_{\mathbb{Z}_p}} & & \downarrow{\scriptstyle \text{res}_{\mathbb{Z}_p'}} \\ MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\iota_p} & MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p')) \end{array}$$

commute. So we only need to show that $\text{res}_{\mathbb{Z}_p'}$ is injective. Consider the decomposition

$$\mathbb{P}_1(\mathbb{Q}) = B_0 \sqcup B_1 \sqcup ... \sqcup B_{p_1} \sqcup \mathbb{Z}_p',$$

where B_j is the mod p residue disk of $-j$. $SL_2(\mathbb{Z})$ acts transitively on this decomposition, and therefore if we have a measure $\mu \in MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))$ satisfying $\mu\{r,s\}|_{\mathbb{Z}_p'} = 0$ for all $r, s \in \mathbb{P}_1(\mathbb{Q})$, then by Γ^p-invariance, the same is true for every element of the above decomposition, and therefore for all of $\mathbb{P}_1(\mathbb{Q})$. This concludes the proof. □

There is a natural action of the U_p operator on the space of overconvergent modular symbols defined by

$$\int_{\mathbb{Z}_p} h(t)d(U_p\mu)\{r,s\}(t) := \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} h(t)d(\mu|\alpha_j)\{r,s\}(t), \qquad \text{where } \alpha_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$$

and the action of α_p on μ is the usual one on modular symbols. It turns out that the action of U_p is well-behaved under $\text{res}(\mathbb{Z}_p)$, i.e.

$$\text{res}_{\mathbb{Z}_p} \ : \ MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))^{\pm} \longrightarrow MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\pm}$$

is a Hecke-equivariant injection, where $MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))^{\pm}$ are the $\pm$-eigenspaces for the action of ω_p, and $MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\pm}$ are the subspaces where U_p acts as ± 1.

There is also a map $\mathcal{D}(\mathbb{Z}_p) \to \mathbb{C}_p$ which sends a measure μ to its total mass $\mu(1)$. It induces a map on modular symbols which is well understood using the next theorem.

Theorem 2.3.3 (Stevens)**.** *The map $\mathcal{D}(\mathbb{Z}_p) \to \mathbb{C}_p$ induces an isomorphism*

$$\rho \ : \ MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1} \to MS^{\Gamma_0(p)}(\mathbb{C}_p).$$

Proof. See [36]. □

Corollary 2.3.4. *The map*

$$\eta := \rho \circ res_{\mathbb{Z}_p} \circ BD \ : \ MS^{\Gamma^p}(\mathcal{O}_2) \to MS^{\Gamma_0(p)}(\mathbb{C}_p)$$

is injective.

We conclude this section by giving a quick summary of the proof of the surjectivity of η, completing the classification of rigid analytic cocycles.

Definition 2.3.5. *Let Ω be a Γ^p-module. A function $c \ : \ \mathcal{E}^* \to \Omega$ is called harmonic if it satisfies the two relations*

$$c(\overline{e}) = -c(\overline{e}), \text{ for all } e \in \mathcal{E}^*, \quad \text{and} \quad \sum_{s(e)=v} c(e) = 0, \text{ for all } v \in \mathcal{V}, e \in \mathcal{E}^*,$$

where $\overline{e}$ is the edge e with the opposite orientation, and $s(e)$ is the source of e. The set of harmonic functions with values in Ω is denoted $C_{har}(\Omega)$.

Following [17], given $f \in \mathcal{O}_2$, we define a harmonic function with values in $\mathbb{C}_p$ defined by

$$c_f(e) = \partial_e f(z)dz, \quad \text{for } e \in \mathcal{E}^*,$$

where ∂_e is the p-adic annular residue on the oriented annulus $\mathcal{W}_e$. We thus have a map

$$\begin{aligned} \partial \ : \ \mathcal{O}_2 &\to C_{\text{har}}(\mathbb{C}_p) \\ c &\mapsto c_f \end{aligned}$$

called the *residue map*. It induces a map on Γ^p-invariant modular symbols, also denoted ∂. There is also a natural map

$$\begin{aligned} \text{ev}_{e_0} \ : \ C_{\text{har}}(\mathbb{C}_p) &\to \mathbb{C}_p \\ c &\mapsto c(e_0), \end{aligned}$$

where e_0 is the standard edge with stabilizer $\Gamma_0(p)$. It induces a map on modular symbols

$$\mathrm{ev}_{e_0} \ : \ MS^{\Gamma^p}(C_{\mathrm{har}}(\mathbb{C}_p)) \to MS^{\Gamma_0(p)}(\mathbb{C}_p)$$

which fits in the commutative diagram

$$\begin{array}{ccc} MS^{\Gamma^p}(\mathcal{O}_2) & \xrightarrow{\partial} & MS^{\Gamma^p}(C_{\mathrm{har}}(\mathbb{C}_p)) \\ \downarrow{\scriptstyle BD} & \swarrow{\scriptstyle \eta} & \\ MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & & \downarrow{\scriptstyle \mathrm{ev}_{e_0}} \\ \downarrow{\scriptstyle \mathrm{res}_{\mathbb{Z}_p}} & & \\ MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\rho} & MS^{\Gamma_0(p)}(\mathbb{C}_p) \end{array}$$

So to prove the surjectivity of η, it suffices to prove the surjectivity of ev_{e_0} and of ∂. The surjectivity of ev_{e_0} will be proved in Lemma 3.5.3 and is in fact an isomorphism in this case. The map ∂ on modular symbols is also an isomorphism: its injectivity comes from the fact that η is injective, and its surjectivity will also be shown in Lemma 3.5.5. This gives us the following result.

Theorem 2.3.6. *The map*

$$\eta := \rho \circ \mathit{res}_{\mathbb{Z}_p} \circ BD \ : \ MS^{\Gamma^p}(\mathcal{O}_2) \to MS^{\Gamma_0(p)}(\mathbb{C}_p)$$

is a Hecke-equivariant isomorphism.

The inverse of this isomorphism is called the *Schneider-Teitelbaum* lift. Using the Eichler-Shimura isomorphism, this gives a characterization of $MS^{\Gamma^p}(\mathcal{O}_2)$ in terms of cusp forms of weight 2 and level p.

2.4 Multiplicative rigid meromorphic cocycles

We now turn our attention to the study of the Γ^p-module $H^1_f(\Gamma^p, \mathcal{M}^\times)$ using the results of the last section and the natural logarithmic derivative map

$$\mathrm{dlog} \ : \ H^1_f(\Gamma^p, \mathcal{M}^\times) \to H^1_{par}(\Gamma^p, \mathcal{M}_2).$$

As in the additive case, we can define multiplicative couterparts to the elements Φ_τ by

$$\overline{J}_\tau\{r,s\} := \prod_{w \in \Sigma_\tau(r,s)} t_w(z)^{\delta_{r,s}(w)}$$

where

$$t_w(z) := \begin{cases} z - w & \text{if } |w| \leqslant 1 \\ z/w - 1 & \text{if } |w| \geqslant 1 \\ 1 & \text{if } w = \infty. \end{cases}$$

They converge to rigid meromorphic cocycles satisfying

$$\text{dlog}\, \overline{J}_\tau\{r,s\} = \Phi_\tau\{r,s\}$$

and $\Phi_\tau\{r,s\}$ completely determine $\overline{J}_\tau\{r,s\}$ up to multplication by a non-zero scalar in $K_p^\times$, where K_p is the completion of $\mathbb{Q}(\tau)$ at the unique prime above p. Therefore $\overline{J}_\tau$ is an element of $MS^{\Gamma^p}(\mathcal{M}^\times/K_p^\times) = H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)$. $\overline{J}_\tau$ is called the projective rigid meromorphic cocyle attached to τ. Similarly to the additive case, given a projective rigid meromorphic cocycle $\overline{J} \in H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)$, we use the lower-case letter j to denote $\overline{j} := \overline{J}\{0,\infty\}$.

Therefore, since we have a full characterization of $H^1_{\text{par}}(\Gamma^p, \mathcal{M}_2)$ and $H^1_{\text{par}}(\Gamma^p, \mathcal{O}_2)$, to understand $H^1_f(\Gamma^p, \mathcal{M}^\times)$ it suffices to study the exact sequence

$$0 \to H^1(\Gamma^p, K_p^\times) \to H^1_f(\Gamma^p, \mathcal{M}^\times) \to H^1_{par}(\Gamma, \mathcal{M}^\times/K_p^\times) \xrightarrow{\delta} H^2(\Gamma^p, K_p^\times)$$

coming from the short exact sequence

$$0 \to K_p^\times \to \mathcal{M}^\times \to \mathcal{M}^\times/K_p^\times \to 0.$$

The space $H^2(\Gamma^p, K_p^\times)$ is called the *space of obstructions*, since if $\delta(\overline{J}) = 0$, then $\overline{J}$ lifts to a genuine rigid meromorphic cocycle $J \in H^1_f(\Gamma^p, \mathcal{M}^\times)$.

Let $\mathbb{T} := \mathbb{Z}[T_2, T_3, T_5, ...]$ be the Hecke algebra generated by the Hecke operators T_l for $l \neq p$ prime. The following lemma characterizes the spaces $H^1(\Gamma^p, K_p^\times)$ and $H^2(\Gamma^p, K_p^\times)$.

Lemma 2.4.1. *(1) The group $H^1(\Gamma^p, K_p^\times)$ is finite of exponent 12.*
(2) There is a natural map

$$\eta \ : \ H^1(\Gamma_0(p), K_p^\times) \to H^2(\Gamma^p, K_p^\times)$$

whose kernel and cokernel are of exponent 12.
(3) The module $H^2(\Gamma^p, K_p^\times)$ is a torsion $\mathbb{T}$-module which is annihilated by $12 \cdot \mathcal{I}_p$, where

$$\mathcal{I}_p := Ann_{\mathbb{T}}(M_2(\Gamma_0(p), \mathbb{Z})$$

is the subalgebra of $\mathbb{T}$ which annihilates the space of modular forms of weight 2 and level p. with integer coefficients.

Proof. This is Lemma 3.11 of [16], which we recall the proof as it will be useful later.

Let $\mathcal{F}(\mathcal{V}, K_p^\times)$ and $\mathcal{F}(\mathcal{E}, K_p^\times)$ be the Γ^p-modules of functions from $\mathcal{V}$ (resp. $\mathcal{E}$) to $K_p^\times$. Given a function $f \in \mathcal{F}(\mathcal{V}, K_p^\times)$, define

$$\begin{aligned} d \ : \ \mathcal{F}(\mathcal{V}, K_p^\times) &\to \mathcal{F}(\mathcal{E}, K_p^\times) \\ f &\mapsto df \end{aligned}$$

where $df(e) = f(v_-) - f(v_+)$. Here, v_+ is the even vertex connected to e and v_- is the odd one. The map d fits into the short exact sequence

$$1 \to K_p^\times \to \mathcal{F}(\mathcal{V}, K_p^\times) \to \mathcal{F}(\mathcal{E}, K_p^\times) \to 1.$$

Using the following isomorphisms

$$\begin{aligned} &\mathcal{F}(\mathcal{V}, K_p^\times) = \mathcal{F}(\mathcal{V}^+, K_p^\times) \oplus \mathcal{F}(\mathcal{V}^-, K_p^\times) = \mathrm{Ind}_{SL_2(\mathbb{Z})}^{\Gamma^p} K_p^\times \oplus \mathrm{Ind}_{SL_2'(\mathbb{Z})}^{\Gamma^p} K_p^\times \\ &\mathcal{F}(\mathcal{E}, K_p^\times) = \mathrm{Ind}_{\Gamma_0(p)}^{\Gamma^p} K_p^\times, \end{aligned}$$

where

$$SL_2'(\mathbb{Z}) = P^{-1} SL_2(\mathbb{Z}) P, \qquad \Gamma_0(p) = SL_2(\mathbb{Z}) \cap SL_2'(\mathbb{Z}).$$

and taking the Γ^p cohomology of the above short exact sequence, we get the long exact sequence

$$\begin{aligned} 1 \to H^1(\Gamma_0^p, K_p^\times) &\to H^1(SL_2(\mathbb{Z}), K_p^\times) \oplus H^1(SL_2'(\mathbb{Z}), K_p^\times) \\ \to H^1(\Gamma_0(p), K_p^\times) &\xrightarrow{\eta} H^2(\Gamma_0^p, K_p^\times) \to H^2(SL_2(\mathbb{Z}), K_p^\times) \oplus H^2(SL_2'(\mathbb{Z}), K_p^\times) \to ... \end{aligned}$$

The first statement follows from the fact that $H^1(SL_2(\mathbb{Z}), K_p^\times)$ is finite of exponent 12 and the second from the fact that $H^2(SL_2(\mathbb{Z}), K_p^\times)$ has also exponent 12. The third statement follows from the Eichler-Shimura isomorphism since $H^2(SL_2(\mathbb{Z}), K_p^\times)$ has exponent 12 and $\mathcal{I}_p$ annihilates $H^1(\Gamma_0(p), K_p^\times)$. □

Using the classification of the space of additive rigid meromorphic cocycles $H^1_{\text{par}}(\Gamma^p, \mathcal{M}_2)$, we can obtain a similar result for $H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)$.

Lemma 2.4.2. *The quotient space*

$$H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)/H^1_{par}(\Gamma^p, \mathcal{O}^\times/K_p^\times)$$

is torsion-free over $\mathbb{T}$. The cocycles $\overline{J}_\tau$ as τ ranges over fundamental elements of $\Gamma^p \backslash \mathcal{H}_p^{RM}$ are multiplicatively independent over $\mathcal{T}$.

Proof. The logarithmic derivative map

$$\mathrm{dlog} \ : \ \mathcal{M}^\times \to \mathcal{M}_2$$

has image in $\mathcal{M}_2^{\mathbb{Z}}$, the set of differentials of the third kind, i.e. of rigid meromorphic differentials having at worst simple poles and integer residues. Now consider $\overline{J} \in H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)$ and $\theta \in \mathbb{T}$. If $\overline{J}|\theta$ is regular, then so is $\Phi|\theta$ where $\Phi := \mathrm{dlog}\overline{J} \in H^1_{par}(\Gamma^p, \mathcal{M}_2)$. Corollary 2.2.5 then tells us that Φ is regular and thus so is $\overline{J}$. The second part is a direct consequence of Lemma 2.2.4. □

Theorem 2.4.3. *The group $H^1_f(\Gamma^p, \mathcal{M}^\times)$ is of infinite rank over $\mathbb{Z}$.*

Proof. This follows from considering the exact sequence

$$0 \to H^1(\Gamma^p, K_p^\times) \to H^1_f(\Gamma^p, \mathcal{M}^\times) \to H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times) \xrightarrow{\delta} H^2(\Gamma^p, K_p^\times),$$

since $H^1(\Gamma^p, K_p^\times)$ and $H^2(\Gamma^p, K_p^\times)$ are $\mathbb{T}$-torsion modules, and $H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)/H^1_{par}(\Gamma^p, \mathcal{O}^\times/K_p^\times)$ is an infinitely generated $\mathbb{T}$-torsion free module. □

The second statement of Lemma 2.4.1 tells us that given $\overline{J} \in H^1_{par}(\Gamma^p, \mathcal{M}^\times/K_p^\times)$, there exists $\kappa_J \in H^1(\Gamma_0(p), K_p^\times)$ such that

$$\delta(\overline{J}^{12}) = \eta(\kappa_J).$$

This class κ_J is called the *lifting obstruction associated to* J and is well-defined up to $\ker\eta$ which is of exponent 12 by the same result.

For the sake of simplicity, we will only consider parabolic cohomology / modular symbols up to 12-torsion for the rest of this subsection. Therefore, the equivalent exact sequence defining the space of obstructions is

$$0 \to MS^{\Gamma^p}(\mathcal{M}^\times) \to MS^{\Gamma^p}(\mathcal{M}^\times/\mathbb{C}_p^\times) \to H^1(\Gamma^p, MS(\mathbb{C}_p^\times)).$$

By the second statement of Lemma 2.4.1 and the Eichler-Shimura isomorphism, $H^1(\Gamma^p, MS(\cdot))$ is a torus of rank $2g+1$ over $\mathbb{Q}_p$ whose K_p-valued points are $H^1(\Gamma^p, MS(K_p^\times))$, where g is the genus of $X_0(p)$. One could ask what the image of the obstruction map

$$\delta_{\mathcal{O}^\times} \ : \ MS^{\Gamma^p}(\mathcal{O}^\times/\mathbb{C}_p^\times) \to H^1(\Gamma^p, MS(\mathbb{C}_p^\times))$$

is when considering analytic theta cocycles. It turns out that $\Pi := \mathrm{im}(\delta_{\mathcal{O}^\times})$ is a lattice in $H^1(\Gamma^p, MS(\mathbb{C}_p^\times))$ which follows from the fact that the composition

$$MS^{\Gamma_0(p)}(\mathbb{Z}) \xrightarrow{ST^\times} MS^{\Gamma^p}(\mathcal{O}^\times/\mathbb{C}_p^\times) \xrightarrow{\delta_{\mathcal{O}^\times}} H^1(\Gamma^p, MS(\mathbb{C}_p^\times)) \to MS^{\Gamma_0(p)}(\mathbb{C}_p^\times) \xrightarrow{\mathrm{ord}_p} MS^{\Gamma_0(p)}(\mathbb{Z})$$

is the identity. This gives us the following result (See [15])

Proposition 2.4.4. *Let T be the $\mathbb{Q}_p$-torus whose K_p-valued points are $H^1(\Gamma^p, MS(K_p^\times))$. Then Π is a lattice in T. Also, the character group*

$$X(T) := \mathit{hom}(T, \mathbb{G}_m)$$

and the lattice Π are isomorphic to $\mathbb{Z}^{2g+1}$ up to torsion.

2.5 Borcherds Lifts

We conclude this chapter with an overview of the analogue of Borcherds lifts, but in the real quadratic universe. In the CM-case, Borcherds lifts gives a correspondance between modular forms on $O(2,l)$, the orthogonal group of signature $(2,l)$, and modular forms of weight $1-l/2$. As seen in the earlier sections, rigid analytic cocycles are closely connected to modular forms on $O(2,1)$ of level p and therefore we find a similar kind of lift between rigid analytic cocycles and modular forms of weight $1/2$.

We first start with some definitions.

Definition 2.5.1. *Let $-d_1, -d_2$ be a pair of negative discriminants satisfying*
(1) $-d_1$ is fundamental and prime to p,
(2) $ord_p(-d_2) \leqslant 1$,
(3) $\left(\frac{d_1 d_2}{p}\right) \neq 1$,
and let $D = d_1 d_2$. Consider the character ω_{-d_1} on the narrow class group $Cl^+(D)$ defined by

$$\omega_{-d_1}(Q) = \begin{cases} 0 & \text{if } gcd(a,b,c,d_1) > 1 \\ \left(\frac{-d_1}{r}\right) & \text{if } gcd(a,b,c,d_1) = 1, \end{cases}$$

where $Q = [a,b,c]$ is a binary quadratic form and r is any element represented by Q such that $gcd(r, d_1) = 1$. Given a pair $(-d_1, -d_2)$ satisfying these conditions, we define the RM-divisors

$$\mathbb{D}_{-d_1,-d_2} := \sum_{Q \in Cl^+(D)} \omega_{-d_1}(Q) \cdot [\tau_Q],$$

$$\mathbb{D}^+_{-d_1,-d_2} := \sum_{Q \in Cl^+(D)} \omega_{-d_1}(Q) \cdot ([\tau_Q] + [p\tau_Q]),$$

$$\mathbb{D}^-_{-d_1,-d_2} := \sum_{Q \in Cl^+(D)} \omega_{-d_1}(Q) \cdot ([\tau_Q] - [p\tau_Q]),$$

where τ_Q is the positive root of $Q(z,1)$. Here $[\tau_Q]$ represents a class in $\Gamma^p \backslash \mathcal{H}^p$ and $\mathbb{D}^{\pm}_{-d_1,-d_2}$ are seen as

elements in $Div^{\dagger}(\Gamma^p \backslash \mathcal{H}^p)$. *We can then extend the above to arbitrary discriminants* $-d_1$ *by putting for* $m \geqslant 1$

$$\mathbb{D}^{+}_{-d_1,-d_2p^{2m}} := \mathbb{D}^{+}_{-d_1,-d_2}, \qquad \mathbb{D}^{-}_{-d_1,-d_2p^{2m}} := (-1)^m \mathbb{D}^{-}_{-d_1,-d_2}.$$

Let $M^{!}_{k}(N)$ be the space of weakly holomorphic modular forms of weight k and level N and let $M^{!!}_{k}(N) \subset M^{!}_{k}(N)$ be the subset of those modular forms regular at all cusps except possibly ∞. Values of rigid analytic cocycles are intimately linked to these spaces of modular forms as shown in the following theorem (See [15]):

Theorem 2.5.2. *Let* $J \in H^1(\Gamma^p, \mathcal{O}^{\times}/\mathbb{C}_p^{\times})$.

(1) If $\left(\frac{-d_1}{p}\right) = 1$, *then the series*

$$\Theta_{-d_1,J}(q) := \sum_{d_2 \geqslant 0} \log_p J[\mathbb{D}^{+}_{-d_1,-d_2}] q^{d_2}$$

belongs to $M_{3/2}(4p) \otimes \mathbb{Q}_p$.

(2) If $\left(\frac{-d_1}{p}\right) = -1$, *then the series*

$$\Theta_{-d_1,J}(q) := \sum_{d_2 \geqslant 0} \log_p J[\mathbb{D}^{-}_{-d_1,-d_2}] q^{d_2}$$

belongs to $M_{3/2}(4p) \otimes \mathbb{Q}_p$.

In fact, the statement is a bit more precise, giving subspaces of $M_{3/2}(4p) \otimes \mathbb{Q}_p$ where each generating series lives depending on the value of $\left(\frac{-d_1}{p}\right)$.
Conversely, an anologue of the Borcherds lift for RM-points is also proven:

Theorem 2.5.3. *Let* $\phi := \sum_{n \gg -\infty} c_\phi(n) q^n \in M^{!!}_{1/2}(4p)$ *be a weakly holomorphic modular form regular at all cusps except* ∞, *with integer Fourier coefficients.*
(1) If $\left(\frac{-d_1}{p}\right) = 1$, *there exists a rigid meromorphic cocycle* $J_{-d_1,\phi} \in H^1(\Gamma^p, \mathcal{M}^{\times}) \otimes \mathbb{Q}$ *satisfying*

$$Div(J_{-d_1,\phi}) = \sum_{d_2 \cong 0,3 \ mod\ 4} c_\phi(-d_2) \mathbb{D}^{+}_{-d_1,-d_2},$$

where the sum runs over positive integers d_2 *satisfying* $\left(\frac{-d_2}{p}\right) \neq 1$.
(2) If $\left(\frac{-d_1}{p}\right) \neq 1$, *there exists a rigid meromorphic cocycle* $J_{-d_1,\phi} \in H^1(\Gamma^p, \mathcal{M}^{\times}) \otimes \mathbb{Q}$ *satisfying*

$$Div(J_{-d_1,\phi}) = \sum_{d_2 \cong 0,3 \ mod\ 4} c_\phi(-d_2) \mathbb{D}^{-}_{-d_1,-d_2},$$

where the sum runs over positive integers d_2 *satisfying* $\left(\frac{-d_2}{p}\right) \neq -1$.

The rigid meromorphic cocycle $J_{-d_1,\phi}$ is called the real quadratic Borcherds lift associated to the pair $(-d_1, \phi)$.

A slightly different statement and approach can be seen in [35]. In this article, two kernels are defined for weight $2k$ with k odd, giving a $\overline{\mathbb{Q}}$-linear map between a subset of $S_{2k}(\Gamma_0(p))$ and $MS^{\Gamma^p}(\mathcal{O}_{2k})$, akin to some Shimura-Shintani lift. Generalizations of these kernels and the lift for $\Gamma_0^p(N)$ are given in Chapter 4.

Chapter 3

The case $\Gamma_0^p(N)$

We will now generalize the results of the last chapter in the case where the matrix group is $\Gamma_0^p(N)$, i.e. the subgroup of $SL_2(\mathbb{Z}[1/p])$ with lower-left entry divisible by N. As mentioned before, we assume that $p \nmid N$. We will sometimes assume that N is square-free, as this will help us simplify some arguments. This will be mentionned when used.

This chapter can be thought as divided into two main parts: the first concerns modular symbols and the particular case of rigid meromorphic period functions when the $\Gamma_0^p(N)$-module is $\mathcal{M}_2$ or $\mathrm{Div}^\dagger(\mathcal{H}_p)$. This part will be particularly useful in Chapter 5 when discussing efficient computations of rigid meromorphic cocycles. In particular, we prove in Theorem 3.3.5 that the support of a rigid meromorphic period function consists only of RM-points.

The second part concerns the classification of rigid analytic/meromorphic cocycles in the additive and multiplicative case. We mention how certain divisors can be lifted to additive rigid meromorphic cocycles in Theorem 3.4.1, which will be proved in Section 4.5.

3.1 Modular symbols

This subsection gives general results concerning $\Gamma_0(N)$-invariant modular symbols that will be useful later when computing rigid meromorphic period functions and when giving the classification of rigid analytic cocycles. Similarly to the case $N = 1$, modular symbols are again useful tools in computing the parabolic cohomology of a $\Gamma_0^p(N)$-module.

Lemma 3.1.1. *Let Ω be a $\Gamma_0^p(N)$-module and c_i be representatives of $\Gamma_0^p(N)\backslash\mathbb{P}_1(\mathbb{Q})$. Then there is a natural exact sequence of $\Gamma_0^p(N)$-modules*

$$0 \to \Omega^{\Gamma_0^p(N)} \to \bigoplus_{i=1}^{r} \Omega^{\Gamma_0^p(N)_{c_i}} \to MS^{\Gamma_0^p(N)}(\Omega) \to H^1(\Gamma_0^p(N), \Omega) \xrightarrow{f} \bigoplus_{i=1}^{r} H^1(\Gamma_0^p(N)_{c_i}, \Omega)$$

where $\Gamma_0^p(N)_{c_i}$ is the stabilizer of the cusp c_i in $\Gamma_0^p(N)$.

Proof. It is essentially the same proof as Lemma 2.1.2. Consider $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega)$, the $\Gamma_0^p(N)$-module of Ω-valued functions on $\mathbb{P}_1(\mathbb{Q})$ with action $f|\gamma = f(\gamma^{-1}z)$. Then we have a short exact sequence

$$0 \to \Omega \to \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega) \xrightarrow{d} MS(\Omega) \to 0,$$

where $df\{r, s\} = f(s) - f(r)$. Splitting $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega)$ in terms of cusps and using Shapiro's lemma gives us an isomorphism

$$H^i(\Gamma_0^p(N), \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega)) \simeq \bigoplus_{i=1}^{r} H^i(\Gamma_0^p(N)_{c_i}, \Omega)$$

and therefore the result follows by taking the $\Gamma_0^p(N)$-cohomology of the short exact sequence above. □

Define the parabolic cohomology $H^i_{par}(\Gamma_0^p(N), \Omega)$ to be the kernel of f. Then this result implies that we have a surjective morphism

$$\delta \ : \ MS^{\Gamma_0^p(N)}(\Omega) \twoheadrightarrow H^1_{par}(\Gamma_0^p(N), \Omega)$$

which is an isomorphism if $\Omega^{\Gamma_0^p(N)} \simeq \bigoplus_{i=1}^{r} \Omega^{\Gamma_0^p(N)_{c_i}}$.

Restricting our attention to the particular cases where $\Omega = \mathcal{M}^\times$ or $\mathcal{M}_k$ as seen in the last chapter, we also have the following equalities

$$H^0(\Gamma_0^p(N)_{c_i}, \mathcal{M}^\times) = \mathbb{C}_p^\times, \quad H^0(\Gamma_0^p(N)_{c_i}, \mathcal{M}_k) = \begin{cases} \mathbb{C}_p, & \text{if } k = 0; \\ 0, & \text{if } k > 0. \end{cases}$$

The argument for these equalities are as in Proposition 2.1.3, i.e. any $\Gamma_0^p(N)_{c_i}$-invariant rigid meormorphic function must be constant on the standard affinoid by the Weierstrass preparation theorem, and thus everywhere by analytic continuation. This gives isomorphisms for $k > 0$

$$MS^{\Gamma_0^p(N)}(\mathcal{M}_k) \xrightarrow{\sim} H^1_{par}(\Gamma_0^p(N), \mathcal{M}_k), \qquad MS^{\Gamma_0^p(N)}(\mathcal{O}_k) \xrightarrow{\sim} H^1_{par}(\Gamma_0^p(N), \mathcal{O}_k),$$

whereas in the multiplicative case and in the case $k = 0$, we have exact sequences

$$0 \to \left(\mathbb{C}_p^\times\right)^{l-1} \to MS^{\Gamma_0^p(N)}(\mathcal{M}^\times) \to H^1_{par}(\Gamma_0^p(N), \mathcal{M}^\times) \to 0.$$

and

$$0 \to (\mathbb{C}_p)^{l-1} \to MS^{\Gamma_0^p(N)}(\mathcal{M}) \to H^1_{par}(\Gamma_0^p(N), \mathcal{M}) \to 0$$

Here $l-1$ is equal to the dimension of the Eisenstein subspace of modular forms of weight 2 for $\Gamma_0(N)$. This is no accident since, as we will see, the spaces $MS^{\Gamma_0^p(N)}(\mathcal{M}^\times)$ and $H^1_{par}(\Gamma_0^p(N),\mathcal{M}^\times)$ are closely related to the space of modular forms for $\Gamma_0(N)$ using the Eichler-Shimura isomorphism.
As in the last chapter, in the special case $\mathcal{M}^\times$, it will be interesting to define $H^1_f(\Gamma_0^p(N),\mathcal{M}^\times)$ to be the submodule of $H^1(\Gamma_0^p(N),\mathcal{M}^\times)$ which maps to $\bigoplus_{i=1}^r H^1(\Gamma_0^p(N)_{c_i},\mathbb{C}_p^\times)$ under the map

$$H^1(\Gamma_0^p(N),\mathcal{M}^\times) \xrightarrow{f} \bigoplus_{i=1}^{r} H^1(\Gamma_0^p(N)_{c_i},\mathcal{M}^\times).$$

If we apply the same definition for the module $H^1(\Gamma_0^p(N),\mathbb{C}_p^\times)$, we see that we trivially have $H^1_f(\Gamma_0^p(N),\mathbb{C}_p^\times) = H^1(\Gamma_0^p(N),\mathbb{C}_p^\times)$.

3.2 p-new modular forms

This section will give a quick introduction on modular forms on the Bruhat-Tits tree $\mathcal{T}$ which will be useful later. We will follow closely Section 1 of [12].

The motivation for studying modular forms on $\mathcal{H}_p\times\mathcal{H}$ comes from similar constructions in the classical and the p-adic setting. In the classical setting, the action of $\Gamma_0(N)$ on $\mathcal{H}$ is discontinuous and discrete and this gives a uniformization $X_0(N)(\mathbb{C})\cong\Gamma_0(N)\backslash\mathcal{H}^*$ where modular forms can be seen as global sections of a certain space of sheaves over the modular curve $X_0(N)$.
In the p-adic setting, consider a definite quaternion algebra B over $\mathbb{Q}$ such that p splits B, let $\mathcal{O}$ be a maximal $\mathbb{Z}[1/p]$-order in B and let Γ be a congruence subgroup of $\mathcal{O}^\times$. Then a famous result by Cerednik and Drinfeld (See [8] and [19]) gives a p-adic uniformization of $\mathcal{H}_p/\Gamma$, i.e., there exists a Shimura curve X such that $X(\mathbb{C}_p)\cong\mathcal{H}_p/\Gamma$. Similarly as in the classical setting, we can consider global sections of certain sheaves of differentials on this curve which can be seen as cusp forms on it.
In the case where B is an indefinite quaternion algebra (take $B\cong M_2(\mathbb{Q})$ for example), a typical example of a congruence subgroup of a maximal $\mathbb{Z}[1/p]$-order in B is $\Gamma_0^p(N)$. Unfortunately, the action of $\Gamma_0^p(N)$ is not discrete or discontinuous on $\mathcal{H}$ nor $\mathcal{H}_p$, but it is on the product $\mathcal{H}\times\mathcal{H}_p$. It then seems natural to consider the quotient $\Gamma_0^p(N)\backslash(\mathcal{H}\times\mathcal{H}_p)$ and certain kinds of functions on it.

Definition 3.2.1. *A cusp form of weight 2 on $(\mathcal{T}\times\mathcal{H})/\Gamma_0^p(N)$ is a function*

$$f\ :\ \mathcal{E}^*\times\mathcal{H}\to\mathbb{C}$$

such that

(1) $f(\gamma e,\gamma z)=(cz+d)^2f(e,z)$, for all $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0^p(N)$.

(2) The function f is harmonic, i.e. f satisfies

$$\sum_{s(e)=v} f(e,z) = 0$$

where $s(e)$ is the source of the vertex v, and $f(\overline{e},z) = -f(e,z)$.
(3) For each edge e, the function $f_e(z) := f(e,z)$ is a cusp form of weight 2 for the subgroup $\Gamma_e :=$ Stab$_\Gamma(e)$.

We denote the set of such functions by $S_2(\mathcal{T}, \Gamma_0^p(N))$.

We see that a cusp form on $(\mathcal{T} \times \mathcal{H})/\Gamma_0^p(N)$ is nothing else than family of cusp forms indexed by the ordered edges of $\mathcal{T}$ satisfying compatibility relations and harmonicity. The action of $\Gamma_0^p(N)$ being transitive on $\mathcal{E}$, each subgroup Γ_e is conjugate to $\Gamma_0(pN) = \Gamma_{e_0}$, the stabilizer of the standard edge. Therefore, each f_e is a cusp form on a subgroup conjugate to $\Gamma_0(pN)$.

$S_2(\mathcal{T}, \Gamma_0^p(N))$ is closely related to other spaces of cusp forms on $\mathcal{T} \times \mathcal{H}$. To that end, let $\Gamma_0^{p,+}(N)$ be the set of matrices of $GL_2^+(\mathbb{Z}[1/p])$ with lower-left entry divisible by N.

Definition 3.2.2. *A cusp form on $(\mathcal{E}^* \times \mathcal{H})/\Gamma_0^{p,+}(N)$ is a function*

$$f \ : \ \mathcal{E}^* \times \mathcal{H} \to \mathbb{C}$$

satisfying
(1) $f(\gamma e, \gamma z) = \frac{(cz+d)^2}{\det(\gamma)} f(e,z)$, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{p,+}(N)$.
(2) For each edge e, the function $f_e(z) := f(e,z)$ is a cusp form of weight 2 for Stab$(e) \cap \Gamma_0^{p,+}(N)$.

Similarly, a cusp form on $(\mathcal{V} \times \mathbb{H})/\Gamma_0^{p,+}(N)$ is a function

$$f \ : \ \mathcal{V} \times \mathcal{H} \to \mathbb{C}$$

satisfying
(1) $f(\gamma v, \gamma z) = \frac{(cz+d)^2}{\det(\gamma)} f(v,z)$, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{p,+}(N)$.
(2) For each vertex v, the function $f_v(z) := f(v,z)$ is a cusp form of weight 2 for Stab$(v) \cap \Gamma_0^{p,+}(N)$.

We denote these spaces $S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N))$ and $S_2(\mathcal{V}, \Gamma_0^{p,+}(N))$ respectively.

Let $\beta \in \Gamma_0^{p,+}(N)$ be such that $\mathrm{ord}_p(\det(\beta)) = 1$. Then β is parity reversing on $\mathcal{T}$, i.e. β sends odd vertices to even vertices, even vertices to odd vertices, and similarly for edges. Given $f \in S_2(\mathcal{T}, \Gamma_0^p(N))$, define a cusp form $\widetilde{f}$ by setting

$$\widetilde{f}(e,z)dz = f(e,z)dz \text{ if } e \text{ is even;}$$
$$\widetilde{f}(e,z)dz = f(\beta e, \beta z)d\beta z \text{ if } e \text{ is odd.}$$

Note that $\widetilde{f} \in S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N))$ and that the map $i : f \mapsto \widetilde{f}$ is a monomorphism from $S_2(\mathcal{T}, \Gamma_0^p(N))$ to $S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N))$, since if $\widetilde{f}_e = 0$ for all edges $e \in \mathcal{T}_1^*$, then $f_e = 0$ for all $e \in \mathcal{E}^+$ and the same goes for all edges by the harmonicity condition.

We can define two natural degeneracy maps

$$\pi_s, \pi_t : S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N)) \to S_2(\mathcal{V}, \Gamma_0^{p,+}(N))$$

by

$$\pi_s(f)(v,z) := \sum_{s(e)=v} f(e,z), \qquad \pi_t(f)(v,z) := \sum_{t(e)=v} f(e,z),$$

where each sum is over edges with source v and target v respectively. These maps give an exact sequence

$$0 \longrightarrow S_2(\mathcal{T}, \Gamma_0^p(N)) \longrightarrow S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N)) \xrightarrow{\pi_s \oplus \pi_t} S_2(\mathcal{V}, \Gamma_0^{p,+}(N)) \oplus S_2(\mathcal{V}, \Gamma_0^{p,+}(N)).$$

Similar maps also arise in the case of classical modular forms. First let $\gamma_1, ..., \gamma_{p+1}$ be representatives of $\Gamma_0(N)/\Gamma_0(pN)$ and define maps $\varphi_s, \varphi_t : S_2(\Gamma_0(pN)) \to S_2(\Gamma_0(N))$ by

$$\varphi_s(f)(z)dz = \sum_{j=1}^{p+1} f(\gamma_j^{-1}z)d(\gamma_j^{-1}z), \qquad \varphi_t(f)(z)dz = \sum_{j=1}^{p+1} f(\beta\gamma_j^{-1}z)d(\beta\gamma_j^{-1}z).$$

These maps come from the natural projection of modular curves $X_0(pN) \to X_0(N)$ and the one twisted by W_p, the Atkin-Lehner involution at p. The kernel of

$$\varphi_s \oplus \varphi_t : S_2(\Gamma_0(pN)) \to S_2(\Gamma_0(N)) \oplus S_2(\Gamma_0(N))$$

is called the space of p-new cusp forms, denoted by $S_2^{\text{new}-p}(\Gamma_0(pN))$. It turns out that $\varphi_s \oplus \varphi_t$ and $\pi_s \oplus \pi_t$ are basically the same map.

Lemma 3.2.3. *Let v_0 be the standard vertex of $\mathcal{T}$ and e_0 the edge whose source is v_0 and with stabilizer equal to $\Gamma_0(pN)$. Then*
(1) The function $f(e,z) \mapsto f_{e_0}$ induces an isomorphism from $S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N))$ to $S_2(\Gamma_0(pN))$.
(2) The function $f(v,z) \mapsto f_{v_0}$ induces an isomorphism from $S_2(\mathcal{V}, \Gamma_0^{p,+}(N))$ to $S_2(\Gamma_0(N))$.

(3) The function $f(e,z) \mapsto f_{e_0}$ induces an isomorphism from $S_2(\mathcal{T}, \Gamma_0^p(N))$ to $S_2^{new-p}(\Gamma_0(pN))$.
(4) We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_2(\mathcal{T}, \Gamma_0^p(N)) & \xrightarrow{i} & S_2(\mathcal{E}^*, \Gamma_0^{p,+}(N)) & \xrightarrow{\pi_s \oplus \pi_t} & S_2(\mathcal{V}, \Gamma_0^{p,+}(N)) \oplus S_2(\mathcal{V}, \Gamma_0^{p,+}(N)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S_2^{new-p}(\Gamma_0(pN)) & \xrightarrow{i} & S_2(\Gamma_0(pN)) & \xrightarrow{\varphi_s \oplus \varphi_t} & S_2(\Gamma_0(N)) \oplus S_2(\Gamma_0(N)) \end{array}$$

Proof. See Lemma 1.3 of [12]. □

3.3 Rigid meromorphic period functions

As in the Γ^p case, rigid meromorphic cocycles for $\Gamma_0^p(N)$ can be related to rigid meromorphic period functions, the main difference being that the $\mathbb{Z}[\Gamma_0(N)]$-module Δ_0, i.e. divisors of degree 0 on $\mathbb{P}_1(\mathbb{Q})$, can not be generated by a single element. Therefore, a rigid meromorphic period function can not be determined by computing the single value $m\{0,\infty\}$. This section gives results concerning how we can compute and describe the relevant spaces of rigid meromorphic period functions for $\Gamma_0^p(N)$ and how similar results on the divisors of rigid meromorphic cocycles can be obtained.

In all of our cases, we consider Ω to be a $\Gamma_0^p(N)$-module. Let's consider the induced module $\mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega)$, i.e. the set of functions

$$f \ : \ \Gamma^p/\Gamma_0^p(N) \to \Omega$$

with Γ^p-action given by

$$f|\gamma(A) = f(A\gamma^{-1})|\gamma.$$

By Shapiro's lemma, we have $H^1(\Gamma_0^p(N), \Omega) \simeq H^1(\Gamma^p, \mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega))$, and more specifically

$$H^1_{\mathrm{par}}(\Gamma_0^p(N), \Omega) \simeq H^1_{\mathrm{par}}(\Gamma^p, \mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega)).$$

So it will make sense to study the latter group to have access to the full Γ^p-action.

Now let

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

be the usual matrices of order 4 and 6 in $\mathrm{SL}_2(\mathbb{Z})$. We denote by $(\Omega^Q)^\dagger \subset \mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega)$ the submodule of elements satisfying

$$\{\omega \in \mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega) \ : \ \omega|(1+S) = \omega|(1+U+U^2) = 0\}.$$

The notation $(\Omega^Q)^\dagger$ is used since the induced module $\mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega)$ can be related to Ω^Q by the correspondence

$$\begin{aligned}\mathrm{Ind}_{\Gamma_0^p(N)}^{\Gamma^p}(\Omega) &\to \Omega^Q \\ f &\mapsto (f(\alpha_i))_{i=1}^Q,\end{aligned}$$

where Q is the number of representatives of $\Gamma^p/\Gamma_0^p(N)$. For the sake of simplicity, the module Ω^Q will be used from now on, remembering that the Γ^p action is the one that comes from the induced module, and not the one acting on each coordinate.

Lemma 3.3.1. *Let $m \in MS^{\Gamma_0(N)}(\Omega)$ and $\alpha_1, ..., \alpha_Q$ a set of representatives of $SL_2(\mathbb{Z})/\Gamma_0(N)$. Then the function*

$$f_m(\alpha_i) = \alpha_i^{-1} \cdot m\{\alpha_i \cdot 0, \alpha_i \cdot \infty\}$$

is an element of $(\Omega^Q)^\dagger$.

Proof. Fix α_i. Then

$$\begin{aligned}f_m(\alpha_i) + S \cdot f_m(\alpha_i \cdot S^{-1}) &= \alpha_i^{-1} \cdot m\{\alpha_i \cdot 0, \alpha_i \cdot \infty\} + S \cdot (S^{-1}\alpha_i^{-1}) m\{\alpha_i \cdot \infty, \alpha_i \cdot 0\} \\ &= \alpha_i^{-1}(m\{\alpha_i \cdot 0, \alpha_i \cdot \infty\} + m\{\alpha_i \cdot \infty, \alpha_i \cdot 0\}) \\ &= 0\end{aligned}$$

Exactly the same argument works for $f_m|(1 + U + U^2)$.

□

Now we prove a small lemma that will be useful in associating modular symbols with rigid meromorphic period functions.

Lemma 3.3.2. *The map*

$$\begin{aligned}f \ : \ \left\{ \left(\frac{a}{c}, \frac{b}{d}\right) \ | \ ad - bc = 1 \right\} &\to \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \\ \left(\frac{a}{c}, \frac{b}{d}\right) &\mapsto (c \ : \ d).\end{aligned}$$

induces a bijection between $\Gamma_0(N)\backslash\{\left(\frac{a}{c}, \frac{b}{d}\right) \ | \ ad - bc = 1\}$ and $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

Proof. Clearly, the set $\{\left(\frac{a}{b}, \frac{c}{d}\right) \ | \ ad - bc = 1\}$ can be identified with $SL_2(\mathbb{Z})$, and we consider the bijection

$$\begin{aligned}f \ : \ \Gamma_0(N)\backslash SL_2(\mathbb{Z}) &\to \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (c \ : \ d).\end{aligned}$$

This map is surjective since every element $(c\ :\ d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ can be represented by taking c and d to be coprime integers or one of them to be 0 and the other one to be 1. In the first case we can find $a, b \in \mathbb{Z}$ such that $ad - bc = 1$, and in the second case we take $(a, b) = (0, 1)$ or $(0, -1)$ depending on whether $c = 0$ or $d = 0$ respectively.
Now if two matrices γ_1, γ_2 are mapped to the same element $(c\ :\ d)$ under f, then modulo N we have

$$\gamma_1 \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma_2 \equiv \begin{pmatrix} au^{-1} & bu^{-1} \\ cu & du \end{pmatrix},$$

where u is an invertible integer modulo N. Therefore, $\gamma_1\gamma_2^{-1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ modulo N and γ_1, γ_2 are $\Gamma_0(N)$-equivalent.

□

In the literature (See [10] for example), the name M-symbol is used to denote an equivalence class $(c\ :\ d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. As we will see, these M-symbols can be used to describe succintly $\Gamma_0(N)$ and $\Gamma_0^p(N)$-invariant modular symbols.

Proposition 3.3.3. *The assignment*

$$m \mapsto f_m$$

identifies $MS^{\Gamma_0(N)}(\Omega)$ *with* $(\Omega^Q)^\dagger$.

Proof. By the lemma above, $\Gamma_0(N)\backslash\{\left(\frac{a}{b}, \frac{c}{d}\right) \mid ad - bc = 1\}$ is in bijection with $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, which is naturally $\Gamma/\Gamma_0(N)$. Since a modular symbol is determined by its values on the set $\{\left(\frac{a}{b}, \frac{c}{d}\right) \mid ad - bc = 1\}$ and that it has $|\Gamma/\Gamma_0(N)|$ orbits under $\Gamma_0(N)$, the injectivity of the assignment follows.

Now for surjectivity, let $f \in (\Omega^Q)^\dagger$ and $\left(\frac{a}{b}, \frac{c}{d}\right)$ be a unimodular pair which is associated to α_i under the above correspondence. Define

$$m\{\frac{a}{b}, \frac{c}{d}\} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} f(\alpha_i),$$

where the matrix appearing on the right is in $\alpha_i\Gamma_0(N)$. By the two and three term relation, m satisfies the modular symbol relations and can be extended uniquely to an element of $MS^{\Gamma_0(N)}(\Omega)$. □

This bijection is not "optimal", meaning we don't need all the Q values $m\{\alpha_i \cdot 0, \alpha_i \cdot \infty\}$ to completely determine the modular symbol m. For example, since a modular symbol satisfies $m\{r, s\} = -m\{s, r\}$, the

value $f_m(\alpha_i^{-1})$ can be determined from $f_m(\alpha_i)$. Even more specific, we can easily see that a modular symbol $m \in MS^{\Gamma_0(2)}(\Omega)$ is still completely determined by $m\{0,\infty\}$ by using the 2 and 3-terms relation.

Interesting things can be said about going from $SL_2(\mathbb{Z})$-invariant modular symbols to $\Gamma_0(N)$-invariant modular symbols, and vice-versa. One can easily see that if $f \in (\Omega^Q)^\dagger$, then $\hat{f}; = \sum_{i=1}^Q f(\alpha_i)$ is in $\Omega^\dagger$. Conversely, as is proven in [24], given $f \in \Omega^\dagger$, there exists matrices $\{H_i\}_{i=1}^Q$ such that

$$(f|H_i)_{i=1}^Q \in (\Omega^Q)^\dagger.$$

In particular, if $N = q$ is a prime number (and so $Q = q+1$), one finds that taking $m\{0,\infty\}(z) \in \mathcal{M}_k^\dagger$ associated to a modular symbol $m \in MS^\Gamma(\mathcal{M}_k)$, then the corresponding element in $(\mathcal{M}_k^{q+1})^\dagger$ is given by

$$m\{\frac{-i}{q},\infty\}\left(\frac{z-i}{q}\right), \text{ for } i = 1...q, \quad m\{0,\infty\}(qz), \text{ for } i = q+1.$$

In other words, let's say that

$$\varphi_\tau(z) := \sum_{w\in\Sigma_\tau} \delta_{0,\infty}(w)\frac{1}{z-w}$$

is the rigid meromorphic period function for Γ^p coming from the rigid meromorphic cocycle associated to an RM-point τ. Then the associated element in $(\mathcal{M}_k^{q+1})^\dagger$ is the Mittag-Leffler decomposition (relative to q) of the rigid meromorphic period function associated to the RM-point $q\tau$, which closely ressembles how rigid meromorphic cocycles are computed in Chapter 3 of [16] by use of the U_p-operator. Methods to lift solutions of particular equations from $SL_2(\mathbb{Z})$ to $\Gamma_0(N)$ are commonly called transfer operators and are studied in (among others) [21], [24], [30], [31] and [33].

Definition 3.3.4. *A rigid meromorphic period function of weight k for $\Gamma_0^p(N)$ is the image of a modular symbol $m \in MS^{\Gamma_0^p(N)}(\mathcal{M}_k)$ under the assignment*

$$m \mapsto f_m.$$

Similarly, a rigid period divisor for $\Gamma_0^p(N)$ is the image of an element $m \in MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p))$ under the same assignment. These spaces are denoted $(\mathcal{M}_k)^\ddagger_{\Gamma_0^p(N)}$ and $(Div(\mathcal{H}_p))^\ddagger_{\Gamma_0^p(N)}$ respectively.

If Ω is a Γ^p-module, we can similarly define the Γ^p-module $(\Omega)^\ddagger_{\Gamma_0^p(N)}$. Let D be the matrix

$$D = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}.$$

Since $p \nmid N$, D acts trivially on the cusps and thus a given rigid meromorphic period function f (or divisor) is invariant under D. However, nothing guarantees us that invariance under D characterizes $(\Omega)^\ddagger_{\Gamma_0^p(N)}$. In general, its elements may need to satisfy more relations.

There is a natural map

$$(\mathcal{M}_k)^{\ddagger}_{\Gamma_0^p(N)} \to (\mathrm{Div}(\mathcal{H}_p))^{\ddagger}_{\Gamma_0^p(N)}$$
$$(\varphi_i)_{i=1}^Q \mapsto (div(\varphi_i))_{i=1}^Q.$$

The divisor or support of a rigid meromorphic period function of weight k for $\Gamma_0^p(N)$ is defined to be the union $\bigcup_{i=1}^Q div(\varphi_i)$.

Now, we can use pretty much the same argument as in the case $N = 1$ seen in [9] and [16] to prove that a rigid meromorphic period function for $\Gamma_0^p(N)$ has zeroes and poles at RM-points. Namely, let $\Sigma_{(\varphi_i)_{i=1}^Q}$ be the set of zeroes and poles of a rigid meromorphic period function $(\varphi_i)_{i=1}^Q$. Then since $(\varphi_i)_{i=1}^Q$ satisfies the two and three-terms relation, we have

$$w \in \Sigma_{(\varphi_i)_{i=1}^Q} \Rightarrow S(w) \in \Sigma_{(\varphi_i)_{i=1}^Q} \quad \text{and} \quad U(w) \in \Sigma_{(\varphi_i)_{i=1}^Q} \text{ or } U^2(w) \in \Sigma_{(\varphi_i)_{i=1}^Q}.$$

It is seen in the articles mentioned above that a finite set satisfying the above relations must contain only RM-points. Since $\Sigma_{(\varphi_i)_{i=1}^Q} \cap \mathcal{H}_p^{\leqslant M}$ is finite for all M, we deduce that $\Sigma_{(\varphi_i)_{i=1}^Q}$ must contain only RM-points. This is in accordance to the definition of support of a modular symbol $m \in MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p))$ given in [22], i.e. we have a decomposition of $\Gamma_0^p(N)$-modules

$$MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p)) \simeq \bigoplus_{\Gamma_0^p(N)x \in \Gamma_0^p(N)\backslash\mathcal{H}_p} MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\Gamma_0^p(N)x)).$$

A point $x \in \mathcal{H}_p$ is said to be in the support of $m \in MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p))$ if its projection in $MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\Gamma_0^p(N)x))$ is non-zero. Indeed, the two definitions of support are equivalent if we consider the commutative diagram

$$\begin{array}{ccc}
MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p)) & \longrightarrow & MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\Gamma_0^p(N)x)) \\
\downarrow & & \downarrow \\
MS^{\Gamma^p}((\mathrm{Div}^\dagger(\mathcal{H}_p))^Q) & \longrightarrow & MS^{\Gamma^p}(\mathrm{Div}^\dagger((\Gamma_0^p(N)x)^Q)) \\
\downarrow & & \downarrow \\
(\mathrm{Div}(\mathcal{H}_p))^{\ddagger}_{\Gamma_0^p(N)} & \longrightarrow & (\mathrm{Div}(\Gamma_0^p(N)x)^{\ddagger}_{\Gamma_0^p(N)}.
\end{array}$$

The arrows from left to right represent the projection to the orbit of $x \in \mathcal{H}_p$, and the arrows from top to bottom are isomorphisms coming from respectively Shapiro's Lemma and the association modular symbol / period divisors. Therefore the image of an element in the top row is non-zero if and only if the corresponding element in the bottom row is non-zero. We collect all of the above in the following theorem.

Theorem 3.3.5. *The support of a rigid period divisor $(\Delta_i)_{i=1}^Q \in (Div(\mathcal{H}_p))^{\ddagger}_{\Gamma_0^p(N)}$ (or similarly a rigid divisor cocycle $m \in MS^{\Gamma_0^p(N)}(Div^{\dagger}(\mathcal{H}_p))$) consist only of RM-points. The same is true for a rigid meromorphic period function $(\varphi_i)_{i=1}^Q \in (\mathcal{M}_k)^{\ddagger}_{\Gamma_0^p(N)}$ and a rigid meromorphic cocycle $m \in MS^{\Gamma_0^p(N)}(\mathcal{M}_k)$.*

3.4 Additive cocycles

This section will give a classification of additive rigid meromorphic cocycles, similar to that of rational period functions given in [25] and [9] in the case $N = 1$, and in [1] in the general congruence subgroup case. In this last article, the author proves the existence of a short exact sequence

$$0 \to \mathbb{C} \to H^1_{par}(\Gamma_0(N), M) \to \bigoplus_{w \in O_{irr}} \mathbb{C}_w \to S_2(\Gamma_0(N)) \otimes_{\mathbb{R}} \mathbb{C} \to 0,$$

where M is the $\mathbb{C}$-vector space of rational functions on the Riemann sphere, O_{irr} is the set of RM points in the classical upper-half plane $\mathcal{H}$, $\mathbb{C}_w$ is a 1-dimensional $\mathbb{C}$-vector space and $S_2(\Gamma_0(N))$ is the space of cusp forms of weight 2 and level N. This is strikingly similar to the short exact sequence obtained after Theorem 2.2.3 with the exception that we now have a space of "obstructions" given by cusp forms. The goal of this section will be to obtain a similar result in the rigid meromorphic case.

We use the same notation as in Chapter 2 with $\Gamma_0^p(N)$ instead of Γ^p. For example, $\Sigma_\tau(r,s)$ is the set of RM points w in the $\Gamma_0^p(N)$-orbit of τ such that w and w' lie in two distinct connected components of $\overline{\mathcal{H}} - \gamma(r,s)$. Again, $\Sigma_\tau(r,s)$ is endowed with a sign function $\delta_{r,s}(w)$.

Similarly to the $N = 1$ case, for $\tau \in \mathcal{H}_p^{RM}$, we define divisors

$$\Delta_\tau\{r,s\} = \sum_{w \in \Sigma_\tau(r,s)} \delta_{r,s}(w)[w].$$

We also have a natural morphism $MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \to MS^{\Gamma_0^p(N)}(Div^{\dagger}(\mathcal{H}_p))$ given by $f \mapsto \sum res_w(f)[w]$, where res is the residue of the rigid meromorphic function f at w.

Proposition 10 in [22] gives the isomorphism

$$MS^{\Gamma_0^p(N)}(Div^{\dagger}(\mathcal{H}_p)) \cong \bigoplus_{\Gamma_0^p(N)\backslash\mathcal{H}_p^{RM}} \mathbb{C}_p \cdot \Delta_\tau\{r,s\}.$$

So the set $\{\Delta_\tau \ : \ \tau \in \Gamma_0^p(N)\backslash\mathcal{H}_p^{RM}\}$ gives a basis of $MS^{\Gamma_0^p(N)}(Div^{\dagger}(\mathcal{H}_p))$. We will try to understand this module in more detail for the remainder of this section.

Let $v, \overline{v}$ two adjacent vertices of the Bruhat-Tits tree $\mathcal{T}$ (v even and $\overline{v}$ odd) and e the connecting edge. Then $\{v, \overline{v}, e\}$ is a fundamental set for the action of $\Gamma_0^p(N)$ on $\mathcal{T}$ (See [39]). We will typically take $\{v_0, v_0', e_0\}$ as

the fundamental set, where v_0 is the standard vertex, e_0 is the edge whose image under the reduction map is

$$W_{e_0} = \{z \in \mathbb{C}_p \ : \ 1 < |z| < p\},$$

and v_0' is the vertex connected to v_0 by e_0. Let $\mathcal{A}_{v_0}, \mathcal{A}_{v_0'}, \mathcal{W}_{e_0}$ be their respective affinoid / annulus. They give rise to natural morphisms

$$\begin{aligned}
\deg_{\mathcal{A}_{v_0}} &: MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p)) \to MS^{\Gamma_{v_0}}(\mathbb{C}_p) \\
\deg_{\mathcal{A}_{v_0'}} &: MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p)) \to MS^{\Gamma_{v_0'}}(\mathbb{C}_p) \\
\deg_{\mathcal{W}_e} &: MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p)) \to MS^{\Gamma_{e_0}}(\mathbb{C}_p)
\end{aligned}$$

where $\Gamma_{v_0} \simeq \Gamma_{v_0'} \simeq \Gamma_0(N)$ and $\Gamma_{e_0} \simeq \Gamma_0(pN)$. By Eichler-Shimura, these last modules are isomorphic to two copies of a certain space of cusp forms plus a space of Eisenstein series. We will make use of that when needed.

Now let's study the sequence

$$MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \xrightarrow{res} MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p)) \xrightarrow{deg} MS^{\Gamma_v}(\mathbb{C}_p) \oplus MS^{\Gamma_{\overline{v}}}(\mathbb{C}_p) \oplus MS^{\Gamma_e}(\mathbb{C}_p)$$

An element $\Delta \in MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p))$ can be lifted to an element in $MS^{\Gamma_0^p(N)}(\mathcal{M}_2)$ if its degree on each affinoid $\mathcal{A}_v$ and standard wide open subset $\mathcal{W}_v$ is 0 (the same argument as in the case $N = 1$ can be applied). Since $\{v_0, v_0', e_0\}$ is a fundamental set for the action of $\Gamma_0^p(N)$ on $\mathcal{T}$, this comes down to $\deg_{\mathcal{A}_{v_0}}(\Delta) = \deg_{\mathcal{A}_{v_0'}}(\Delta) = 0$ (the degree is 0 on every vertex affinoid) and

$$\sum_{i=1}^{p+1} \gamma_i \deg_{\mathcal{W}_{e_0}}(\Delta) = 0, \quad \sum_{i=1}^{p+1} \beta \cdot \gamma_i \deg_{\mathcal{W}_{e_0}}(\Delta) = 0$$

where β is a matrix that acts on the Bruhat-Tits tree by reversing the direction of the edges (This is equivalent to have degree 0 on $\mathcal{W}_v \backslash \mathcal{A}_v$). In fact, we will show that a less restrictive condition than "being of degree 0 on each affinoid" is enough to guarantee that a divisor-valued modular symbol Δ can be lifted to a rigid meromorphic cocycle.

Theorem 3.4.1. *There is an exact sequence*

$$\begin{aligned}
0 \to & MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \to MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \xrightarrow{div} MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p)) \\
& \xrightarrow{(\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}})} \frac{H^1_{par}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{par}(\Gamma_0'(N), \mathbb{C}_p)}{\mathrm{im}(\varphi_s \oplus \varphi_t)} \to 0
\end{aligned}$$

We will prove this result at the end of Chapter 4 (Theorem 4.5.2) when we have more tools at our disposition. This theorem is in accordance with the case $N = 1$ where the rightmost module is trivial, which implies that every element of $MS^{\Gamma^p}(Div^\dagger(\mathcal{H}_p))$ lifts genuinely to a rigid meromorphic cocycle.

In particular, the isomorphism $MS^{\Gamma_0^p(N)}(\mathcal{O}) \xrightarrow{\sim} MS^{\Gamma_0^p(N)}(\mathcal{M})$ is also true in this case with the same argument, i.e. the composition

$$MS^{\Gamma_0^p(N)}(\mathcal{M}) \to MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \to MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p))$$

is zero, so elements of $MS^{\Gamma_0^p(N)}(\mathcal{M})$ must be analytic.

3.5 Additive rigid analytic cocycles

The classification of elements of $MS^{\Gamma_0^p(N)}(\mathcal{O}_2)$ is similar to that of $MS^{\Gamma^p}(\mathcal{O}_2)$ with some small subtleties related to p-new modular forms. Also, the discrepancy between modular symbols and the parabolic cohomology of $\mathcal{O}_2$ is not as simple as the Γ^p-case since the Eisenstein subspace of $\Gamma_0(pN)$ is not 1-dimensional in general. This discrepancy will be studied at the end of this section. Finally, we will also describe how the classification of rigid analytic cocycles generalizes for the non-parabolic part by introducing Dedekind-Rademacher cocycles and "modified Dedekind-Rademacher cocycles" in the style of [32] and [13].

We can see that some arguments and proofs of the last chapter generalizes directly to the case of congruence subgroups. First, recall the following maps:
1)

$$BD \ : \ \mathcal{O}_2 \xrightarrow{\sim} \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)).$$

This map again induces an isomorphism on $\Gamma_0^p(N)$-invariant modular symbols, which we also denote BD.

2)

$$res_{\mathbb{Z}_p} \ : \ MS^{\Gamma^p}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \to MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)),$$

which again induces a map from $MS^{\Gamma_0^p(N)}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))$ to $MS^{\Gamma_0(pN)}(\mathcal{D}(\mathbb{Z}_p))$ since the stabilizer in $\Gamma_0^p(N)$ of $\mathbb{Z}_p$ is $\Gamma_0(pN)$.
3)

$$\rho \ : \ MS^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \longrightarrow MS^{\Gamma_0(p)}(\mathbb{C}_p),$$

which also gives a map between $\Gamma_0(pN)$-invariant modular symbols since ρ come from the total measure map from $\mathcal{D}(\mathbb{Z}_p)$ to $\mathbb{C}_p$.

Proposition 3.5.1. *The map*

$$\eta := \rho \circ res_{\mathbb{Z}_p} \circ BD \ : \ MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \hookrightarrow MS^{\Gamma_0(pN)}(\mathbb{C}_p)$$

is injective.

Proof. First of all, BD is an isomorphism since it comes from the isomorphism $\mathcal{O}_2 \xrightarrow{\sim} \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$.
Second, as seen before when studying the action of $\Gamma_0^p(N)$ on the Bruhat-Tits tree, if we decompose $\mathbb{P}_1(\mathbb{Q})$ as a union of $p+1$ open balls

$$\mathbb{P}_1(\mathbb{Q}) = B_0 \sqcup ... \sqcup B_{p-1} \sqcup \mathbb{Z}_p',$$

then we know that $\Gamma_0(N)$ acts transitively on this decomposition. Therefore the proof of the injectivity of $res_{\mathbb{Z}_p}$ transposes to the $\Gamma_0^p(N)$-case.
Finally, the map

$$\rho \ : \ MS^{\Gamma_0(pN)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1} \longrightarrow MS^{\Gamma_0(pN)}(\mathbb{C}_p)$$

is now only injective by Theorem 1.1 of [36]. In fact it is an isomorphism if we restrict the codomain to the ordinary subspace of $MS^{\Gamma_0(pN)}(\mathbb{C}_p)$, i.e. the direct summand of the subspace such that U_p acts with slope 0 (U_p^n acts as 0 for some n) , which corresponds to the p-new part of the module.
Since all three maps are injective, we conclude that η is injective. □

Similarly to the space of p-new modular forms, we define the space of p-new modular symbols.

Definition 3.5.2. *Let $\gamma_1, ..., \gamma_{p+1}$ be representatives of $\Gamma_0(N)/\Gamma_0(pN)$ and consider the maps*

$$\varphi_s, \varphi_t \ : \ MS^{\Gamma_0(pN)}(\mathbb{C}_p) \to MS^{\Gamma_0(N)}(\mathbb{C}_p)$$

defined by

$$\varphi_s(m) := \sum_{j=1}^{p+1} m|\gamma_j, \quad \varphi_t(m) := \sum_{j=1}^{p+1} m|\beta\gamma_j,$$

where β is again a "parity-reversing" matrix. Then the kernel of

$$\varphi_s \oplus \varphi_t \ : \ MS^{\Gamma_0(pN)}(\mathbb{C}_p) \longrightarrow MS^{\Gamma_0(N)}(\mathbb{C}_p) \oplus MS^{\Gamma_0(N)}(\mathbb{C}_p)$$

is called the space of p-new $\Gamma_0(pN)$-invariant modular symbols with values in $\mathbb{C}_p$ and is denoted by $MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$.

Looking back at the definition of cusp forms on the Bruhat-Tits tree and their relation to p-new cusp form, we see the resemblance between harmonic cocycles and the p-new modular symbols. Indeed, we have:

Lemma 3.5.3. *The map $ev_0 \ : \ MS^{\Gamma_0^p(N)}(C_{har}(\mathbb{C}_p)) \to MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$ is an isomorphism.*

Proof. The injectivity is proven as in the Γ^p-case: let c be an element in the kernel of ev_0, i.e.

$$c\{r,s\}(e_0) = 0 \quad \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}).$$

Since $\Gamma_0^p(N)$ acts transitively on $\mathcal{E}^+$, then

$$c\{r,s\}(e) = 0 \quad \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}),\ e \in \mathcal{E}^+.$$

By harmonicity, we conclude the same to be true for all $e \in \mathcal{E}^*$.

Now for surjectivity, let $c_0 \in MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-\text{new}}$, and define $c \in MS^{\Gamma_0^p(N)}(C_{\text{har}}(\mathbb{C}_p))$ by setting

$$c\{r,s\}(e) := c_0\{\gamma r, \gamma s\}$$

where $e = \gamma^{-1}e_0$ for some $\gamma \in \Gamma_0^p(N)$. Since the stabilizer of e_0 in $\Gamma_0^p(N)$ is $\Gamma_0(pN)$, the values $c\{r,s\}(e)$ don't depend on γ. One readily sees that $ev_0(c) = c_0$, but we still need to check that c takes its values in the space of harmonic cocycles.

Since c_0 is a p-new modular symbol, it satisfies

$$\sum_{j=1}^{p+1} c_0|\gamma_j = 0, \quad \sum_{j=1}^{p+1} c_0|\beta\gamma_j = 0.$$

Since the matrices $\gamma_1, ..., \gamma_{p+1}$ send the standard edge to all the edges surrounding v_0, translating this in the case of harmonic cocycles, one gets using the first relation

$$\sum_{s(e)=v} c(e) = 0 \quad \text{for all } v \in \mathcal{V}^+,$$

and similarly for $\mathcal{V}^-$ using the second relation. Therefore c takes its values in $C_{\text{har}}(\mathbb{C}_p)$. □

3.5.1 Schneider-Teitelbaum lift

We now turn our attention to the Schneider-Teitelbaum lift for $\Gamma_0^p(N)$. This will finally give an isomorphism

$$MS^{\Gamma_0^p(N)}(C_{\text{har}})(\mathbb{C}_p) \xrightarrow{\sim} MS^{\Gamma_0^p(N)}(\mathcal{O}_2),$$

a crucial step in the classification of rigid analytic cocycles for $\Gamma_0^p(N)$.

Let's recall the subspace of bounded harmonic cocycles denoted by $C_{\text{har}}^b(\mathbb{C}_p)$, i.e. harmonic cocycles that take values in a bounded subset of $\mathbb{C}_p$ as well as its associated measure given by

$$\int_{U_e} 1 d\mu_c := c(e).$$

Lemma 3.5.4. *The natural inclusion $MS^{\Gamma_0^p(N)}(C_{har}^b(\mathbb{C}_p)) \hookrightarrow MS^{\Gamma_0^p(N)}(C_{har}(\mathbb{C}_p))$ is an isomorphism.*

Proof. This is basically the same proof as in the Γ^p-case, i.e. let $c \in MS^{\Gamma_0^p(N)}(C_{\text{har}}(\mathbb{C}_p))$ and consider $\text{ev}_{e_0}(c) \in MS^{\Gamma_0(pN)}(\mathbb{C}_p)$. Since $\Gamma_0(pN)$ is finitely generated, there exists a bounded subset $\Omega \subset \mathbb{C}_p$ such that $\text{ev}_{e_0}(c) \in MS^{\Gamma_0(pN)}(\Omega)$. Now if we consider the commutative diagram

$$\begin{array}{ccc} MS^{\Gamma_0^p(N)}(C_{\text{har}}(\Omega)) & \xrightarrow{\text{ev}_{e_0}} & MS^{\Gamma_0(pN)}(\Omega) \\ \downarrow & & \downarrow \\ MS^{\Gamma_0^p(N)}(C_{\text{har}}(\mathbb{C}_p)) & \xrightarrow{\text{ev}_{e_0}} & MS^{\Gamma_0(pN)}(\mathbb{C}_p), \end{array}$$

and knowing that ev_{e_0} is an isomorphism in both cases, we readily see that $c \in MS^{\Gamma_0^p(N)}(C_{\text{har}}(\Omega))$. □

The map

$$\text{ST}(c)(z) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_c(t)$$

turns out to be a $\Gamma_0^p(N)$-equivariant splitting of the residue map ∂ on $C^b_{\text{har}}(\mathbb{C}_p)$ (See [42] for more details). Therefore, one gets the following result:

Lemma 3.5.5. *The residue map*

$$\partial \ : \ MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \longrightarrow MS^{\Gamma_0^p(N)}(C_{har}(\mathbb{C}_p))$$

is an isomorphism.

Proof. The injectivity of ∂ follows from the same argument as in the Γ^p-case since we have seen that ρ is injective. Now given $c \in MS^{\Gamma_0^p(N)}(C_{\text{har}}(\mathbb{C}_p))$, since $c\{r,s\}$ takes values in a bounded subset of $\mathbb{C}_p$ by the last result, we can define

$$f\{r,s\} := \text{ST}(c\{r,s\}) \in \mathcal{O}_2.$$

f is an element of $MS^{\Gamma_0^p(N)}(\mathcal{O}_2)$ and $\partial(f) = c$, proving surjectivity. □

Theorem 3.5.6. *The map*

$$\eta \ : \ MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \longrightarrow MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$$

is an isomorphism.

Proof. This follows from Lemma 3.5.3 and 3.5.5. □

Using the Eichler-Shimura isomorphism, this gives an isomorphism between rigid analytic cocycles for $\Gamma_0^p(N)$ and the direct sum of two copies of p-new cusp forms of level pN with an "Eisenstein subspace".

3.5.2 Relation between $H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p)^{p-new}$ and $MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$

The goal of this small section is to give the discrepancy between the two modules $H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p)^{p-new}$ and $MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$, akin to the definition of the "universal" or "boundary" cocycle given in section 2.6 of [16] or section 3.7 of [15]. We will start by looking at the whole module, i.e. not only the part new at p. Let $c_1, ...c_l$ be the cusps of $\Gamma_0(N)$. By abuse of notation, we will denote by $c_1, ..., c_l, c_{l+1}, ..., c_{2l}$ the cusps of $\Gamma_0(pN)$ in such a way that c_i and c_{i+l} are equivalent cusps under $\Gamma_0(N)$ for $i = 1, ..., l$. Such a classification is possible since $p \nmid N$ and N is square-free, which we assume in this section and the next. As in Section 3.1, we take the $\Gamma_0(pN)$-cohomology of the short exact sequence

$$0 \to \mathbb{C}_p \to \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \mathbb{C}_p) \xrightarrow{d} MS(\mathbb{C}_p) \to 0,$$

to obtain the exact sequence

$$0 \to \mathbb{C}_p^{2l-1} \to MS^{\Gamma_0(pN)}(\mathbb{C}_p) \to H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p) \to 0.$$

The kernel of $MS^{\Gamma_0(pN)}(\mathbb{C}_p) \to H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p)$ is generated by the "boundary" modular symbols

$$m^i_{\#}\{r, s\} := \begin{cases} 1 & \text{if } [c_i, c_{i+1}] \text{ and } [r, s] \text{ intersect positively,} \\ -1 & \text{if } [c_i, c_{i+1}] \text{ and } [r, s] \text{ intersect negatively,} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, ..., 2l - 1$.

Now we inspect what the Schneider-Teitelbaum lift of $m^i_{\#}$ might be. First, given two cusps $c_i = \frac{r_i}{s_i}, c_{i+1} = \frac{r_{i+1}}{s_{i+1}}$, we define the matrix

$$A_i := \begin{pmatrix} s_i & -r_i \\ s_{i+1} & -r_{i+1}. \end{pmatrix}$$

Lemma 3.5.7. *Given $r, s \in \mathbb{P}_1(\mathbb{Q})$, the function*

$$f^i_{\#}\{r, s\} := \textit{dlog}\left(\frac{z - A \cdot s}{z - A \cdot r}\right) = \frac{dz}{z - A \cdot s} - \frac{dz}{z - A \cdot r}$$

is an element of $MS^{\Gamma_0^p(N)}(\mathcal{O}_2)$ (with the conventions $z - \infty := 1$ and $\frac{dz}{z-\infty} := 0$).

Proof. We first begin by looking at the multiplicative counterpart of $f^i_{\#}$, i.e.

$$(f^{i,\times}_{\#})\{r, s\}(z) := \left(\frac{z - A \cdot s}{z - A \cdot r}\right) \in \mathcal{O}^\times / \mathbb{C}_p^\times.$$

It is clear that $(f_\#^{i,\times})$ satisfies the 2 and 3-terms relation (without even taking the function modulo scalars), and thus is a modular symbol. Also, it is the unique rational function, up to multiplication by $\mathbb{C}_p^\times$ that has divisor $A\cdot((s)-(r))$. For any $\gamma\in\Gamma_0^p(N)$, $(f_\#^{i,\times})\{\gamma r,\gamma s\}(\gamma z)$ and $(f_\#^{i,\times})\{r,s\}(z)$ have the same divisor, and hence their quotient is a constant. Therefore $(f_\#^{i,\times})\in MS^{\Gamma_0^p(N)}(\mathcal{O}^\times/\mathbb{C}_p^\times)$ and its logarithmic derivative $f_\#^i$ is an element of $MS^{\Gamma_0^p(N)}(\mathcal{O}_2)$. □

Lemma 3.5.8. *$f_\#^i$ is the Schneider-Teitelbaum lift of $m_\#^i$.*

Proof. Let $\mathcal{W}_{e_0}$ be the standard annulus. It divides $\mathbb{P}_1(\mathbb{C}_p)$ into two components, and their intersection with $\mathbb{P}_1(\mathbb{Q}_p)$ is $\mathbb{Z}_p$ and $\mathbb{Z}_p'$. We see that $r\in\mathbb{Z}_p$ and $s\in\mathbb{Z}_p'$ (respectively $s\in\mathbb{Z}_p$ and $r\in\mathbb{Z}_p'$) if and only if $[r,s]$ intersects $[0,\infty]$ positively (resp. negatively).

We see that the matrix $A_i:=\begin{pmatrix}s_i & -r_i\\ s_{i+1} & -r_{i+1}\end{pmatrix}$ attached to a cusp c_i sends c_i to 0 and c_{i+1} to ∞. Therefore

$$\partial_{\mathcal{W}_{e_0}}(f_\#^i)=\sum_{t\in\mathbb{Z}_p'\cap\{r,s\}}\partial_t(f_\#^i)=\begin{cases}1 & \text{if } [c_i,c_{i+1}] \text{ and } [r,s] \text{ intersect positively,}\\ -1 & \text{if } [c_i,c_{i+1}] \text{ and } [r,s] \text{ intersect negatively,}\\ 0 & \text{otherwise}\end{cases}$$

□

Now our goal is to find which of these boundary symbols generate the p-new part of $MS^{\Gamma_0(pN)}(\mathbb{C}_p)$ so the have a characterization of the relation between $H^1_{\text{par}}(\Gamma_0(pN),\mathbb{C}_p)^{p-new}$ and $MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$.

First, we see that the action of a matrix $\gamma\in\Gamma_0(N)$ on $m_\#^i$ only change the associated cusp in the definition, i.e.

$$\gamma\cdot m_\#^i=\begin{cases}1 & \text{if } [\gamma^{-1}c_i,\gamma^{-1}c_{i+1}] \text{ and } [r,s] \text{ intersect positively,}\\ -1 & \text{if } [\gamma^{-1}c_i,\gamma^{-1}c_{i+1}] \text{ and } [r,s] \text{ intersect negatively,}\\ 0 & \text{otherwise.}\end{cases}$$

So if we look at the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}_p & \longrightarrow & \mathbb{C}_p^{2l} & \longrightarrow & MS^{\Gamma_0(pN)}(\mathbb{C}_p) & \longrightarrow & H^1_{\text{par}}(\Gamma_0(pN),\mathbb{C}_p) & \longrightarrow & 0\\
 & & \downarrow g & & \downarrow f & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}_p\oplus\mathbb{C}_p & \longrightarrow & \mathbb{C}_p^l\oplus\mathbb{C}_p^l & \longrightarrow & MS^{\Gamma_0(N)}(\mathbb{C}_p)\oplus MS^{\Gamma_0(N)}(\mathbb{C}_p) & \longrightarrow & H^1_{\text{par}}(\Gamma_0(N),\mathbb{C}_p)\oplus H^1_{\text{par}}(\Gamma_0(N),\mathbb{C}_p) & \longrightarrow & 0,
\end{array}$$

where the arrows from top to bottom are induced by the map $\varphi_s\oplus\varphi_t$ in Section 3.2. We see that g is injective and since c_i and c_{i+l} are equivalent cusps under $\Gamma_0(N)$, the kernel of f is generated by the modular symbols

$$m_\flat^i\{r,s\}:=m_\#^i-m_\#^{l+i}$$

for $i = 1, ..., l$. Therefore, taking the kernel of the maps from the top row to the bottom row, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C}_p^l & \longrightarrow & MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new} & \longrightarrow & H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p)^{p-new} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}_p & \longrightarrow & \mathbb{C}_p^{2l} & \longrightarrow & MS^{\Gamma_0(pN)}(\mathbb{C}_p) & \longrightarrow & H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p) & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow f & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}_p \oplus \mathbb{C}_p & \longrightarrow & \mathbb{C}_p^l \oplus \mathbb{C}_p^l & \longrightarrow & MS^{\Gamma_0(N)}(\mathbb{C}_p) \oplus MS^{\Gamma_0(N)}(\mathbb{C}_p) & \longrightarrow & H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p) & \longrightarrow & 0,
\end{array}$$

where the kernel of f is generated by $m_\flat^i$ for $i = 1, ..., l$. In the case $N = 1$, we would have $l = 1$ and $m_\flat^i = 2 \cdot m_\#^i$, which is in accordance to the original results.

Corollary 3.5.9. *The Schneider-Teitelbaum lift induces an isomorphism*

$$L_{par}^{ST} \ : \ H^1_{par}(\Gamma_0(pN), \mathbb{C}_p)^{p-new} \to MS^{\Gamma_0^p(N)}(\mathcal{O}_2)/ \oplus_{i=1}^l \mathbb{C}_p f_\flat^i,$$

where

$$f_\flat^i := f_\#^i - f_\#^{l+i}.$$

Proof. This comes directly from the above commutative diagram, the fact that $\{m_\flat^i\{r, s\}\}_{i=1}^l$ generate the kernel of f, Theorem 3.5.6 and Lemma 3.5.8. □

3.5.3 Dedekind-Rademacher cocycles and non-parabolic part of $H^1(\Gamma_0^p(N), \mathcal{O}_2)$

We now conclude the section on rigid analytic cocycles by studying the non-parabolic part of $H^1(\Gamma_0^p(N), \mathcal{O}_2)$. Unsurprisingly, the non-parabolic part $H^1(\Gamma_0^p(N), \mathcal{O}_2)$ is closely related to the non-parabolic part of $H^1(\Gamma_0(pN), \mathbb{C}_p)$, which by Eichler-Shimura, is isomorphic to the space of Eisenstein series of level pN.

First, we see that the short exact sequence induced by the residue map (See [17])

$$0 \to \mathcal{O}_0/\mathbb{C}_p \to \mathcal{O}_2 \xrightarrow{\partial} C_{\text{har}}(\mathbb{C}_p) \to 0$$

gives rise to the exact sequence

$$0 \to \mathbb{C}_p^l \to H^1(\Gamma_0^p(N), \mathcal{O}_2) \to H^1(\Gamma_0(pN), \mathbb{C}_p)^{p-\text{new}} \to 0,$$

where again l is the number of cusps of $\Gamma_0(N)$. Similarly as in the last subsection, we give a classification of the non-parabolic part of $H^1(\Gamma_0(pN), \mathbb{C}_p)^{p-\text{new}}$, which will complete the full classification of $H^1(\Gamma_0^p(N), \mathcal{O}_2)$.

By Eichler-Shimura, the non-parabolic part of $H^1(\Gamma_0(N), \mathbb{C}_p)$ (respectively $H^1(\Gamma_0(pN), \mathbb{C}_p)$) is of dimension $l-1$ (respectively $2l-1$). More precisely, given a divisor

$$\delta = \sum_{d|N} n_d [d],$$

we can define the modified Dedekind sum

$$D^\delta(x) := \sum_{d|N} n_d D(dx),$$

where

$$D\left(\frac{a}{m}\right) := \sum_{x=1}^{m} B_1\left(\frac{x}{m}\right) B_1\left(\frac{ax}{m}\right), \quad \text{for } gcd(a,m) = 1, m > 0$$

and $B_1(x) = x - [x] - 1/2$ is the first Bernoulli polynomial. Given a divisor δ with $\deg(\delta) = 0$, we define the Dedekind-Rademacher cocycle $\Phi_\delta^{(N)} \in H^1(\Gamma_0(N), \mathbb{C}_p)$ associated to δ to be

$$\Phi_\delta^{(N)}\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} := \begin{cases} [\delta] b/d & \text{if } c = 0 \\ \frac{[\delta](a+d)}{Nc} - 12 \cdot \text{sign}(c) \cdot D^\delta(\frac{a}{N|c|}) & \text{otherwise,} \end{cases}$$

where $[\delta] = \sum_{d|N} n_d \cdot d$. If $\delta = [N] - [1]$, this corresponds to the cocycle introduced in [32], and when $[\delta] = 0$, this corresponds to the "modified Dedekind-Rademacher cocycles" introduced in [13]. One sees using the fact that $\Phi_{[d]-[1]}^{(N)}$ is a cocycle for any $d|N$ (see [32]), that $\Phi_\delta^{(N)}$ are also cocycles for any δ. The Dedekind-Rademacher cocycle studied in [16] Section 5.3 corresponds to the divisor $[p] - [1]$. Also, given a divisor δ (not necessarily of degree 0), we define $\Phi_{\delta_p}^{pN} \in H^1(\Gamma_0(pN), \mathbb{C}_p)$ to be

$$\Phi_{\delta_p}^{(pN)}\begin{pmatrix} a & b \\ pNc & d \end{pmatrix} := \begin{cases} ([p\delta] - [\delta]) b/d & \text{if } c = 0 \\ \frac{([p\delta]-[\delta])(a+d)}{pNc} - 12 \cdot \text{sign}(c) \cdot \left(D^\delta(\frac{a}{N|c|}) - D^\delta(\frac{a}{pN|c|})\right) & \text{otherwise.} \end{cases}$$

Even if $\delta = \sum_{d|N} n_d [d]$ is not of degree 0, $\delta_p = \sum_{d|N} n_d [pd] - \sum_{d|N} n_d [d]$ is. One can see that

$$\Phi_{\delta_p}^{(pN)} = \Phi_{\delta^{(p)} - \delta}^{(pN)} = \Phi_{\delta^{(p)}}^{(pN)} - \Phi_\delta^{(pN)} = \Phi_{\delta^{(p)}}^{(pN)} - P^{-1}\Phi_{\delta^{(p)}}^{(pN)} P,$$

where $\delta^{(p)} := \sum_{d|N} n_d [pd]$ if $\delta = \sum_{d|N} n_d [d]$. The Dedekind-Rademacher cocycle studied in [16] Section 5.3 corresponds to $\Phi_{[1]_p}^{(p)}$ since $[1]_p = [p] - [1]$.

So the goal of this section is to classify the Eisenstein or non-parabolic part of $H^1(\Gamma_0(pN), \mathbb{C}_p)^{p-\text{new}}$, i.e. the module

$$H^1(\Gamma_0(pN), \mathbb{C}_p)^{p-\text{new}} / H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p)^{p-\text{new}} \simeq \left(H^1(\Gamma_0(pN), \mathbb{C}_p) / H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p)\right)^{p-\text{new}}.$$

In that regard, let E_2 be the usual weight 2 Eisenstein series and given $\delta = \sum_{d|N} n_d[d]$ of degree 0, define

$$E_2^\delta := \sum_{d|N} n_d \cdot dE_2(dz).$$

It is a weight 2 modular form for $\Gamma_0(N)$. Let $\omega_{E_2^\delta} := 48\pi i E_2^\delta dz$ be the associated regular differential on $Y_0(N)$ (See [13] Chapter 2 and 3 for an explanation of the constant $48\pi i$). Using results of [32] and additivity of the integral, the cocycle $\Phi_\delta^{(N)}$ encodes the periods of $\omega_{E_2^\delta}$, i.e.

$$\Phi_\delta^{(N)}(\gamma) = \frac{1}{2\pi i}\int_{z_0}^{\gamma z_0} \omega_{E_2^\delta},$$

where $z_0 \in \mathcal{H}$ is an arbitrary point.

So now the question boils down to finding Eisenstein series generating the p-new part of

$$H^1_{\text{eis}}(\Gamma_0(pN), \mathbb{C}_p) := H^1(\Gamma_0(pN), \mathbb{C}_p)/H^1_{\text{par}}(\Gamma_0(pN), \mathbb{C}_p).$$

It is well-known that the space of Eisenstein series for $\Gamma_0(pN)$ has dimension $2l-1$ and a explicit isomorphism (of vector spaces) can be given by

$$\begin{aligned}\text{Eis}_2(\Gamma_0(pN)) := M_2(\Gamma_0(pN))/S_2(\Gamma_0(pN)) &\to \{(v_{c_1}, ..., v_{c_{2l}}) \in \mathbb{C}_p^{2l} \ : \ \sum_{i=1}^{2l} v_i = 0\} \\ E &\mapsto (\text{constant coefficient of the expansion of } E \text{ at cusp } i)_{i=1}^{2l},\end{aligned}$$

where we have numbered the $2l$ cusps $c_1, ..., c_{2l}$ as in the last section. As before, let $\{\gamma_i\}_{i=1}^{p+1}$ be a set of representatives of $\Gamma_0(N)/\Gamma_0(pN)$, and let $\varphi_s \oplus \varphi_t$ be the associated map

$$\varphi_s \oplus \varphi_t \ : \ \text{Eis}_2(\Gamma_0(pN)) \to \text{Eis}_2(\Gamma_0(N)) \oplus \text{Eis}_2(\Gamma_0(N)').$$

By abuse of notation, we also denote by $\varphi_s \oplus \varphi_t$ the map on $\mathbb{C}_p^{\text{cusps}}$ by using the aforementioned isomorphism. Since the sets $\{c_i, c_{i+l}\}$ for $i = 1, ..., l$ are the orbits of cusps of $\Gamma_0(pN)$ for the action of $\Gamma_0(N)$, we can see as in the last section that the elements

$$(0, ..., 0, v_{c_i}, 0, ..., 0), \quad \text{and} \quad (0, ..., 0, v_{c_{i+l}}, 0, ..., 0)$$

have the same image under $\varphi_s \oplus \varphi_t$ and that the kernel of this map is generated by the elements

$$(0, ..., 0, v_{c_i}, 0, ..., 0, -v_{c_{i+l}}, 0, ..., 0) \text{ with } v_{c_i} = v_{c_{i+l}} = 1, \quad i = 1, ..., l.$$

In terms of Eisenstein series, E_2^δ with $\delta = [pd] - [d]$ for $d|N$ forms a basis of this subspace, since these gives l independent ordinary Eisenstein series, i.e. they satisfy $U_p(E_2^\delta) = E_2^\delta$ with the usual U_p-operator. This gives the following theorem.

Theorem 3.5.10. *There is an exact sequence*

$$0 \to \mathbb{C}_p^l \to H^1(\Gamma_0^p(N), \mathcal{O}_2) \to H^1(\Gamma_0(pN), \mathbb{C}_p)^{p-new} \to 0,$$

where the image of $\mathbb{C}_p^l$ in $H^1(\Gamma_0^p(N), \mathcal{O}_2)$ is generated by the $f_\flat^i$ of the last section and $H^1(\Gamma_0(pN), \mathbb{C}_p)^{p-new}$ is generated, using the Eichler-Shimura isomorphism, by two spaces of the p-new cusp forms for $\Gamma_0(pN)$ with the Eisenstein series E_2^δ with $\delta = [pd] - [d]$ for $d|N$. Therefore $H^1(\Gamma_0^p(N), \mathcal{O}_2)$ is of dimension $2g + 2l$, where l = number of cusps of $\Gamma_0(N)$ = number of divisors of N, and g is the genus of $X_0(pN)^{p-new}$.

We conclude this section with a small remark on Eisenstein cocycles and their relation to partial modular symbols.

Let c_i be a cusp of Γ, where Γ is any congruence or p-arithmetic congruence subgroup. The module of *partial modular symbols* $MS_{c_i}(\Omega)$ associated to c_i with values in Ω is defined exactly the same way as the usual modular symbols, except that they can only be evaluated at pairs $(r, s) \in \Gamma c_i \times \Gamma c_i$. Similarly as the usual modular symbols, we get an exact sequence

$$0 \to \Omega \to \mathcal{F}(\Gamma c_i, \Omega) \to MS_{c_i}(\Omega) \to 0,$$

where $\mathcal{F}(\Gamma c_i, \Omega)$ is the Γ-module of functions on Γc_i with values in Ω. Taking the Γ-cohomology one obtains

$$0 \to \Omega^\Gamma \to \Omega^{\Gamma_{c_i}} \to MS_{c_i}^\Gamma(\Omega) \to H^1(\Gamma, \Omega) \to H^1(\Gamma_{c_i}, \Omega),$$

where Γ_{c_i} is the stabilizer of c_i in Γ. Taking $\Omega = \mathbb{C}_p$ gives the isomorphism

$$MS_{c_i}^\Gamma(\mathbb{C}_p) \simeq H_{c_i}^1(\Gamma, \mathbb{C}_p),$$

where $H_{c_i}^1(\Gamma, \mathbb{C}_p)$ are the cocycles being sent to 0 in $H^1(\Gamma_{c_i}, \mathbb{C}_p)$. Under Eichler-Shimura, these corresponds to modular form having 0 constant term at c_i. For example, if we take $c_i = \infty$ and $\Gamma = \Gamma_0(N)$, then $E_2^\delta(z) \in H_{c_i}^1(\Gamma, \mathbb{C}_p)$ if $\sum_{d|N} n_d \cdot d = 0$ (See [13]).
By the above, clearly one has that the groups $H_{c_i}^1(\Gamma_0^p(N), \mathcal{O}_2)$ generate $H^1(\Gamma_0^p(N), \mathcal{O}_2)$ and therefore given $\Phi \in H^1(\Gamma_0^p(N), \mathcal{O}_2)$, one can find a (non-unique) decomposition

$$\Phi = \sum_{c_i} \Phi_{c_i},$$

with $\Phi_{c_i} \in H_{c_i}^1(\Gamma_0^p(N), \mathcal{O}_2)$ and c_i is running through the cusps of $\Gamma_0^p(N)$. Since $H_{c_i}^1(\Gamma_0^p(N), \mathcal{O}_2) \simeq MS_{c_i}^\Gamma(\mathcal{O}_2)$, this describes Φ as a sum of modular symbols.

We remark that the point of view adopted here regarding the non-parabolic part of H^1 differs slightly from the literature. For example, in [18], the *p-new quotient* is typically considered. More precisely, consider as usual the degeneracy maps

$$\varphi_s \ : \ X_0(pN) \to X_0(N)$$

and $\varphi_t = \varphi_s \circ W_p$. Then the pullback / restriction of these maps gives a morphism

$$\varphi^* \ : \ H^1(\Gamma_0(N), \mathbb{C}_p)^{\oplus 2} \to H^1(\Gamma_0(pN), \mathbb{C}_p).$$

The cokernel of this map is called the *p-new quotient*, in contrast with the *p-new kernel* which is given by the pushforward / corestriction

$$\varphi_* \ : \ H^1(\Gamma_0(pN), \mathbb{C}_p) \to H^1(\Gamma_0(N), \mathbb{C}_p)^{\oplus 2}.$$

One can show that the composition of the restriction with the corestriction is given by

$$\varphi_* \circ \varphi^* = \begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix},$$

and since the eigenvalues of cusp forms are $< 2\sqrt{p}$ in absolute value, this map is invertible on the parabolic part of H^1 and the p-new quotient and p-new kernel are isomorphic there.
However, this matrix is clearly non-invertible for the Eisenstein part of H^1, which explains the discrepency with the literature. When considering the p-new quotient, one can see that the corresponding map on cusp

$$\text{cusps}(\Gamma_0(N)) \times \text{cusps}(\Gamma_0'(N)) \to \text{cusps}(\Gamma_0(pN))$$

is injective, since above each cusp of $\Gamma_0(N)$ lies two cusps of $\Gamma_0(pN)$, one unramified and one with ramification index p. The injectivity of this map implies that the Eisenstein part of the p-new quotient of $H^1(\Gamma_0(pN), \mathbb{C}_p)$ has dimension 1 since $\dim(E_2(\Gamma_0(pN))) - 2\dim(E_2(\Gamma_0(N))) = 1$. This is different than when one considers the p-new kernel, as we have seen in this section.

3.6 Multiplicative rigid meromorphic cocycles

We now turn our head to the classification of multiplicative rigid meromorphic cocycles, i.e. of the abelian group $H^1(\Gamma_0^p(N), \mathcal{M}^\times)$. As is the case for $N = 1$, the main problem in classifying comes from the non-triviality of the kernel of

$$\text{dlog} \ : \ H^1_f(\Gamma_0^p(N), \mathcal{M}^\times) \to H^1(\Gamma_0^p(N), \mathcal{M}_2),$$

which is $H^1(\Gamma_0^p(N), \mathbb{C}_p^\times)$. The motivation comes from studying multiplicative lifts of elements of $H^1(\Gamma_0^p(N), \mathcal{M}_2)$ which lives a priori in $H^1(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times)$. In that regard, we then consider the $\Gamma_0^p(N)$-cohomology of the short exact sequence

$$0 \to \mathbb{C}_p^\times \to \mathcal{M}^\times \to \mathcal{M}^\times/\mathbb{C}_p^\times \to 0,$$

which gives the exact sequence

$$0 \to H^1(\Gamma_0^p(N), \mathbb{C}_p^\times) \to H^1_f(\Gamma_0^p(N), \mathcal{M}^\times) \to H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times) \xrightarrow{\delta} H^2(\Gamma_0^p(N), \mathbb{C}_p^\times).$$

Therefore, both $H^1(\Gamma_0^p(N), \mathbb{C}_p^\times)$ and $H^2(\Gamma_0^p(N), \mathbb{C}_p^\times)$ are important to understand if we want to construct cocycles that live in $H^1_f(\Gamma_0^p(N), \mathcal{M}^\times)$.

The cohomology of arithmetic groups and S-arithmetic groups has been widely studied in the last 50 years. Two fundamental articles by Borel and Serre ([4], [5]) study the cohomology of such groups Γ by considering an appropriate space $\overline{X}$ such that Γ acts freely on $\overline{X}$ where the quotient $\overline{X}/\Gamma$ is compact, and then study the homology / cohomology of the given compact space. In the S-arithmetic case, $\overline{X}$ is constructed by taking the product of some "classical space" with the Bruhat-Tits buildings associated to the non-archimedean primes of S, exactly as $\mathcal{H} \times \mathcal{T}$ seems to be the natural candidate if one wants to study $\Gamma_0^p(N)$.

As seen before, the Eichler-Shimura isomorphism tells us that $H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p)$ is isomorphic to some space of weight 2 cusp form of level N. In general, if Γ is some S-arithmetic group, $H^d(\Gamma, \mathbb{C}_p)$ is related to the space of functions on $\overline{X}/\Gamma$, where d is the dimension of $\overline{X}$. A particular case of this can be seen in [3] Section 1.2 where we see the isomorphism

$$H^d_{\text{par}}(SL_2(\mathbb{Z}[1/N]), \mathbb{C}) \cong S_2^{\text{new}}(\Gamma_0(N)) \oplus \overline{S_2^{\text{new}}(\Gamma_0(N))}.$$

Here N is square-free, $S_2^{\text{new}}(\Gamma_0(N))$ is the space of weight 2 newforms of level N and

$$d = \#\text{prime divisors of } N + 1$$

(Remark that d is indeed equal to the cardinality of $S = \{\text{prime divisors of } N, \infty\}$). This is in accordance with what is seen in [16], where $H^2_{\text{par}}(SL_2(\mathbb{Z}[1/p]), \mathbb{C}_p)$ is related to the space of weight 2 cusp forms of level p (which are all new).

Therefore, it stands to reason that $H^2_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p)$ should be related to the space of p-new cusp forms of level pN, which are the natural functions to consider on $\mathcal{H} \times \mathcal{T}$, which we will see in this section.

First of all, some care needs to be taken since contrary to the case $N = 1$, the abelianization of $\Gamma_0(N)$ is not finite. The article [45] computes the cohomology groups $H^i(\Gamma_0(N), \mathbb{Z})$, but the same arguments can be used to compute $H^i(\Gamma_0(N), K_p^\times)$.

Using notation from the aforementioned article, let $a(N)$ be the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$, $b(N)$ the number of solutions of x^2+x+1 mod N and $c(n)$ the number of solutions of x^2+1 mod N. Finally, let

$$r(N) := 1 + \frac{a(N)-4b(N)}{6} - \frac{c(N)}{2}, \quad v(N) = \frac{a(N)+2b(N)}{3} + c(N), \quad c(N) = \frac{a(N)+c(N)}{2}.$$

$r(N)$ is twice the dimension of the space of cusp forms of weight 2 and level N plus the dimension of the associated Eisenstein space. This gives us the exact sequence

$$\begin{aligned} 0 \to (\mathbb{C}_p^\times)^{r(N)} \to H^1(\Gamma_0(N), \mathbb{C}_p^\times) \to \mu_2^{v(N)-c(N)} \oplus \mu_3^{b(N)} \oplus \mu_4^{c(N)} \\ \to H^2(\Gamma_0(N), \mathbb{C}_p^\times) \to \left(\mathbb{C}_p^\times/(\mathbb{C}_p^\times)^2\right)^{v(N)-c(N)} \oplus \left(\mathbb{C}_p^\times/(\mathbb{C}_p^\times)^3\right)^{b(N)} \oplus \left(\mathbb{C}_p^\times/(\mathbb{C}_p^\times)^4\right)^{c(N)}, \end{aligned}$$

where μ_i is the group of i-th roots of unity in $\mathbb{C}_p^\times$. This gives us two short exact sequences

$$0 \to (\mathbb{C}_p^\times)^{r(N)} \to H^1(\Gamma_0(N), \mathbb{C}_p^\times) \to \text{finite group of exponent } 12 \to 0$$

$$0 \to \text{finite group of exponent } 12 \to H^2(\Gamma_0(N), \mathbb{C}_p^\times) \to \text{finite group of exponent } 12 \to 0.$$

(The proof relies on $\Gamma_0(N)$ acting on the cubic tree and then using a theorem of groups acting on trees by Serre.) We infer from these two short exact sequences that modulo 12-torsion, $H^1(\Gamma_0(N), \mathbb{C}_p^\times)$ is isomorphic to two spaces of cusp forms plus an Eisenstein space (Eichler-Shimura) and $H^2(\Gamma_0(N), \mathbb{C}_p^\times)$ is zero.

Using the fact that $\{v, \overline{v}, e\}$ is a fundamental domain for $\mathcal{T}$ under the action of $\Gamma_0^p(N)$ yields the exact sequence

$$\begin{aligned} 1 \to H^1(\Gamma_0^p(N), \mathbb{C}_p^\times) \to H^1(\Gamma_0(N), \mathbb{C}_p^\times) \oplus H^1(\Gamma_0'(N), \mathbb{C}_p^\times) \\ \xrightarrow{\Phi} H^1(\Gamma_0(pN), \mathbb{C}_p^\times) \xrightarrow{\eta} H^2(\Gamma_0^p(N), \mathbb{C}_p^\times) \to H^2(\Gamma_0(N), \mathbb{C}_p^\times) \oplus H^2(\Gamma_0'(N), \mathbb{C}_p^\times) \to ... \end{aligned}$$

So the map

$$\eta \ : \ H^1(\Gamma_0(pN), \mathbb{C}_p^\times) \to H^2(\Gamma_0^p(N), \mathbb{C}_p^\times)$$

does not have a finite exponent kernel, but the cokernel is finite of exponent 12. So if we define the lifting obstruction $\kappa_J \in H^1(\Gamma_0(pN), \mathbb{C}_p^\times)$ associated to $J \in H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times)$ to be the an element that satisfies

$$\delta(J^{12}) = \eta(\kappa_J),$$

then κ_J is well-defined up to the finitely generated group $\ker \eta$.

If we look at the exact sequence defining the space of p-new cusp forms

$$0 \to S_2^{new-p}(\Gamma_0(pN)) \xrightarrow{i} S_2(\Gamma(pN)) \xrightarrow{\varphi_s \oplus \varphi_t} S_2(\Gamma_0(N)) \oplus S_2(\Gamma_0(N)) \\ \to \operatorname{coker}(\varphi_s \oplus \varphi_t) \to 0,$$

we see a striking resemblance with the above exact sequence relating cohomology groups where the arrows are reversed. In particular, we have (heuristically for now) up to 12-torsion and Eisenstein subspace (which can be deleted by taking the parabolic cohomology)

$$\ker \eta \simeq \operatorname{coker} i = S_2^{old-p}(\Gamma_0(pN)), \quad H^1(\Gamma_0^p(N), \mathbb{C}_p^\times) \simeq \operatorname{coker}(\varphi_s \oplus \varphi_t).$$

This again is in accordance with the case $N = 1$ since in that instance $S_2^{old-p}(\Gamma_0(pN)) = \operatorname{coker}(\varphi_s \oplus \varphi_t) = 0$.

More precisely, we will see that the dual of the map Φ induced by the map d of the exact sequence

$$1 \to \mathbb{C}_p^\times \to \mathcal{F}(\mathcal{T}_0, \mathbb{C}_p^\times) \xrightarrow{d} \mathcal{F}(\mathcal{T}_1, \mathbb{C}_p^\times) \to 1$$

is exactly $\varphi_s \oplus \varphi_t$. Using the usual Eichler-Shimura isomorphism

$$H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}) \simeq S_2(\Gamma_0(N)) \oplus \overline{S_2(\Gamma_0(N))},$$

we get

$$\ker \eta \simeq \operatorname{im} \Phi \simeq \operatorname{coim} \varphi_s \oplus \varphi_t \simeq \operatorname{coker} i.$$

and

$$\operatorname{coker}(\varphi_s \oplus \varphi_t) \simeq \ker \Phi \simeq H^1_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p^\times).$$

Since η is surjective up to 12-torsion and again using Eichler-Shimura, we have

$$\mathcal{I}_{Np} := \text{Ann}_{\mathbb{T}}(H^1(\Gamma_0(pN)), \mathbb{C}_p^\times) = \text{Ann}_{\mathbb{T}}(M_2(\Gamma_0(pN), \mathbb{Z})),$$

$H^2(\Gamma_0^p(N), \mathbb{C}_p^\times)$ is still a torsion $\mathbb{T}$-module annihilated by $12 \cdot \mathcal{I}_{Np}$. We put all of this in the following theorem.

Theorem 3.6.1. *The two exact sequences*

$$1 \to H^1_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times) \to H^1_{par}(\Gamma_0(N), \mathbb{C}_p^\times) \oplus H^1_{par}(\Gamma_0'(N), \mathbb{C}_p^\times) \\ \xrightarrow{\Phi} H^1_{par}(\Gamma_0(pN), \mathbb{C}_p^\times) \xrightarrow{\eta} H^2_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times) \to 1$$

and

$$0 \to S_2^{new-p}(\Gamma_0(pN)) \otimes \mathbb{C}_p^\times \xrightarrow{\iota} S_2(\Gamma(pN)) \otimes \mathbb{C}_p^\times \xrightarrow{\varphi_s \oplus \varphi_t} (S_2(\Gamma_0(N)) \otimes \mathbb{C}_p^\times) \oplus (S_2(\Gamma_0'(N)) \otimes \mathbb{C}_p^\times) \\ \to \operatorname{coker}(\varphi_s \oplus \varphi_t) \otimes \mathbb{C}_p^\times \to 0$$

are dual up to 12-torsion. More precisely, if we denote by V' the algebraic $\mathbb{C}_p$-dual of V and every space of cusp forms is understood to be tensored with $\mathbb{C}_p$, we have the following commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & H^1_{par}(\Gamma_0^p(N),\mathbb{C}_p^\times) & \longrightarrow & H^1_{par}(\Gamma_0(N),\mathbb{C}_p^\times)\oplus H^1_{par}(\Gamma_0'(N),\mathbb{C}_p^\times) & \xrightarrow{\Phi} & H^1_{par}(\Gamma_0(pN),\mathbb{C}_p^\times) & \xrightarrow{\eta} & H^2_{par}(\Gamma_0^p(N),\mathbb{C}_p^\times) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow{\scriptstyle ES^N\oplus ES^N} & & \downarrow{\scriptstyle ES^{pN}} & & \downarrow & & \\
1 & \longrightarrow & (\mathrm{coker}(\varphi_s\oplus\varphi_t))' & \longrightarrow & (S_2(\Gamma_0(N))\oplus S_2(\Gamma_0'(N)))' & \xrightarrow{(\varphi_s\oplus\varphi_t)'} & S_2(\Gamma(pN))' & \xrightarrow{\iota'} & S_2^{new-p}(\Gamma_0(pN))' & \longrightarrow & 1,
\end{array}$$

where vertical maps are isomorphisms, and isomorphisms are up to 12*-torsion.*

Proof. First we see that the maps $ES^N\oplus ES^N$ and ES^{pN} are isomorphisms by Eichler-Shimura. The implied isomorphism between a space of cusp form and its dual is the one given by the Petersson inner product, i.e.

$$\begin{aligned} S_2(\Gamma_0(N)) &\xrightarrow{\sim} S_2(\Gamma_0(N))'\\ f &\mapsto \langle\cdot, f\rangle, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the Petersson inner product. Also, we define a bilinear form on $S_2(\Gamma_0(N))\oplus S_2(\Gamma_0'(N))$ to be

$$\langle (f_1,f_2),(g_1,g_2)\rangle = \frac{1}{p+1}(\langle f_2,g_2\rangle - \langle f_1,g_1\rangle),$$

which again gives an isomorphism between $S_2(\Gamma_0(N))\oplus S_2(\Gamma_0'(N))$ and its dual.
Now let $(\varphi_1,\varphi_2)\in H^1_{\text{par}}(\Gamma_0(N),\mathbb{C}_p^\times)\oplus H^1_{\text{par}}(\Gamma_0'(N),\mathbb{C}_p^\times)$ be associated to $(f_1,f_2)\in S_2(\Gamma_0(N))\oplus S_2(\Gamma_0'(N))$ under the Eichler-Shimura isomorphism. The corresponding element in the dual is $\frac{1}{p+1}(\langle f_2,\cdot\rangle - \langle f_1,\cdot\rangle)$, and if we apply $(\varphi_s\oplus\varphi_t)'$ to it, we get the element

$$\frac{1}{p+1}(\langle f_2,\varphi_t(\cdot)\rangle - \langle f_1,\varphi_s(\cdot)\rangle)\in S_2(\Gamma(pN))'.$$

Using basic propreties of the Petersson inner product and the fact that $(\varphi_s(f_1),\varphi_t(f_2))) = (p+1)(f_1,f_2)$, we see that the above element is equal to $\langle f_2-f_1,\cdot\rangle$.
Now clearly $\Phi((\varphi_1,\varphi_2)) = \varphi_2-\varphi_1$ which corresponds to $\langle f_2-f_1,\cdot\rangle\in S_2(\Gamma(pN))'$ under Eichler-Shimura. So in all, we see that the diagram

$$\begin{array}{ccc}
H^1_{\text{par}}(\Gamma_0(N),\mathbb{C}_p^\times)\oplus H^1_{\text{par}}(\Gamma_0'(N),\mathbb{C}_p^\times) & \xrightarrow{\Phi} & H^1_{\text{par}}(\Gamma_0(pN),\mathbb{C}_p^\times)\\
\downarrow{\scriptstyle ES^N\oplus ES^N} & & \downarrow{\scriptstyle ES^{pN}}\\
(S_2(\Gamma_0(N))\oplus S_2(\Gamma_0'(N)))' & \xrightarrow{(\varphi_s\oplus\varphi_t)'} & S_2(\Gamma(pN))'
\end{array}$$

is commutative, and even that Φ and $(\varphi_s\oplus\varphi_t)'$ are equal under Eichler-Shimura. Therefore, we can also say that $\ker\Phi\cong\ker(\varphi_s\oplus\varphi_t)'\cong(\mathrm{coker}(\varphi_s\oplus\varphi_t))'$ and $\mathrm{coker}\,\Phi\cong\mathrm{coker}((\varphi_s\oplus\varphi_t)')\cong(\ker(\varphi_s\oplus\varphi_t))'$. This finishes the proof.

□

Corollary 3.6.2. *Consider the following two exact sequences*

$$0 \to H^1_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times) \to H^1_{par}(\Gamma_0^p(N), \mathcal{M}^\times) \to H^1_{par}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times) \to H^2_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times) \tag{3.6.1}$$

and

$$0 \to H^1_{par}(\Gamma_0^p(N), \mathcal{O}^\times/\mathbb{C}_p^\times) \to H^1_{par}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times) \to H^1_{par}(\Gamma_0^p(N), \mathit{Div}^\dagger(\mathcal{H}_p)). \tag{3.6.2}$$

We have the following:
(1) $H^1_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times)$ *is isomorphic to* $\mathrm{coker}(\varphi_s \oplus \varphi_t)$.
(2) There is a natural map

$$\eta \ : \ H^1_{par}(\Gamma_0(pN), \mathbb{C}_p^\times)^{p-new} \to H^2_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times)$$

whose kernel and cokernel are finite of exponent 12.
(3) The module $H^2_{par}(\Gamma_0^p(N), \mathbb{C}_p^\times)$ *is a torsion* $\mathbb{T}$*-module annihilated by* $12 \cdot \mathcal{I}_{pN}^{p-new}$*, where*

$$\mathcal{I}_{pN}^{p-new} := \mathit{Ann}_{\mathbb{T}}(M_2(\Gamma_0(pN), \mathbb{Z})^{p-new}).$$

(4) $H^1_{par}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times)/H^1_{par}(\Gamma_0^p(N), \mathcal{O}^\times/\mathbb{C}_p^\times)$ *is torsion-free over* $\mathbb{T}$.
(5) $H^1_{par}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times)/H^1_{par}(\Gamma_0^p(N), \mathcal{O}^\times/\mathbb{C}_p^\times)$ *has countably infinite rank over* $\mathbb{Z}$.

Proof. (1), (2) and (3) are consequences of Theorem 3.6.1.
(4) comes from the exact sequence (3.6.2) and the fact that $H^1_{\text{par}}(\Gamma_0^p(N), \mathrm{Div}^\dagger(\mathcal{H}_p))$ is torsion-free over $\mathbb{T}$ by exactly the same argument as [16] Corollary 1.26.
(5) comes from the fact that $H^1_{\text{par}}(\Gamma_0^p(N), \mathrm{Div}^\dagger(\mathcal{H}_p))$ is torsion-free over $\mathcal{T}$ and of infinite rank by Section 3.4, and that $H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{O}^\times/\mathbb{C}_p^\times)$ is a torsion-module for $\mathcal{I}_{pN}$ by Theorem 3.5.6. □

Remark 3.6.3. (4) and (5) still hold true when replacing $H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{O}^\times/\mathbb{C}_p^\times)$ and $H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times)$ with $H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{O}^\times)$ and $H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{M}^\times)$ respectively by using (3.6.1) and seeing that $H^1_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p^\times)$ and $H^2_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p^\times)$ are torsion $\mathcal{I}_{pN}$-modules.

Chapter 4

Shimura-Shintani correspondance

The goal of this Chapter is now to use the characterization of rigid analytic cocycles for $\Gamma_0^p(N)$ and derive a Shimura-Shintani lift between $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$ and the space of vector-valued cusp forms for $\Gamma_0(N)$ of weight $1/2+k$, generalizing results of [35]. This is given in Theorem 4.3.8 and in the remark afterwards. The edge case $k=1$ is also dealt with in Section 4.4.

4.1 The Weil representation and vector-valued modular forms

Let (V,Q) be a non-degenerate quadratic form where V is the vector space and Q the quadratic form, and L an even lattice with dual lattice L'. Let $\mathbb{C}[L'/L]$ be the group algebra of $\mathbb{C}$-linear combinations of elements of L'/L with standard basis denoted by $\{\mathfrak{e}_g\}_{g\in L'/L}$. This comes equipped with a Hermitian scalar product where $(\mathfrak{e}_{g_1},\mathfrak{e}_{g_2}) := \delta_{g_1,g_2}$ and extended sesquilinearly to $\mathbb{C}[L'/L]$. The Weil representation of the metaplectic group $Mp_2(\mathbb{Z})$ acting on $\mathbb{C}[L'/L]$ is then defined by

$$\rho_L(T)(\mathfrak{e}_g) := e^{Q(g)}\mathfrak{e}_g, \quad \rho_L(S)(\mathfrak{e}_g) := \frac{e^{(s-r)/8}}{\sqrt{|L'/L|}} \sum_{g'\in L'/L} e^{-(g,g')}\mathfrak{e}_{g'},$$

where T and S are the standard generators of $Mp_2(\mathbb{Z})$ associated to $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ respectively and (r,s) is the signature of V.

Given any representation ρ of the metaplectic group $Mp_2(\mathbb{Z})$ and $k\in\frac{1}{2}\mathbb{Z}$, we define the space of vector-valued automorphic forms of weight k and level (Γ,ρ) (with Γ a subgroup of $Mp_2(\mathbb{Z})$) to be the vector space of functions $f : \mathbb{H}\to\mathbb{C}[L'/L]$ satisfying

$$f(\gamma\tau) = \phi(\tau)^{2k}\rho(\gamma,\phi)f(\tau),$$

for all $(\gamma, \phi) \in \Gamma$. This space is denoted $A_k(\Gamma, \rho)$. In general, unless specified otherwise, the representation ρ in question will be assumed to be the Weil representation ρ_L of a given lattice L. As usual, we denote by $M_k(\Gamma, \rho)$ and $S_k(\Gamma, \rho)$ the subspaces of modular forms and cusp forms respectively.
Given a vector-valued automorphic form $f \in A_k(\Gamma, \rho)$, we write

$$f = \sum_{g \in L'/L} f_g \mathfrak{e}_g$$

for its decomposition under the canonical basis of $\mathbb{C}[L'/L]$. Now given an arbitrary lattice L and $g \in L'/L$ we can define a *Siegel theta function*

$$\theta_g := \sum_{x \in L+g} p(x) e^{2\pi i Q(x)\tau},$$

where $p(x)$ is a function on the coset $L + g$ satisfying certain properties. For example, if p is homogeneous of weight λ and spherical for Q, i.e. its Laplacian is zero and it satisfies $Ep(x) = \lambda p(x)$ where E is the Euler operator $E := \sum x_i \partial/\partial x_i$, then θ_g is an automorphic form of weight $k = \frac{r-s}{2} + \lambda$. More generally, we can consider all these Siegel theta functions together for every coset $g + L$, i.e.

$$\theta_L := \sum_{g \in L'/L} \theta_g \mathfrak{e}_g,$$

and $\theta_L \in A_k(\Gamma, \rho_L)$. (See [44], [41], [6] and [7])

4.2 Setup and definitions

Using the terminology of theta functions, let $V = \{\lambda \in M_2(\mathbb{Q}) \ : \ tr(\lambda) = 0\}$ with quadratic form $Q(\lambda) = -N \cdot det(\lambda)$ and consider the lattice

$$L := \left\{ \begin{bmatrix} -b & -c/N \\ a & b \end{bmatrix} \ : \ a, b, c \in R \right\},$$

where R could be $\mathbb{Z}$ or $\mathbb{Z}[1/p]$. One can compute that its dual lattice is

$$L' := \left\{ \begin{bmatrix} -b/2N & -c/N \\ a & b/2N \end{bmatrix} \ : \ a, b, c \in R \right\}.$$

L is a lattice of level $4N$, discriminant $2N$, and L'/L has order $2N$. Also, V is a quadratic space of signature $(2, 1)$.

As in [35], the goal of this section will be to construct a theta kernel $\widehat{\Omega(z, \tau)}$, which will be a cusp form of weight $1/2 + k$ and level ρ_L in τ, realizing a correspondance between $S_{1/2+k,\rho}$ and $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$. In fact,

$\widehat{\Omega(z,\tau)}$ will be an element of $S_{1/2+k,\rho}$ with coefficients in $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$, i.e. an element of $\mathbb{C}[L'/L](q) \otimes MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$ where $q = e^{2\pi i\tau}$.

A binary quadratic form $Q(x,y) = Ax^2 + Bxy + Cy^2$ over a ring R is said to be a Heegner form of level N if $N|A$. We define the sets

$$\mathcal{F}^{(N)}(R) := \{[A,B,C] \ : \ Ax^2 + Bxy + Cy^2 \text{ is a primitive Heegner form over } R \text{ of level } N\},$$
$$\mathcal{F}_D^{(N)}(R) := \{[A,B,C] \ : \ [A,B,C] \in \mathcal{F}^{(N)}(R) \text{ and } Ax^2 + Bxy + Cy^2 \text{ has discriminant D}\}.$$

As above, R will either be $\mathbb{Z}$ or $\mathbb{Z}[1/p]$. Note that elements of $\mathcal{F}^{(N)}(R)$ can be seen as elements of L' under the correspondance

$$\begin{bmatrix} -b/2N & -c/N \\ a & b/2N \end{bmatrix} \longrightarrow Nax^2 + bxy + cy^2$$

Note that if we let $F := Nax^2 + bxy + cy^2$ and $\lambda := \begin{bmatrix} -b/2N & -c/N \\ a & b/2N \end{bmatrix}$, then $Q(\lambda) = \mathrm{disc}(F)/4N$.

Given a quadratic form Q and a fixed square root $\sqrt{D}$, we will let τ_1 and τ_2 be the "positive" and "negative" roots of $Q(z,1)$, i.e.

$$\tau_1 = \frac{-B + \sqrt{D}}{2A}, \quad \tau_2 = \frac{-B - \sqrt{D}}{2A},$$

and γ_Q will be the geodesic in $\mathcal{H}$ joining τ_1 and τ_2. As before, given $(r,s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, $\gamma_Q \cdot (r,s)$ will denote the intersection pairing between geodesics.

It will be assumed that $\left(\frac{D}{p}\right) = 1$ and that D is not a square and we will for now consider only the case $k \geqslant 2$, so that the relevant rigid analytic cocycle and modular form converge. In [35], the condition that k is odd is added so that the cusp form defined below is non-zero. However, it will not be important here as we will be working with vector-valued modular forms, and so, even though the sum of all the components of the vector-valued modular form will be 0 for k odd, the individual components might be non-zero.

4.3 Rigid analytic cocycle and p-new modular form for $\Gamma_0(pN)$

In the hope of generalizing the theta functions $\widehat{\Omega(z,\tau)}$ and $\overline{\Omega(z,\tau)}$ of [35], given a discriminant D and an integer k as above, we define a modular symbol

$$J_{k,D}^{(N)}\{r,s\}(z) := \sum_{Q \in \mathcal{F}_D^{(N)}(\mathbb{Z}[1/p])} \frac{(\gamma_Q \cdot (r,s))}{Q(z,1)^k}.$$

and a modular form

$$f_{k,D}^{(pN)} := \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{Q(z,1)^k},$$

where for $i = 0, ..., p$

$$\gamma_Q \cdot e_i = \begin{cases} 1 & \text{if } \tau_1 \in \mathcal{W}_{e_i} \text{ and } \tau_2 \notin \mathcal{W}_{e_i} \\ -1 & \text{if } \tau_1 \notin \mathcal{W}_{e_i} \text{ and } \tau_2 \in \mathcal{W}_{e_i} \\ 0 & \text{otherwise} \end{cases}$$

In the hope that the Schneider-Teitelbaum lift for $\Gamma_0(N)$ sends $f_{k,D}^{(pN)}$ to $J_{k,D}^{(N)}$, we will need to show that $f_{k,D}^{(pN)}$ is a p-new cusp form and that the sum defining $J_{k,D}^{(N)}$ actually converges and defines a modular symbol.

The following is Lemma 1.4 of [35] and is a generalization of the proof of Theorem 1.22 of [16].

Lemma 4.3.1. *Let $Q = [a, b, c]$ be a primitive binary quadratic form of positive discriminant $D = D_0 p^m$ where $p \nmid D_0$. For $z \in \mathcal{H}_p^{\leqslant n}$ with $n < m/2$, we have*

$$\left| \frac{1}{Q(z,1)^k} \right| < p^{2nk}.$$

Proposition 4.3.2. *For $k \geqslant 1$ and $(r, s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, let*

$$J_{k,D}^{(N)}\{r, s\}(z) := \sum_{Q \in \mathcal{F}_D^{(N)}(\mathbb{Z}[1/p])} \frac{(\gamma_Q \cdot (r, s))}{Q(z, 1)^k}.$$

Then as a function of $z \in \mathcal{H}_p$, $J_{k,D}^{(N)}\{r, s\}$ converges to a rigid meromorphic function which is analytic if $\left(\frac{D}{p}\right) = 1$. Moreover $J_{k,D}$ is a $\Gamma_0^p(N)$-invariant modular symbol with values in $\mathcal{M}_{2k}$.

Proof. By clearing the powers of p in the denominators, we can write

$$J_{k,D}^{(N)}\{r, s\}(z) = \sum_{n=0}^{\infty} \left(\sum_{Q \in \mathcal{F}_{Dp^{2n}}^{(N)}(\mathbb{Z}[1/p])} \frac{(\gamma_Q \cdot (r, s))}{Q(z, 1)^k} \right) p^{nk}.$$

Since the inner sum is over quadratic forms such that $\gamma_Q \cdot (r, s) \neq 0$, we know that this sum is finite (See for example [16] Lemma 1.21 with the fact that $\mathcal{F}_{Dp^{2n}}^{(N)} \subset \mathcal{F}_{Dp^{2n}}^{(1)}$).
Let $z \in \mathcal{H}_p^h$. Then by the last lemma, we see that the general term is bounded in absolute value by $p^{k(2h-n)}$, which tends to 0 when $n \to \infty$. Therefore the sequence of the general term is Cauchy and the sum converges to a rigid meromorphic function. In addition, when $\left(\frac{D}{p}\right) = 1$, $\sqrt{D} \notin \mathcal{H}_p$ and thus $J_{k,D}^{(N)}\{r, s\}(z)$ is a rigid analytic function since it has no poles in $\mathcal{H}_p$.

Since $\Gamma_0^p(N)$ fixes $\mathcal{F}_{Dp^{2n}}^{(N)}$, taking $\gamma \in \Gamma_0^p(N)$, we get

$$\begin{aligned} J_{k,D}^{(N)}\{r,s\}(z)|\gamma &= \sum_{Q \in \mathcal{F}_D^{(N)}(\mathbb{Z}[1/p])} \frac{(\gamma_Q \cdot (\gamma r, \gamma s))}{(Q|\gamma)(z,1)^k} \\ &= \sum_{Q \in \mathcal{F}_D^{(N)}(\mathbb{Z}[1/p])} \frac{(\gamma_{Q|\gamma^{-1}} \cdot (\gamma r, \gamma s))}{Q(z,1)^k} \\ &= \sum_{Q \in \mathcal{F}_D^{(N)}(\mathbb{Z}[1/p])} \frac{(\gamma_Q \cdot (r, s))}{Q(z,1)^k}. \end{aligned}$$

Now the modular symbol properties come from the observation that

$$\gamma_Q \cdot (r,s) = -\gamma_Q \cdot (s,r), \quad \text{and} \quad \gamma_Q \cdot (r,t) = \gamma_Q \cdot (r,s) + \gamma_Q \cdot (s,t).$$

□

Proposition 4.3.3. *The function*

$$f_{k,D}^{(pN)} := \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{Q(z,1)^k},$$

is a p-new cusp form of weight $2k$ for $\Gamma_0(pN)$.

Proof. As usual, let $\gamma_1, ..., \gamma_{p+1}$ be representatives of $\Gamma_0(N)/\Gamma_0(pN)$ and β the parity-reversing matrix. First, note that for $\gamma \in SL_2(\mathbb{Z})$, we have

$$(f_{k,D}^{(pN)}|\gamma)(z) = \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{(Q|\gamma)(z,1)^k}.$$

Recall that the coefficient $\gamma_Q \cdot e_i$ is non-zero if and only if Q has a root in the annulus $\mathcal{W}_{e_i}$ and one outside this annulus and that the matrices $\gamma_1, ..., \gamma_{p+1}$ mix these annulus. So when we consider the sum

$$\begin{aligned} \sum_{i=1}^{p+1} (f_{k,D}^{(pN)}|\gamma_i)(z) &= \sum_{i=1}^{p+1} \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{(Q|\gamma_i)(z,1)^k} \\ &= \sum_{i=1}^{p+1} \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})|\gamma_i} \frac{\gamma_{Q|\gamma_i^{-1}} \cdot e_0}{Q(z,1)^k} \\ &= \sum_{i=1}^{p+1} \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})|\gamma_i} \frac{\gamma_Q \cdot e_i}{Q(z,1)^k}, \end{aligned}$$

we see that if a quadratic form Q appears in the sum with roots in annulus $\mathcal{W}_{e_i}$ and $\mathcal{W}_{e_j}$ (with $i \neq j$), then it appears twice: once with the coefficient $\gamma_Q \cdot e_i$ and once with the coefficient $\gamma_Q \cdot e_j$. Since these coefficients

cancel out, we see that the whole sum is 0.

The same argument applies when considering the matrices $\gamma_i \cdot \beta$ instead.

Now modularity comes from the fact that $\Gamma_0(pN)$ fixes Heegner forms of level pN: if $\gamma \in \Gamma_0(pN)$, then as above

$$\begin{aligned}(f_{k,D}^{(pN)}|\gamma)(z) &= \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{(Q|\gamma)(z,1)^k} \\ &= \sum_{Q \in \mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_{Q|\gamma^{-1}} \cdot e_0}{Q(z,1)^k},\end{aligned}$$

and $\gamma_{Q|\gamma^{-1}} \cdot e_0 = \gamma_Q \cdot e_0$ for $\gamma \in \Gamma_0(pN)$. □

Now let L be the lattice defined in the last section and L' its dual. We can define the vector-valued modular form and modular symbol associated to $f_{k,D}^{(pN)}$ and $J_{k,D}^{(N)}$ respectively as follows. Let $h \in L'/L \cong \mathbb{Z}/2N\mathbb{Z}$ such that $h^2 \equiv D \pmod{4N}$. We define

$$\begin{aligned}\mathcal{F}_D^h(R) :&= \{Q = [A,B,C] \in \mathcal{F}_D^{(N)}(R) \ : \ [A,B,C] \in h \cdot L'/L\} \\ &= \{Q = [A,B,C] \in \mathcal{F}_D^{(N)}(R) \ : \ B \equiv h \pmod{4N}\}.\end{aligned}$$

These sets partition $\mathcal{F}_D^{(N)}(R)$ into $2N$ disjoint subsets, i.e.

$$\mathcal{F}_D^{(N)}(R) = \bigsqcup_h \mathcal{F}_D^h(R)$$

where the union runs over the different square roots of D modulo $4N$. Under the correspondance

$$\begin{bmatrix} -b/2N & -c/N \\ a & b/2N \end{bmatrix} \longrightarrow Nax^2 + bxy + cy^2,$$

$\mathcal{F}_D^h(R)$ correspond to the subset $L_{D,h}$ of Section 2.4 of [6].

Now, the vector-valued modular form and modular symbol can be defined in terms of $\mathcal{F}_D^h(R)$, i.e.

$$f_{k,D,h}^{(pN)} := \sum_{Q \in \mathcal{F}_D^h(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{Q(z,1)^k},$$

and

$$J_{k,D,h}^{(N)}\{r,s\}(z) := \sum_{Q \in \mathcal{F}_D^h(\mathbb{Z}[1/p])} \frac{(\gamma_Q \cdot (r,s))}{Q(z,1)^k},$$

where we have a slight abuse of notation as the first sum is related to the lattice of level $4pN$ and the second to the one of level $4N$. To circumvent this problem, let

$$L_1 := L'^{pN}(\mathbb{Z})/L^{pN}(\mathbb{Z}), \qquad L_2 := L'^{N}(\mathbb{Z}[1/p])/L^{N}(\mathbb{Z}[1/p])$$

where we have used exponents to distinguish the lattices of level $4pN$ and $4N$ respectively. Then we have a natural projection

$$p \ : \ L_1 \to L_2. \tag{4.3.1}$$

We will make use of this projection latter in this section to relate the modular form $f_{k,D,h}^{(pN)}$ with the modular symbol $J_{k,D,h}^{(N)}$.

Proposition 4.3.4. *$f_{k,D,h}^{(pN)}$ is a modular form of weight $2k$ and level $\Gamma_0(pN)$*

Proof. This is the same argument as the last part of the proof of Proposition 4.3.2 since $\mathcal{F}_D^h(\mathbb{Z})$ is fixed by $\Gamma_0(N)$. □

Since we have functions $f_{k,D,h}^{(pN)}$ for every $h \in L'/L$, a "natural" question seems to be if we can package these together to have an interesting vector-valued modular form. Unfortunately, since each $f_{k,D,h}^{(pN)}$ already satisfy modularity properties for the same subgroup $\Gamma_0(pN)$, they can trivially be packaged into a vector-valued cusp form for the trivial representation. Therefore, no fascinating results can really be derived out of this construction.

We now relate $J_{k,D,h}^{(N)}$ and $f_{k,D,h}^{(pN)}$ under the Schneider-Teitelbaum lift. In that regard, recall from Section 2.1.3 that the Eichler-Shimura isomorphism associates to a cusp form f of weight k and level N a modular symbol κ_f defined by

$$\kappa_f\{r,s\} := \int_r^s f(z)(x-z)^{k-2}dz \in MS^{\Gamma_0(pN)}(\mathcal{P}_{k-2}).$$

Similarly, if f is a cusp form of level pN and weight k, then we define

$$\overline{\kappa}_f\{r,s\} := \kappa_f\{r,s\} - (\kappa|\widetilde{I})\{r,s\} = \kappa_f\{r,s\} - \kappa_f\{-r,-s\}|\widetilde{I},$$

where

$$\widetilde{I} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This gives the odd part of the modular symbol associated to f. More specifically, $\overline{\kappa}_f\{0,\infty\}$ corresponds to the odd cycle associated to f.

Proposition 4.3.5. *With notation as above, let*

$$\kappa_{k,D,h}^{(pN)}\{r,s\}(z) := \binom{2k-2}{k-1}\frac{1}{D^{k-1}\sqrt{D}}\sum_{Q\in\mathcal{F}_D^h(\mathbb{Z})}(\gamma_Q\cdot e_0)\cdot(\gamma_Q\cdot(r,s))\cdot Q(z,1)^{k-1}$$

where the associated lattice L is the one of level $4pN$. Then $\kappa_{k,D,h}^{(pN)} \in MS^{\Gamma_0(pN)}(\mathcal{P}_{2k-2})$ and we have the relations

$$ST\left(\sum_{g\in p^{-1}(h)}\kappa_{k,D,g}^{(pN)}\right) = J_{k,D,h}^{(N)}, \quad \overline{\kappa}_{f_{k,D,h}^{(pN)}} = -2\pi i\cdot\kappa_{k,D,h}^{(pN)},$$

where p is the projection map in (5.1).

Proof. This is given in Proposition 4.11, 5.3 and Theorem 4.12 in [35], the only difference being that the results are with $N = 1$. The proofs carry on in exactly the same way since the set $\mathcal{F}_D^h(\mathbb{Z})$ is a subset of $\mathcal{F}_D^{(p)}(\mathbb{Z})$ (and similarly for $\mathbb{Z}[1/p]$). □

Now we are ready to define the kernel functions $\widehat{\Omega_k(z,\tau)}$ and $\overline{\Omega_k(z,\tau)}$ (or more precisely kernel functions $\widehat{\Omega_{k,h}(z,\tau)}$ and $\overline{\Omega_{k,h}(z,\tau)}$ with $h \in L'/L$) realizing the correpondance between $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$ and $S_{k+1/2}(\Gamma_0(pN),\rho_L)$, where ρ_L is the Weil representation associated to the lattice L in Section 4.2.

Definition 4.3.6. *Let $L_1 := L^{pN}(\mathbb{Z})$ and $L_2 := L^N(\mathbb{Z}[1/p])$ and let $g \in L_1'/L_1$ and $h \in L_2'/L_2$. Then we define the functions*

$$\overline{\Omega_{k,g}(z,\tau)} := \sum_{D>0} D^{k-1/2} f_{k,D,g}^{(pN)} q^{\frac{D}{4pN}} \in \mathbb{C}[[q]]\otimes S_{2k}(\Gamma_0(pN))$$

and

$$\widehat{\Omega_{k,h}(z,\tau)} := \sum_{D>0} D^{k-1/2} J_{k,D,h}^{(N)} q^{\frac{D}{4N}} \in \mathbb{C}[[q]]\otimes MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$$

where in both sums D ranges over discriminants satisfying $\left(\frac{D}{p}\right) = 1$, and $q = e^{2\pi i\tau}$. More generally, we define vector-valued functions

$$\overline{\Omega_k(z,\tau)} := \sum_{g\in L_1'/L_1}\overline{\Omega_{k,g}(z,\tau)}\mathfrak{e}_g \in \mathbb{C}[L'/L][[q]]\otimes S_{2k}(\Gamma_0(pN))$$

and

$$\widehat{\Omega_k(z,\tau)} := \sum_{h\in L_2'/L_2}\widehat{\Omega_{k,h}(z,\tau)}\mathfrak{e}_h \in \mathbb{C}[L'/L][[q]]\otimes MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k}).$$

Proposition 4.3.7. $\overline{\Omega_k(z,\tau)} \in S_{k+1/2}(\Gamma_0(4pN),\rho_{L_1})\otimes S_{2k}(\Gamma_0(pN))$*, i.e. $\overline{\Omega_k(z,\tau)}$ is a $\Gamma_0(pN)$- vector-valued cusp form for the Weil representation associated to L_1 with coefficients in $S_{2k}(\Gamma_0(pN))$. More precisely $\overline{\Omega_k(z,\tau)}$ is a vector-valued cusp form of level $4pN$ and weight $k+1/2$ in the variable τ, with coefficients in $S_{2k}(\Gamma_0(pN))$.*

Proof. Consider $\overline{\Omega_{k,g}(z,\tau)}$ associated to $g \in L_1'/L_1$. Using the usual correspondance

$$\begin{bmatrix} -b/2N & -c/N \\ a & b/2N \end{bmatrix} \longrightarrow Nax^2 + bxy + cy^2$$

between $g + L_1$ and $\mathcal{F}^g(\mathbb{Z})$, we can rewrite this sum as

$$\begin{aligned}
\overline{\Omega_{k,g}(z,\tau)} &= \sum_{D>0} D^{k-1/2} f_{k,D,g}^{(pN)} q^{\frac{D}{4pN}} \\
&= \sum_{D>0} D^{k-1/2} \sum_{Q \in \mathcal{F}_D^g(\mathbb{Z})} \frac{\gamma_Q \cdot e_0}{Q(z,1)^k} q^{\frac{D}{4pN}} \\
&= \pm \sum_{\lambda=[a,b,c] \in L_1+g} \frac{(b^2-4Nac)^{k-1/2}}{(Naz^2+bz+c)^k} q^{Q(\lambda)}.
\end{aligned}$$

The $\pm$ appears because when we are fixing a class $g \in L_1'/L_1$, the square root of b modulo $4pN$ is fixed and therefore $\gamma_Q \cdot e_0$ is fixed for all $Q \in L_1 + g$. If we let $p_z(\lambda) := \pm\frac{(b^2-4Nac)^{k-1/2}}{(Naz^2+bz+c)^k}$, we see that

$$\overline{\Omega_{k,g}(z,\tau)} = \sum_{\lambda \in L_1+g} p_z(\lambda) q^{Q(\lambda)},$$

where $p_z(\lambda)$ is a spherical homogenous polynomial in λ of degree $k-1$. Therefore, $\overline{\Omega_{k,g}(z,\tau)}$ is a cusp form of weight $k - 1 + 3/2 = k + 1/2$ and level $4pN = \text{level}(L)$.
Now by the theory of automorphic forms and the Weil representation laid out in Section 4.1, $\overline{\Omega_{k,g}(z,\tau)}$ can be seen as a theta function in τ for the lattice L_1 since

$$\overline{\Omega_k(z,\tau)} = \sum_{g \in L_1'/L_1} \overline{\Omega_{k,g}(z,\tau)} \mathfrak{e}_g, \quad \text{where} \quad \overline{\Omega_{k,g}(z,\tau)} = \sum_{\lambda \in L_1+g} p_z(\lambda) q^{Q(\lambda)}.$$

Thus, $\overline{\Omega_k(z,\tau)}$ is a vector-valued cusp form of level $4pN$ and weight $k + 1/2$ under the Weil representation associted to L_1. □

Remark: In [35], the lattices L and L' are a bit different, i.e. the quadratic form is $Q([a,b,c]) = \text{disc}([a,b,c])$ instead of $\text{disc}([a,b,c])/4pN$ in our case. When computing the level of the corresponding lattices, this multiplies the level of the cusp form $\overline{\Omega_{k,g}(z,\tau)}$ by p (Only the case $N = 1$ is studied in [35]).

Now we can use the Schneider-Teitelbaum lift defined earlier to obtain a similar result for rigid analytic cocycles.

Theorem 4.3.8. *$\widehat{\Omega_k(z,\tau)} \in S_{k+1/2}(\Gamma_0(4pN), \rho_{L_2}) \otimes MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$, i.e. $\widehat{\Omega_k(z,\tau)}$ is a $\Gamma_0(pN)$- vector-valued cusp form for the Weil representation associated to L_2 with coefficients in $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$.*

Proof. Let ST be the Schneider-Teitelbaum lift from $MS^{\Gamma_0(pN)}(\mathcal{P}_{2k-2})^{p-\text{new}}$ to $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$ and let ES be the Eichler-Shimura map

$$\begin{aligned} ES \ : \ S_{2k}(\Gamma_0(pN)) &\to MS^{\Gamma_0(pN)}(\mathcal{P}_{2k-2}) \\ f &\mapsto \frac{\overline{\kappa}_f}{-2\pi i}. \end{aligned}$$

Then $\iota := ES \circ ST$ is a $\mathbb{C}$-linear map and by Proposition 4.3.5, $\iota\left(\sum_{g\in p^{-1}(h)} f_{k,D,g}^{(pN)}\right) = J_{k,D,h}^{(N)}$, where p is the projection map in (5.1).
Using the $\mathbb{C}$-linear map ι we see that

$$\begin{aligned} \widehat{\Omega_{k,h}(z,\tau)} &= \sum_{D>0} D^{k-1/2} J_{k,D,h}^{(N)} q^{\frac{D}{4N}} \\ &= \sum_{D>0} \iota\left(\sum_{g\in p^{-1}(h)} D^{k-1/2} f_{k,D,g}^{(pN)}\right) q^{\frac{D}{4N}} \\ &= \sum_{g\in p^{-1}(h)} \iota\left(\overline{\Omega_g(z,\tau)}\right). \end{aligned}$$

Since $\overline{\Omega_{k,g}(z,\tau)} \in S_{k+1/2}(\Gamma_0(4pN))$ for all $g \in L_1'/L_1$, we conclude that the same is true for $\widehat{\Omega_{k,h}(z,\tau)}$ and similarly as the end of the proof of Proposition 4.3.7, $\widehat{\Omega_k(z,\tau)}$ is a vector-valued cusp form of level $4pN$ and weight $k+1/2$ in τ under the Weil repersentation associted to L_2. □

Remark. As mentioned in the beginning of this chapter and in Chapter 6 of [35], the construction of $\widehat{\Omega_k(z,\tau)}$ gives us a correspondance between $S_{k+1/2}(\Gamma_0(4pN), \rho_{L_2})$ and $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$. More precisely if we write

$$\widehat{\Omega_k(z,\tau)} = \sum_{i=1}^{r} f_i \otimes m_i,$$

where $\{f_1, ..., f_r\}$ is a basis of $S_{k+1/2}(\Gamma_0(4pN), \rho_{L_2})$ and $m_i \in MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$, then such a description of $\widehat{\Omega_k(z,\tau)}$ gives us a correspondence via

$$\begin{aligned} \mathcal{C} \ : \ S_{k+1/2}(\Gamma_0(4pN), \rho_{L_2}) &\to MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k}) \\ \sum_{i=1}^{r} a_i f_i &\mapsto \sum_{i=1}^{r} a_i m_i. \end{aligned}$$

4.4 The case $k = 1$

We conclude this chapter by proving similar results as before, but when $k = 1$, i.e. when the weight of the cusp form is 2. For the rest of the thesis, we also assume N to be square-free.

Unfortunately, the summation defining $f_{1,D}^{(pN)}$ does not converge absolutely, but we can use a well-known trick due to Hecke to define an auxiliary function $f_{1,D}^{(pN)}(z,s)$ such that

$$f_{1,D}^{(pN)}(z) := \lim_{s\to 0} f_{1,D}^{(pN)}(z,s)$$

satisfies properties similar to when $k \geqslant 2$. More precisely, let

$$f_{1,D}^{(pN)}(z,s) := \sum_{Q\in\mathcal{F}_D^{(pN)}(\mathbb{Z})} \frac{\gamma_Q\cdot e_0}{Q(z,1)}\left|\frac{1}{Q(z,1)}\right|^s,$$

and similarly for $f_{1,D,h}^{(pN)}(z,s)$. We have the following.

Proposition 4.4.1. *For every $z\in\mathcal{H}$, $f_{1,D}^{(pN)}(z,s)$ has an holomorphic continuation to $s=0$ If we put*

$$f_{1,D}^{(pN)}(z) := \lim_{s\to 0} f_{1,D}^{(pN)}(z,s),$$

then $f_{1,D}^{(pN)}(z)$ is a cusp form of weight 2 and level N.

Proof. This is Proposition 1 of [26] since N is square-free, and therefore also cube-free. □

Proposition 4.4.2. *Using notation as in Proposition 4.3.5, we have the following identities:*

$$ST\left(\sum_{g\in p^{-1}(h)}\kappa_{1,D,g}^{(pN)}\right) = J_{1,D,h}^{(N)}, \quad \overline{\kappa}_{f_{1,D,h}^{(pN)}} = -2\pi i\cdot\kappa_{1,D,h}^{(pN)}.$$

Proof. The identity

$$ST\left(\sum_{g\in p^{-1}(h)}\kappa_{1,D,g}^{(pN)}\right) = J_{1,D,h}^{(N)}$$

follows exactly the same way as when $k\geqslant 2$ since $\kappa_{1,D,g}^{(pN)}$ and $J_{1,D,h}^{(N)}$ are defined and converge when $k=1$. Now for the second identity, in the style of [35] Theorem 5.1 and [27] Theorem 5, we compute

$$\begin{aligned}
\overline{\kappa}_{f_{1,D,h}^{(pN)}}\{r,s\} &= \int_r^s \lim_{t\to 0} f_{1,D,h}^{(pN)}(z,t)dz - \int_{-r}^{-s}\lim_{t\to 0} f_{1,D,h}^{(pN)}(z,t)dz \\
&= \lim_{t\to 0}\sum_{Q\in\mathcal{F}_D^h(\mathbb{Z})}(\gamma_Q\cdot e_0)\int_r^s \frac{1}{Q(z,1)}\left|\frac{1}{Q(z,1)}\right|^t - \lim_{t\to 0}\sum_{Q\in\mathcal{F}_D^h(\mathbb{Z})}(\gamma_Q\cdot e_0)\int_{-r}^{-s}\frac{1}{Q(z,1)}\left|\frac{1}{Q(z,1)}\right|^t \\
&= \lim_{t\to 0}\sum_{Q\in\mathcal{F}_D^h(\mathbb{Z})}(\gamma_Q\cdot e_0)\left(\int_r^s \frac{1}{Q(z,1)}\left|\frac{1}{Q(z,1)}\right|^t - \int_{-r}^{-s}\frac{1}{\widetilde{Q}(z,1)}\left|\frac{1}{\widetilde{Q}(z,1)}\right|^t\right),
\end{aligned}$$

where $\widetilde{Q} = [-A, B, -C]$ given that $Q = [A, B, C]$.
When $(r, s) = (0, \infty)$, we see that $\int_r^s \frac{1}{Q(z,1)} \left|\frac{1}{Q(z,1)}\right|^t = \int_{-r}^{-s} \frac{1}{Q(z,1)} \left|\frac{1}{Q(z,1)}\right|^t$ and therefore $\overline{\kappa}_{f_{1,D,h}^{(pN)}}\{r, s\} = 0$. On the other hand, since

$$\kappa_{1,D,h}^{(pN)} r, s := \frac{1}{\sqrt{D}} \sum_{Q \in \mathcal{F}_D^h} (\gamma_Q \cdot (r, s))(\gamma_Q \cdot e_0),$$

we see that this sum is 0 since the bijection $[A, B, C] \to [-A, B, -C]$ changes the sign of $\gamma_Q \cdot (r, s)$ but not of $\gamma_Q \cdot e_0$.
We now turn to the case where $r, s \in \mathbb{Q}$. The case $(r, s) = (r, \infty)$ will be a consequence of the other two cases by the modular symbol properties.
First, let's suppose $\gamma_Q \cdot (r, s) = 0$ and let r_1 be the positive root of $Q(z, 1)$, and r_2 the negative one. Consider the closed path consisting of the geodesic between r and s and the real line segment $[r, s]$. If r_1 and r_2 are outside the real line segment $[r, s]$, then by the residue theorem, the integral of $\frac{1}{Q(z,1)} \left|\frac{1}{Q(z,1)}\right|^t$ over this closed path is 0. Therefore, the integral over the geodesic between r and s is equal to the one over the line segment $[r, s]$. Clearly, the limit as $t \to 0$ of the integral converges, and using partial fraction decomposition we obtain

$$\int_r^s \frac{1}{Q(z, 1)} = \frac{1}{\sqrt{D}} \left(\ln \left| \frac{s - r_1}{s - r_2} \right| - \ln \left| \frac{r - r_1}{r - r_2} \right| \right).$$

Similarly for $\widetilde{Q}$, we have

$$\int_{-r}^{-s} \frac{1}{\widetilde{Q}(z, 1)} = \frac{1}{\sqrt{D}} \left(\ln \left| \frac{s - r_1}{s - r_2} \right| - \ln \left| \frac{r - r_1}{r - r_2} \right| \right),$$

and so the integral over the geodesic is 0. If r_1, r_2 are both inside the segment $[r, s]$, we sligthly modify $[r, s]$ by adding two small semicircle of radius ρ in the lower half-plane around r_1 and r_2, so that the segment now constitutes of three line segments and two semicircles (we could take the semi-circles in the upper half-plane instead so that the closed curve doesn't contain r_1 nor r_2, but this means that the orientation of the semi-circles are counterclockwise). Call this segment α. By the residue theorem, the integral of $\frac{1}{Q(z,1)}$ over the closed path is equal to

$$2\pi i \left(\text{Res}_{z=r_1} \frac{1}{Q(z, 1)} + \text{Res}_{z=r_1} \frac{1}{Q(z, 1)} \right) = 0,$$

and the limit as $\rho \to 0$ of the integral over α is equal to

$$\int_{[r,s]} \frac{1}{Q(z, 1)} + \frac{2\pi i}{2} \left(\text{Res}_{z=r_1} \frac{1}{Q(z, 1)} + \text{Res}_{z=r_2} \frac{1}{Q(z, 1)} \right) = \frac{1}{\sqrt{D}} \left(\ln \left| \frac{s - r_1}{s - r_2} \right| - \ln \left| \frac{r - r_1}{r - r_2} \right| \right).$$

As before, this quantity cancels out with the integral over $\frac{1}{\widetilde{Q}(z,1)}$ and thus the contribution to $\overline{\kappa}_{f_{1,D,h}^{(pN)}}\{r, s\}$ is 0.

Now let's suppose that $\gamma_Q \cdot (r, s) \neq 0$. Without loss of generality, r_1 is inside the segment $[r, s]$, and r_2 is outside. As above, consider the path α to be the line segment $[r, s]$ with the modification of having a small

semicircle in the lower half-plane of radius ρ around r_1, and consider the closed path given by the geodesic between r and s with α. Again, we have by the residue theorem that the integral over this closed path is

$$2\pi i \text{Res}_{z=r_1} \frac{1}{Q(z,1)} = \frac{2\pi i}{\sqrt{D}}.$$

Exactly as before, we see that the integral of $\frac{1}{Q(z,1)}$ over α is given by

$$\int_{[r,s]} \frac{1}{Q(z,1)} + \frac{2\pi i}{2} \text{Res}_{z=r_1} \frac{1}{Q(z,1)} = \frac{1}{\sqrt{D}} \left(\ln \left| \frac{s-r_1}{s-r_2} \right| - \ln \left| \frac{r-r_1}{r-r_2} \right| + \pi i \right)$$

Therefore, we have

$$\int_r^s \frac{1}{Q(z,1)} = \frac{-1}{\sqrt{D}} \left(\ln \left| \frac{s-r_1}{s-r_2} \right| - \ln \left| \frac{r-r_1}{r-r_2} \right| + \pi i \right).$$

Similarly for $\frac{1}{\widetilde{Q}(z,1)}$, one can compute

$$\int_r^s \frac{1}{\widetilde{Q}(z,1)} = \frac{-1}{\sqrt{D}} \left(\ln \left| \frac{s-r_1}{s-r_2} \right| - \ln \left| \frac{r-r_1}{r-r_2} \right| - \pi i \right).$$

This means that the contribution of Q to $\overline{\kappa}_{f_{1,D,h}^{(pN)}}\{r,s\}$ is $\frac{-2\pi i}{\sqrt{D}}$ and thus

$$\begin{aligned}
\overline{\kappa}_{f_{1,D,h}^{(pN)}}\{r,s\} &= \lim_{t\to 0} \sum_{Q \in \mathcal{F}_D^h(\mathbb{Z})} (\gamma_Q \cdot e_0) \left(\int_r^s \frac{1}{Q(z,1)} \left| \frac{1}{Q(z,1)} \right|^t - \int_{-r}^{-s} \frac{1}{\widetilde{Q}(z,1)} \left| \frac{1}{\widetilde{Q}(z,1)} \right|^t \right) \\
&= \sum_{Q \in \mathcal{F}_D^h(\mathbb{Z})} (\gamma_Q \cdot e_0)(\gamma_Q \cdot (r,s)) \frac{-2\pi i}{\sqrt{D}} \\
&= -2\pi i \cdot \kappa_{1,D,h}^{(pN)}.
\end{aligned}$$

□

As in the last section, let's consider

$$\overline{\Omega_{1,g}(z,\tau)} := \sum_{D>0} \sqrt{D} \cdot f_{1,D,g}^{(pN)} q^{\frac{D}{4pN}}, \qquad \widehat{\Omega_{1,h}(z,\tau)} := \sum_{D>0} \sqrt{D} \cdot J_{1,D,h}^{(N)} q^{\frac{D}{4N}}$$

as well as their counterpart $\overline{\Omega_1(z,\tau)}$ and $\widehat{\Omega_1(z,\tau)}$. Using the last result and the fact that $f_{1,D}^{(pN)}(z,s)$ has an holomorphic continuation to $s=0$, the proofs of Proposition 4.3.7 and Theorem 4.3.8 can be rewritten verbatim. This gives us the slightly more general theorem:

Theorem 4.4.3 (Theorem 4.3.8 revisited)**.** *For every $k \geqslant 1$, we have $\widehat{\Omega_k(z,\tau)} \in S_{k+1/2}(\Gamma_0(4pN), \rho_{L_2}) \otimes MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$, i.e. $\widehat{\Omega_k(z,\tau)}$ is a $\Gamma_0(pN)$- vector-valued cusp form for the Weil representation associated to L_2 with coefficients in $MS^{\Gamma_0^p(N)}(\mathcal{O}_{2k})$.*

4.5 Proof of Theorem 3.4.1

We now return to the proof of Theorem 3.4.1 with the help of results from the last two chapters.

Recall the two exact sequences

$$0 \to \mathcal{O}^\times/\mathbb{C}_p^\times \to \mathcal{F}(\mathcal{E},\mathbb{Z}) \to \mathcal{F}(\mathcal{V},\mathbb{Z}) \to 0$$

and

$$0 \to \mathcal{O}_2 \to \mathcal{M}_2 \to \mathrm{Div}^\dagger(\mathcal{H}_p) \to 0,$$

which, when taking the $\Gamma_0^p(N)$-cohomology of the modular symbol functor, respectively give

$$0 \to MS^{\Gamma_0^p(N)}(\mathcal{O}^\times/\mathbb{C}_p^\times) \to MS^{\Gamma_0(pN)}(\mathbb{Z}) \to MS^{\Gamma_0(N)}(\mathbb{Z}) \oplus MS^{\Gamma_0(N)}(\mathbb{Z}) \to H^1(\Gamma_0^p(N), MS(\mathcal{O}^\times/\mathbb{C}_p^\times)) \to H^1(\Gamma_0^p(N), MS(\mathcal{F}(\mathcal{E},\mathbb{Z})))$$

and

$$0 \to MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \to MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \to MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p)) \to H^1(\Gamma_0^p(N), MS(\mathcal{O}_2)).$$

When taking the $dlog$ of the first exact sequence, using that $MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \simeq MS^{\Gamma_0(pN)}(\mathbb{C}_p)^{p-new}$ and that $H^1(\Gamma_0^p(N), MS(\mathcal{F}(\mathcal{E},\mathbb{Z})))$ is 12-torsion, that gives a heuristic relation between $H^1(\Gamma_0^p(N), MS(\mathcal{O}_2))$ and $(MS^{\Gamma_0(N)}(\mathbb{C}_p) \oplus MS^{\Gamma_0(N)}(\mathbb{C}_p))/\operatorname{im}(\varphi_s \oplus \varphi_t)$, where as usual we make the abuse of notation to define the natural functions

$$\varphi_s, \varphi_t \ : \ MS^{\Gamma_0(pN)}(\mathbb{C}_p) \to MS^{\Gamma_0(N)}(\mathbb{C}_p).$$

Again, Eichler-Shimura gives us a way to relate $(MS^{\Gamma_0(N)}(\mathbb{C}_p) \oplus MS^{\Gamma_0(N)}(\mathbb{C}_p))/\operatorname{im}(\varphi_s \oplus \varphi_t)$ to some space of modular forms.

Let us now restate the theorem with this new heuristic in mind. We begin with an helpful lemma.

Lemma 4.5.1. *Consider the reduction map red $:\ \mathcal{H}_p \to \mathcal{T}$ and let v_0 and e_0 be the standard vertex and standard edge of $\mathcal{T}$ respectively. Then the compositions*

$$MS^{\Gamma_0(N)}(Div(\mathit{red}^{-1}(v_0))) \xrightarrow{deg} MS^{\Gamma_0(N)}(\mathbb{C}_p) \to H^1_{par}(\Gamma_0(N), \mathbb{C}_p)$$

and

$$MS^{\Gamma_0(pN)}(Div(\mathit{red}^{-1}(e_0))) \xrightarrow{deg} MS^{\Gamma_0(pN)}(\mathbb{C}_p) \to H^1_{par}(\Gamma_0(pN), \mathbb{C}_p)$$

are surjective.

Proof. Recall that $MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p)) \simeq \sum_{\tau\in\mathcal{H}_p^{RM}\backslash\Gamma_0^p(N)} \mathbb{C}_p\ \Phi_\tau\{r,s\}$, where

$$\Phi_\tau\{r,s\} = \sum_{w\in\Sigma_\tau\{r,s\}} \delta_{r,s}(w)\cdot[w] = \sum_{Q\in\Gamma_0^p(N)Q_\tau} (\gamma_Q\cdot(r,s))r_1(Q).$$

Here Q_τ is the quadratic form having τ as its positive root, and $r_1(Q)$ is the positive root of Q. Applying $\deg_{\mathcal{A}_{v_0}}$ to these divisor-valued modular symbols and using results of the last chapter shows that the associated cusp forms are multiple of

$$f_{Q_\tau}^{(N)}(z) := \lim_{s\to 0} f_{Q_\tau}^{(N)}(z,s), \quad f_{Q_\tau}^{(N)}(z,s) := \sum_{Q\in\Gamma_0(N)\cdot Q_\tau} \frac{1}{Q(z,1)}\left|\frac{1}{Q(z,1)}\right|^s.$$

Variants of these cusp forms are studied in detail in [20] and [29]. More precisely, if we define

$$f_{N,D,D_0} := \sum_{Q\in\mathcal{F}_{DD_0}^{(N)}(\mathbb{Z})} w_{D_0}(Q)Q(z,1)^{-1},$$

where w_{D_0} is the genus character associated to D_0 and with the small abuse of notation that Hecke's convergence trick was used to make this sum converge, then f_{N,D,D_0} are linear combinations of $f_{Q_\tau}^{(N)}$. The RM-points in $\mathcal{A}_{v_0}$ are exactly those quadratic points having discriminant D with $p\nmid D$. Therefore, to prove surjectivity of $\deg_{\mathcal{A}_{v_0}}$, it is enough to show that the f_{N,D,D_0} generate the whole space of cusp forms when varying the discriminants D, D_0 with $p\nmid DD_0$. Equivalently, by Corollary 3 of [26], we have to show that for every cusp form $f\in S_2(\Gamma_0(N))$, we can find D, D_0 such that

$$L(f\otimes w_D,1)\cdot L(f\otimes w_{D_0},1)\neq 0.$$

This is a well-known result (see [34] Chapter 6 Theorem 1.1 for example). The result for $\deg_{\mathcal{U}_{e_0}}$ is similar. □

Theorem 4.5.2. *There is an exact sequence*

$$0\to MS^{\Gamma_0^p(N)}(\mathcal{O}_2)\to MS^{\Gamma_0^p(N)}(\mathcal{M}_2)\xrightarrow{res} MS^{\Gamma_0^p(N)}(\mathit{Div}^\dagger(\mathcal{H}_p))$$
$$\xrightarrow{(\deg_{\mathcal{W}_{v_0}},-\deg_{\mathcal{A}_{v_0'}})} \frac{H^1_{par}(\Gamma_0(N),\mathbb{C}_p)\oplus H^1_{par}(\Gamma_0'(N),\mathbb{C}_p)}{\mathrm{im}(\varphi_s\oplus\varphi_t)}\to 0.$$

Proof. To ease notation, we will begin to consider the rightmost module as a quotient of $MS^{\Gamma_0(N)}(\mathbb{C}_p)\oplus MS^{\Gamma_0'(N)}(\mathbb{C}_p)$ instead of $H^1_{\text{par}}(\Gamma_0(N),\mathbb{C}_p)\oplus H^1_{\text{par}}(\Gamma_0'(N),\mathbb{C}_p)$, keeping in mind that the image of $(\deg_{\mathcal{W}_{v_0}},-\deg_{\mathcal{A}_{v_0'}})$ lives in the parabolic part.

[43] constructs a "p-adic residue map" from rigid meromorphic functions to $\mathcal{F}(\mathcal{E},\mathbb{C}_p)$ that commutes with the following diagram:

$$\begin{array}{ccc} \mathcal{M}_2 & \xrightarrow{res} & \mathrm{Div}^\dagger(\mathcal{H}_p) \\ \downarrow & & \downarrow{\scriptstyle(\deg_{\mathcal{W}_v},-\deg_{\mathcal{A}_{\overline{v}}})} \\ \mathcal{F}(\mathcal{E},\mathbb{C}_p) & \longrightarrow & \mathcal{F}(\mathcal{V},\mathbb{C}_p). \end{array}$$

(See also [22] and [38] for more details)
This shows that $\mathrm{im}(res) \subseteq \ker((\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}}))$ in the statement of the theorem. In fact, if we use the isomorphism

$$MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p)) \simeq MS^{\Gamma_0(N)}(\mathrm{Div}(\mathrm{red}^{-1}(v_0)))\oplus MS^{\Gamma_0'(N)}(\mathrm{Div}(\mathrm{red}^{-1}(v_0')))\oplus MS^{\Gamma_0(pN)}(\mathrm{Div}(\mathrm{red}^{-1}(e_0))),$$

where as usual red : $\mathcal{H}_p \to \mathcal{T}$ is the reduction map, we will show that

$$\begin{aligned}\mathcal{R} := \{(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) \ : \Delta_{U_{e_0}} = \Delta_1 + \Delta_2 \text{ with } \Delta_1, \Delta_2 \in MS^{\Gamma_0(pN)}(\mathrm{Div}(\mathrm{red}^{-1}(e_0))),\\ \deg(\Delta_{A_{v_0}} + \varphi_s(\Delta_1)) = 0, \deg(\Delta_{A_{v_0'}} + \varphi_t(\Delta_2)) = 0\}\end{aligned}$$

is equal to $\mathrm{im}(res) = \ker((\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}}))$. This will actually give us a method to lift divisor-valued modular symbols into rigid meromorphic cocycles. In other words, a divisor-valued modular symbol can be lifted to a rigid meromorphic cocycle if and only if its divisor on $\mathrm{red}^{-1}(e_0)$ can be partitioned in two divisors Δ_1, Δ_2 such that the degree of $(\Delta_{A_{v_0}}, \Delta_1)$ is 0 on $\mathcal{W}_{v_0}$, and the degree of $(\Delta_{A_{v_0'}}, \Delta_2)$ is 0 on $\mathcal{W}_{v_0'}$.

First, let's prove that $\mathcal{R} = \ker((\deg_{\mathcal{W}_v}, -\deg_{\mathcal{A}_{\overline{v}}}))$. Let $(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) \in \mathcal{R}$. Then,

$$(\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}})(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) = (\deg(\Delta_{A_{v_0}} + \varphi_s(\Delta_{U_{e_0}})), -\deg(\Delta_{A_{v_0'}})).$$

Consider Δ_2 in the decomposition $\Delta_{U_{e_0}} = \Delta_1 + \Delta_2$. By definition of $\mathcal{R}$ and the fact that $\deg$ is $\Gamma_0(N)$-invariant, one has

$$\begin{aligned}\deg_{\mathcal{W}_{v_0}}(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) &= \deg(\Delta_{A_{v_0}} + \varphi_s(\Delta_{U_{e_0}}))\\ &= \deg(\Delta_{A_{v_0}} + \varphi_s(\Delta_1)) + \deg(\varphi_s(\Delta_2)) = \varphi_s(\deg(\Delta_2)),\end{aligned}$$

and

$$\begin{aligned}-\deg_{\mathcal{A}_{v_0'}}(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) &= -\deg(\Delta_{A_{v_0'}})\\ &= \varphi_t(\deg(\Delta_2)).\end{aligned}$$

This shows that $(\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}})(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) \subseteq \mathrm{im}(\varphi_s \oplus \varphi_t)$ and thus $\mathcal{R} \subseteq \ker((\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}}))$. Conversely, let $(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) \in \ker((\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}}))$. By definition, this means that there exists $f \in MS^{\Gamma_0(pN)}(\mathbb{C}_p)$ satisfying

$$\varphi_s(f) = \deg(\Delta_{A_{v_0}} + \varphi_s(\Delta_{U_{e_0}})), \quad \varphi_t(f) = -\deg(\Delta_{A_{v_0'}}).$$

Now choose Δ_2 with $\deg(\Delta_2) = f$, which exists by the previous lemma. One can see that the decomposition $\Delta_{U_{e_0}} = (\Delta_{U_{e_0}} - \Delta_2) + \Delta_2$ satisfies the condition to be in $\mathcal{R}$. So this shows that $\mathcal{R} = \ker((\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}}))$.

We now show that $\mathcal{R} \subseteq \mathrm{im}(res)$ which will conclude the exactness at $MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p))$.
Let again $(\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}}) \in \mathcal{R}$ with decomposition $\Delta_{U_{e_0}} = \Delta_1 + \Delta_2$, and consider the divisor-valued modular symbols

$$\Delta_{\text{even}} := (\Delta_{A_{v_0}}, 0, \Delta_1), \quad \Delta_{\text{odd}} := (0, \Delta_{A_{v_0'}}, \Delta_2).$$

We now construct explicit rigid meromorphic cocycles satisfying $res(J_{\text{even}}) = \Delta_{\text{even}}$ and $res(J_{\text{odd}}) = \Delta_{\text{odd}}$. Seeing $\Delta_{\text{even}}, \Delta_{\text{odd}}$ as elements of $MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p))$, the condition to be in $\mathcal{R}$ implies that

$$\deg_{\mathcal{W}_v}(\Delta_{\text{even}}) = 0, \quad \text{for all even vertices } v,$$

and similarly for Δ_{odd}. We proceed as in [15] Chapter 4 to construct rigid meromorphic cocycles with associated divisor-valued modular symbols Δ_{even} and Δ_{odd}. We will give the argument for Δ_{even} since the one for Δ_{odd} is exactly the same.
Given $\Delta_{\text{even}} \in MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p))$ and v an even vertex, let

$$\Delta_{\text{even}}^v := \Delta_{\text{even}}|_{\mathcal{W}_v},$$

i.e. Δ_{even}^v is the restriction of Δ_{even} on the affinoid $\mathcal{W}_v$. In the same vein, let

$$\Delta_{\text{even}}^{\leqslant n} := \sum_{v \in \mathcal{V}(\mathcal{T}^{\leqslant n})} \Delta_{\text{even}}^v = \Delta_{\text{even}}|_{\mathcal{H}_p^{\leqslant n}},$$

where the sum ranges over even vertices. The condition to be in $\mathcal{R}$ implies that $\deg(\Delta_{\text{even}}^v) = \deg(\Delta_{\text{even}}^{\leqslant n}) = 0$.
If we define

$$J_{\Delta_{\text{even}}^{\leqslant n}}\{r,s\}(z) := \sum_{w \in \Delta_{\text{even}}^{\leqslant n}\{r,s\}} n_{r,s}(w) \frac{1}{z-w},$$

where

$$\Delta_{\text{even}}^{\leqslant n}\{r,s\} = \sum n_{r,s}(w) \cdot [w],$$

then the same argument as in Lemma 4.4 in [15] shows that the sequence $\{J_{\Delta_{\text{even}}^{\leqslant n}}\{r,s\}(z)\}_{n \text{ even}}$ converges to a rigid meromorphic cocycle J_{even} satisfying $res(J_{\text{even}}) = \Delta_{\text{even}}$.
Doing the same with Δ_{odd} gives us a rigid meromorphic cocycle J_{odd} satisfying

$$res(J_{\text{even}} + J_{\text{odd}}) = (\Delta_{A_{v_0}}, \Delta_{A_{v_0'}}, \Delta_{U_{e_0}})$$

This completes the proof of the exactness at $MS^{\Gamma_0^p(N)}(\mathrm{Div}^\dagger(\mathcal{H}_p))$.

We are now left with the surjectivity of $(\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}})$. Clearly, it is enough to prove that

$$\deg_{\mathcal{A}_{v_0}} : MS^{\Gamma_0(N)}(\mathrm{Div}(\mathrm{red}^{-1}(v_0))) \to H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p)$$

is surjective, since then $\deg_{\mathcal{W}_{v_0}}$ will also be surjective and $-\deg_{\mathcal{A}_{v_0'}}$ by symmetry. This is exactly Lemma 4.5.1.

□

Remark 4.5.3. It is known (See [18] for example), that the exact sequence

$$0 \to S_2(\Gamma_0(pN))^{p-\text{new}} \to S_2(\Gamma_0(pN)) \to S_2(\Gamma_0(N)) \oplus S_2(\Gamma_0'(N))$$

is actually exact to the right, meaning that the module

$$\frac{H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{\text{par}}(\Gamma_0'(N), \mathbb{C}_p)}{\operatorname{im}(\varphi_s \oplus \varphi_t)}$$

is trivial. The reader might (rightly) think that we reveal this crucial information too late, as it would have greatly simplified some arguments, namely the surjectivity of $(\deg_{\mathcal{W}_{v_0}}, -\deg_{\mathcal{A}_{v_0'}})$, but it seems like the "structure" of $H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{\text{par}}(\Gamma_0'(N), \mathbb{C}_p)/\operatorname{im}(\varphi_s \oplus \varphi_t)$ is fundamental to the lifting of divisor-valued modular symbols to rigid meromorphic cocycles. Its triviality just seems like an added bonus that is not required anywhere, and the arguments given above might generalize to other kinds of rigid meromorphic cocycles where this space of obstructions is not 0. Although, it finally turns out that we have an exact sequence

$$0 \to MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \to MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \to MS^{\Gamma_0^p(N)}(\text{Div}^\dagger(\mathcal{H}_p)) \to 0,$$

i.e. that every divisor-valued modular symbol can be lifted to a rigid meromorphic cocycle.

Chapter 5

Computations of rigid meromorphic cocycles

This short chapter will use results seen in previous chapters concerning modular symbols, M-symbols, classification of rigid meromorphic cocycles and the relevant Shimura lift to explain and give an efficient way to numerically compute elements of $MS^{\Gamma_0^p(N)}(\mathcal{M}_2)$ and their values at RM-point. As seen in Chapter 3, the main obstacle to computing these objects lies in two spaces of obstructions. More precisely, recall the following two exact sequences:

$$0 \to MS^{\Gamma_0^p(N)}(\mathcal{O}_2) \to MS^{\Gamma_0^p(N)}(\mathcal{M}_2) \xrightarrow{div} MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p))$$
$$\xrightarrow{deg} (H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{\text{par}}(\Gamma_0'(N), \mathbb{C}_p))/\operatorname{im}(\varphi_s \oplus \varphi_t) \to 0$$

and

$$0 \to H^1_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p^\times) \to H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{M}^\times) \to H^1_{\text{par}}(\Gamma_0^p(N), \mathcal{M}^\times/\mathbb{C}_p^\times) \to H^2_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p).$$

In practice, even if $(H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p) \oplus H^1_{\text{par}}(\Gamma_0'(N), \mathbb{C}_p))/\operatorname{im}(\varphi_s \oplus \varphi_t)$ is trivial, it will be easier to consider divisors Δ that are of degree 0 on the standard affinoid. We also want that the image of the corresponding theta symbol in $H^2_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p)$ to be 0 so that we don't have the constant ambiguity. In general, if we have some operator T that annihilates $M_2(\Gamma_0(pN))$, then both spaces of obstruction will be annihilated and φ will lift to a genuine rigid meromorphic cocycle. Therefore, the strategy we will adopt is to find some Hecke operator T that kills $M_2(\Gamma_0(pN))$ and then compute $T(\varphi)$, in a similar way to [23]. The lifting obstruction in $H^2_{\text{par}}(\Gamma_0^p(N), \mathbb{C}_p)$ can be studied in its own right (as done in [15] for $N = 1$) since it can be seen as a point on $\text{Jac}(X_0(pN))^{p-\text{new}} \oplus \text{Jac}(X_0(pN))^{p-\text{new}}$, but we will not do that in this chapter.

References for computations of modular symbols and M-symbols can be found in [10] and [40]. More specifically, algorithms for computing rigid meromorphic cocyles in different settings can be found in:

- [16] for the case $N = 1$,
- [11] for $N = 2$ and the orthogonal group of signature (3,1),
- [23] for the quaternion algebra case.

We will follow a similar strategy as in [11] and [23] in the following chapter.

5.1 Definitions and cases $N = 1,\ 2$

Instead of working with the modular symbols $\varphi_\tau \in MS^{\Gamma_0^p(N)}(Div^\dagger(\mathcal{H}_p))$ seen in Chapter 2 and 3, we will define a slightly modified modular symbol that will depend on a discriminant D instead of an RM-point τ.

Recall from the last chapter the quadratic space point of view that we took. We take the quadratic space $V = \{\lambda \in M_2(\mathbb{Q}) \,:\, tr(\lambda) = 0\}$ with quadratic form $Q(\lambda) = -N \cdot det(\lambda)$ and consider the lattice

$$L^N(R) := \left\{ \begin{bmatrix} -b & -c/N \\ a & b \end{bmatrix} \,:\, a, b, c \in R \right\}.$$

More generally, we could also consider $V = B^{\text{tr}=0}$ where B is a quaternion algebra over $\mathbb{Q}$, which is the point of view adopted in [23] and [22].

Similarly as the modular symbol $J_{k,D}^{(N)}\{r, s\}(z)$ considered in the last chapter, we define

$$J_{D,\chi}\{r, s\}(z) := \sum_{\substack{v \in L^1(\mathbb{Z}[1/p]) \\ Q(v)=D}} \chi(a)(\gamma_Q \cdot (r, s))\Delta_v(z),$$

where χ is a Dirichlet character and $\Delta_v(z) = \frac{1}{az^2+2bz+c}$. We also define its multiplicative counterpart

$$J_{D,\chi}^\times\{r, s\}(z) := \prod_{\substack{v \in L^1(\mathbb{Z}[1/p]) \\ Q(v)=D}} \Delta_v(z)^{\chi(a)(\gamma_Q \cdot (r,s))}.$$

We will assume from now on that $N = q$ is prime and that $\chi = \chi_q$ is the Legendre symbol if q is odd, or $\chi = \chi_4$ is the odd Dirichlet character of conductor 4 if $q = 2$.

Nothing guarantees us that $J^{\times}_{D,\chi}\{r,s\} \in MS^{\Gamma_0^p(q)}(\mathcal{M}/\mathbb{C}_p^{\times})$ because the product might not converge. So the usual strategy is to choose a divisor $\Delta = \sum n_i D_i$, where the D_i's are discriminants and defining

$$J^{\times}_{\Delta,\chi}\{r,s\}(z) := \prod J^{\times}_{D_i,\chi}\{r,s\}(z)^{n_i},$$

hoping that this product converges. Specifically, if we define

$$\Delta_{D,\chi}\{r,s\} = \sum_{\substack{v\in L^1(\mathbb{Z}[1/p]) \\ Q(v)=D}} \chi(a)(\gamma_Q\cdot(r,s))((\tau_v)+(\tau_v')),$$

where τ_v and τ_v' are the roots of $az^2+2bz+c$, then we want $\Delta_{\Delta,\chi}\{r,s\} := \sum n_i\Delta_{D,\chi}\{r,s\}$ to lie in $MS^{\Gamma_0^p(q)}(Div_0^{\dagger}(\mathcal{H}_p))$, i.e. that the divisors $\Delta_{\Delta,\chi}\{r,s\}\bigcap\mathcal{A}$ are of degree 0 for every affinoid $\mathcal{A}$. It will be enough that the divisors

$$\Delta_{\Delta,\chi}\{r,s\}\bigcap\mathcal{H}_p^{\leqslant n} = \sum_i \sum_{\substack{v\in L^1(\mathbb{Z}) \\ Q(v)=D_i p^{2n}}} n_i\chi(a)(\gamma_Q\cdot(r,s))((\tau_v)+(\tau_v'))$$

are all of degree 0.

By construction, we see that the degree of $\Delta_{D,\chi}\{0,\infty\}\bigcap\mathcal{H}_p^{\leqslant n}$ is the Dp^{2n}-Fourier coefficient of

$$g(t) = \theta(t)E_1(1,\chi),$$

where $\theta(t)=\sum_{n\in\mathbb{Z}}t^{n^2}$ and $E_1(1,\chi)$ is the weight one Eisenstein series attached to the Dirichlet character χ. Therefore $g(t)$ is a modular form of weight $3/2$ and level $4q$ if q is odd, and level 4 if $q=2$. Note the strong ressemblance with the generating series considered in the last chapter where the coefficients were $J^{(N)}_{k,D}\{r,s\}$.

Computations for the case $N=1$ have been carried out in [16] using more refined divisors than $\Delta_{D,\chi}\{r,s\}$, i.e. considering divisors $\Delta_\tau\{r,s\}$ associated to a unique RM-point instead of a binary quadratic form associated to two RM-points. Since $N=1$, the only space of obstruction that is not trivially 0 is $H^2_{\text{par}}(\Gamma_0^p(N),\mathbb{C}_p)$. Therefore, the strategy employed is to restrict to the case where p is a monstruous prime, i.e. where the modular curve $X_0(p)$ modulo the action by the Atkin-Lehner involution at p, w_p, has genus 0. In this case, the operator $(1+w_p)$ annihilates $H^2_{\text{par}}(\Gamma_0^p(N),\mathbb{C}_p)$ and the modular symbol associated to the divisor $\Delta_\tau\{r,s\}$ lifts to a rigid meromorphic cocycle.

Computations for the case $N=2$ have been carried out in [11]. In this case, divisors $\Delta_{\Delta,\chi}\{0,\infty\}$ are chosen so that the corresponding Fourier coefficients cancel out and thus the associated rigid meromorphic period function converges. Fortunately, as seen in Chapter 3, rigid meromorphic period functions for $\Gamma_0^p(2)$ are determined by their value at $\{0,\infty\}$, and therefore their computations boils down to computing $J\{0,\infty\}$.

5.2 Efficient algorithm

We now give an efficient algorithm that allows us to compute rigid meromorphic period functions and their values, similar to the one given in Section 3.5 of [16]. Since we are working with modular symbols and M-symbols, we will also be using techniques seen in [10] and [40]. As before we assume for simplicity's sake that $N = q$ where q is a prime, since in this case, representatives of $SL_2(\mathbb{Z})/\Gamma_0(q)$ are easily described. The algorithm described below could be generalized to non-prime N assuming that we have a good description of the corresponding representatives.

First, we recall some important facts about modular symbols. Let

$$Id, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, ... \begin{bmatrix} 1 & 0 \\ q-1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

be a set of representatives of $SL_2(\mathbb{Z})/\Gamma_0(q)$. As seen in Chapter 3, modular symbols $m\{r, s\}$ for $\Gamma_0(q)$ are determined by their values

$$m\{0, \infty\}, m\{0, 1\}, ...m\{0, \frac{1}{q-1}\}, m\{\infty, 0\}.$$

In fact, we need even less information by using the 2 and 3-term relations, but in practice we will only take out the value $m\{\infty, 0\}$. Therefore, a modular symbol will be described using q values. This is pretty wasteful unfortunately, but works well for small values of N.

To simplify notation, let's write $J\{r, s\} := J^{\times}_{\Delta,\chi}\{r, s\}$ for some divisor Δ. As seen in the previous chapters, we have the decomposition

$$J\{r, s\} = \prod_{n \in \mathbb{N}} J^{(n)}\{r, s\},$$

where $J^{(n)}\{r, s\}$ is the product associated to elements of the lattice L of norm Dp^{2n}. Since the cardinality of this set grows exponentially in n, we will instead compute $J^{(n)}\{r, s\}$ from $J^{(n-1)}\{r, s\}$ using a reccurence relation. In this regard, we decompose each $J^{(n)}\{r, s\}$ in its Mittag-Leffler decomposition, i.e.

$$J^{(n)}\{r, s\} = \prod_{a=0}^{p-1} F_a^{(n)}\{r, s\} \times F_{\infty}^{(n)}\{r, s\},$$

where $F_a^{(n)}\{r, s\}$ is the restricted product on the affinoid $a + p\mathcal{O}_{\mathbb{C}_p}$. Applying the U_p-operator on $F_a^{(n)}\{r, s\}$ gives $F_a^{(n+1)}\{r, s\}$, and therefore, for $t = 0, ..., q-1$ we have the following recurrence relations

$$F_a^{(n+1)}\{0, \frac{1}{t}\}(z) = \prod_{l=0}^{p-1} F_l^{(n)}\{\frac{-a}{p}, \frac{1-a}{tp}\} \left(\frac{z-a}{p}\right) \pmod{\mathbb{C}_p^{\times}}$$

and

$$F_\infty^{(n+1)}\{0,\frac{1}{t}\}(z) = \prod_{l=1}^{p-1} F_l^{(n)}\{0,\frac{p}{t}\}\,(pz) \times F_\infty^{(n)}\{0,\frac{p}{t}\}(pz) \pmod{\mathbb{C}_p^\times}.$$

Before describing the algorithm, we give a small remark regarding the computation of a modular symbol evaluated at some $(r,s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, having computed its values at only $(0,1/t)$ for $t = 0, ..., q-1$. Recall from Chapter 3 that from any $r \in \mathbb{P}_1(\mathbb{Q})$, we can find a unimodular path going from r to ∞, i.e. a path

$$r_0 := r \to r_1 \to ... \to r_n := \infty,$$

where each matrix $(r_i, r_{i+1}) \in SL_2(\mathbb{Z})$. More precisely, we know that each (r_i, r_{i+1}) is $\Gamma_0(q)$-equivalent to $(0,1/t)$ for some $t = 0, ..., q-1$. Therefore, if we want to compute $J\{r,s\}$, we can do as follows:

1) Find unimodular paths $r_0 = r \to ... \to r_n = \infty$ and $s_0 = s \to ... \to s_n = \infty$.

2) Find matrices $\gamma_i, \delta_i \in \Gamma_0(q)$ such that the $(r_i, r_{i+1}) = \gamma_i(0,1/t_i)$ and $(s_i, s_{i+1}) = \delta_i(0,1/t_i)$.

3) Compute $J\{r,\infty\} = \prod_i \gamma_i J\{0,1/t_i\}$ and similarly for s.

4) Compute $J\{r,s\} = J\{r,\infty\} \times J\{s,\infty\}^{-1}$.

With this tool in hand, the general algorithm goes as follows:

1) Compute $J^{(1)}\{0,1/t\}$ for $t = 0, ..., q-1$ as well as their repective Mittag-Leffler decomposition $J^{(1)}\{0,1/t\} = \prod_{a=0}^{p} F_a^{(1)}\{0,1/t\}$ (Here F_p means F_∞). All the $F_a^{(1)}$ are finite products over some binary quadratic forms and can be easily computed. Specifically, the quadratic forms appearing in the support of $J^{(1)}\{0,1/t\}$ are those of discriminant $p\Delta$ satisfying $\gamma_Q \cdot (0,1/t) \neq 0$.

2) Apply the U_p-operator to each $F_a^{(n)}\{0,1/t\}$ using the recurrence relation given above to find $F_a^{(n+1)}\{0,1/t\}$ and therefore $J^{(n+1)}\{0,1/t\}$.

3) Compute the approximations $J^M\{0,1/t\} := \prod_{n=1}^{M} J^{(n)}\{0,1/t\}$ of $J\{0,1/t\}$ to some p-adic precision M.

4) Using the approximations above, compute $J^M\{\gamma_\tau\infty,\infty\}$ for some RM-point τ with automorph γ_τ.

In the spirit of [16], we will compute two cocycles J^{even} and J^{odd} which are defined naturally by

$$J^{\text{even}} := \prod_{\substack{n=1 \\ n \text{ even}}}^{\infty} J^{(n)}, \quad J^{\text{odd}} := \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} J^{(n)}.$$

Generally speaking, the quotient $J^{\text{even}}/J^{\text{odd}}$ seems to give algebraic values of lower height, which makes them easier to recognize. As mentioned in the aforementioned article, this operation can also kill the space of obstructions in some cases.

5.3 Numerical Experiments

We give numerous numerical experiments of values of the rigid meromorphic cocycle $J_{\Delta,\chi}^{\text{even / odd}}\{r,s\}(z)$ for varying N and varying divisors Δ. As mentioned before, in the case that the space of obstructions $H^2_{\text{par}}(\Gamma_0^p(N),\mathbb{C}_p)$ is non-trivial, instead of studying the obstruction in itself, we will act on $J_{\Delta,\chi}^{\text{even / odd}}\{r,s\}(z)$ by various Hecke operators in $\text{Ann}(S_2(pN)^{p-\text{new}})$, obtaining an actual algebraic value, which should live (conjecturally) in the compositum of the ring class field associated with Δ and the RM-point we are evaluating the cocycle at. More precisely, let

$$\mathcal{O}_\tau := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\Gamma_0(N))[1/p] \ : \ a\tau + b = c\tau^2 + d\tau \right\},$$

where $M_2(\Gamma_0(N))[1/p]$ is the set of matrices with $\mathbb{Z}[1/p]$-coefficients with lower-left entry divisible by N. This is a $\mathbb{Z}[1/p]$-order in $\mathbb{Q}(\tau)$. If we let $\text{Pic}^+(\mathcal{O}_\tau)$ to be the narrow Picard group of the order $\mathcal{O}_\tau$, then class field theory identifies $\text{Pic}^+(\mathcal{O}_\tau)$ with an abelian extension $H/\mathbb{Q}(\tau)$, i.e. there exists an isomorphism

$$\text{rec} \ : \ \text{Pic}^+(\mathcal{O}_\tau) \to \text{Gal}(H/\mathbb{Q}(\tau)).$$

We expect that the values of $J_{\Delta,\chi}\{r,s\}(z)$ at τ to live in the compositum of the ring class field $\mathcal{O}$ attached to Δ and τ.

We will also restrict our attention to the case where $\Delta_{\Delta,\chi}\{r,s\} \cap \mathcal{A}$ is of degree 0 for convenience's sake. The reader is encouraged to experiment with more values using the code in the appendix.

As is pretty common with values of these cocycles, their algebraic height tend to be pretty large, making them difficult to recognize in practice, especially when the level is large and the aforementioned Hecke operator is complicated. We will therefore concentrate our efforts to cocycles with relatively small level.

5.3.1 Level $N = 2$

We now give various examples of computations for rigid meromorphic cocycles of level 2. We will consider only cases when the space of obstruction is non-trivial, i.e. when $p \geqslant 7$. Since $N = 2$, it will be assume throughout this section that the implied character χ is the one of conductor 4 and will write Δ_D to mean Δ_{D,χ_4}.

Example 5.3.1. Let $p = 7$ and take $\Delta\{r,s\} = \Delta_5\{r,s\} - 3 \cdot \Delta_3\{r,s\}$ which can be seen to be of degree 0 on $\mathcal{H}_p^{\leqslant n}$.

First consider $\tau = \frac{\sqrt{5}-1}{2}$. Its stabilizer in $\Gamma_0^p(2)$ is generated by $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$. We computed $J_\Delta^{\text{even / odd}}[\tau]$ to a

7-adic precision of 7^{300} and found that

$$J_\Delta^{\text{odd}}[\tau] \text{ satisfies the polynomial } 4096x^8 - 3136x^6 + 192561x^4 - 3136x^2 + 4096,$$
$$\frac{J_\Delta^{\text{odd}}[\tau]}{J_\Delta^{\text{even}}[\tau]} \text{ satisfies the polynomial } 1771561x^2 + 9754137335122x + 1771561.$$

The second polynomial is in a sense "trivial", as its roots lie in $\mathbb{Q}(\sqrt{5})$, but the first one generates the abelian extension $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{-1})$, which indeed lies in the compositum of the ring class fields of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{3})$.

Now consider $\tau = \frac{\sqrt{13}-3}{2}$ which has automorph equal to $\begin{bmatrix} 33 & -10 \\ -10 & 3 \end{bmatrix}$. Up to 7-adic precision of 7^{300}, one obtains that $J_\Delta^{\text{odd}}[\tau]$ satisfies the polynomial

$$\begin{aligned} &1000000000000x^8 - 13606208000000x^7 + 189867125877184x^6 \\ &- 1627066613184320x^5 + 14343573742887121x^4 - 1627066613184320x^3 \\ &+ 189867125877184x^2 - 13606208000000x + 1000000000000. \end{aligned}$$

This polynomial generates the extension $\mathbb{Q}(\sqrt{3}, \sqrt{13}, \sqrt{-1})$ which has Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$ over $\mathbb{Q}(\sqrt{13})$. As expected, it is also a subfield of the compositum of the corresponding ring class fields.
Finally, if we consider $\tau = \sqrt{3} + 1$, we do recognize the value $\frac{J_\Delta^{\text{odd}}[\tau]}{J_\Delta^{\text{even}}[\tau]}$ which satisfies the polynomial

$$\begin{aligned} &466364540475625x^4 - 126397838458791245423000x^3 + 109239000614324977882930084440 6x^2 \\ &- 126397838458791245423000x + 466364540475625. \end{aligned}$$

Its roots lie in $\mathbb{Q}(\sqrt{3}, \sqrt{-1})$, again a subfield of the associated ring class fields.

Example 5.3.2.

Now let $p = 11$ and take $\Delta\{r, s\} = \Delta_8\{r, s\} - \Delta_2\{r, s\}$. First, consider $\tau = \frac{\sqrt{13}-3}{2}$. Unsurprisingly, the values $J_\Delta^{\text{even / odd}}[\tau]$ are not recognized to be algebraic since the space of obstruction $S_2(\Gamma_0(22))$ has dimension 2, but one can compute that $\frac{J_\Delta^{\text{odd}}[\tau]}{J_\Delta^{\text{even}}[\tau]}$ satisfies the polynomial

$$\begin{aligned} &7667408913480382710006 25x^4 - 250446581427162142625906949280390 0x^3 \\ &+ 260858826700350378214032462964919744761070 46x^2 \\ &- 250446581427162142625906949280390 0x + 766740891348038271000625. \end{aligned}$$

The narrow class groups (of level 1) of $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{13})$ are trivial, but the roots of this polynomial lie in $\mathbb{Q}(\sqrt{13}, \sqrt{-1})$, which is different than the compositum of the narrow class fields, highlighting the presence of the level in the construction of the cocycle.

Similarly, consider $\tau = \frac{\sqrt{17}-1}{4}$. The Hecke operator $T_3 + 1$ can be seen to annihilate the space $S_2(\Gamma_0(22))$, and indeed we obtain the algebraic values. The cocycle $\frac{J_\Delta^{\text{odd}}[\tau]}{J_\Delta^{\text{even}}[\tau]}$ satisfies the polynomial

$$403225x^4 - 67915299780x^3 + 7943745124510294x^2 - 67915299780x + 403225,$$

and $(T_3(J_\Delta^{\text{odd}}) \cdot J_\Delta^{\text{odd}})[\tau]$ satisfies

$$185640625x^8 - 233805000x^7 - 1570523652900x^6 - 1017298871160x^5 + 3519248614484326x^4$$
$$- 1017298871160x^3 - 1570523652900x^2 - 233805000x + 185640625.$$

The first polynomial generates $\mathbb{Q}(\sqrt{17}, \sqrt{-1})$ which is again different than the compositum of the narrow class fields for $\mathbb{Q}(\sqrt{17})$ and $\mathbb{Q}(\sqrt{2})$. The second polynomial generates the compositum of the relevant ring class fields, which is $\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-1})$.
If one takes $\tau = \sqrt{2}-1$ and applies the Hecke operator $T_{11}-1$, T_3+1, T_5-1 and T_7+2, which all annihilates the space of obstructions, we obtain that the values satisfy, respectively, the following polynomials

$$60025x^4 + 12379613820x^3 + 1773044082667286x^2 + 12379613820x + 60025$$
$$165197538025x^4 - 2348036016560x^3 + 190666828968594x^2 - 2348036016560x + 165197538025$$
$$14870583025x^4 + 987906687360x^3 + 17217426356834x^2 + 987906687360x + 14870583025$$
$$43099225x^4 + 89654502340x^3 + 57395981365206x^2 + 89654502340x + 43099225.$$

As expected, all these polynomials generate the abelian extension $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$.
Finally, if we take $\tau_1 = \frac{1+\sqrt{7}}{2}, \tau_2 = \frac{3+\sqrt{57}}{6}$ and $\tau_3 = 2 + \sqrt{6}$, which all have narrow class number equal to 2, then $\frac{J_\Delta^{\text{odd}}[\tau_1]}{J_\Delta^{\text{even}}[\tau_1]}$ and $\frac{J_\Delta^{\text{odd}}[\tau_2]}{J_\Delta^{\text{even}}[\tau_2]}$ respectively satisfy the polynomials

$$4684639261069178513025625x^4 - 193140368486475928095766851516100x^3$$
$$+ 20154482130669773207384477243546972635 74x^2$$
$$- 193140368486475928095766851516100x + 468463926106917851302 5625,$$

$$2251875390625x^4 + 184547600444983581596997500x^3 + 5318035885315993754203901525239160448966x^2$$
$$+ 184547600444983581596997500x + 2251875390625,$$

and $(T_3(J_\Delta^{\text{odd}}) \cdot J_\Delta^{\text{odd}})[\tau_3]$ satisfies

$$15675904397818708849x^8 - 3895745513100308914592x^7 + 462590615700798664378078x^6$$
$$- 37415022789430594383315968x^5 + 14462614449202217533224315 07x^4 - 37415022789430594383315968x^3$$
$$+ 462590615700798664378078x^2 - 3895745513100308914592x + 15675904397818708849.$$

These polynomials respectively generate $\mathbb{Q}(\sqrt{7}, \sqrt{-1})$, $\mathbb{Q}(\sqrt{57}, \sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1})$, which all are subfields of the associated ring class fields.

5.3.2 Level $N = 3$

Similarly to last section, we give two examples of computations of Hecke translates of values of $J_{\Delta,\chi_3}^{\text{even / odd}}$ at RM-points, where χ_3 is the Dirichlet character of conductor 3.

Example 5.3.3. Let $p = 2$. In this case, the space of obstructions is trivial, so we expect to obtain actual algebraic values without the need to act by Hecke operators.
Consider $\Delta = \Delta_{11} - \Delta_7$ which is of degree 0. If $\tau_1 = \frac{\sqrt{13}-3}{2}$, then we get that $\frac{J_\Delta^{\text{odd}}[\tau]}{J_\Delta^{\text{even}}[\tau]}$ satisfies the polynomial

$$\begin{aligned}&335544610360680730085659862739201x^4 - 304219445696330469076652656567838 76x^3\\ &+ 92915136960848313304040677184808295 0x^2 - 304219445696330469076652656567838 76x\\ &+ 335544610360680730085659862739201.\end{aligned}$$

This polynomial generates $\mathbb{Q}(\sqrt{13}, \sqrt{-3})$, which underlines the role of the level $N = 3$ in evaluating the cocycle, as it is indeed a subfield of the corresponding ring class fields, without being a subfield of the compositum of the narrow class fields.
When $\tau = \frac{\sqrt{5}-1}{2}$ or $\sqrt{2} - 1$, the values of $J_\Delta^{\text{even / odd}}$ are trivial.

Example 5.3.4. Let $p = 5$ and $\Delta = \Delta_8 - 3\Delta_2$, again of degree 0 on $\mathcal{H}_5^{\leqslant n}$. Applying the Hecke operators $T_5 - 1$, $T_2 + 1$ and T_7 (all annihilating $S_2(\Gamma_0(15))$) to the cocycle J_Δ^{odd} and evaluating it at the RM-point $\tau = \frac{\sqrt{13}-3}{2}$, one obtains respectively the polynomials

$$\begin{aligned}&1774632741745416774514669608241x^4 + 45228546083511569440331111 5964x^3\\ &+ 35788485347021064456165083115 90x^2 + 45228546083511569440331111 5964x\\ &+ 1774632741745416774514669608241\end{aligned}$$

$$211188959809x^4 + 330137340476x^3 + 409653959430x^2 + 330137340476x + 211188959809$$

$$\begin{aligned}&10195085599224205301267894360067854198022529x^4 - 13782713964144202666698338089687448137117444x^3\\ &+ 18262667655621929254990790231541091842349830x^2 - 13782713964144202666698338089687448137117444x\\ &+ 10195085599224205301267894360067854198022529.\end{aligned}$$

The first polynomial generates $\mathbb{Q}(\sqrt{-3}, \sqrt{13})$, whereas the last two polynomials generate $\mathbb{Q}(\sqrt{-3}, \sqrt{26})$.

Remark 5.3.5. We should notice that in all the above examples, the polynomials are reciprocal and that their leading coefficient is highly factorizable by small primes. The fact that these polynomials are symmetric is explained for $N = 1$ in Chapter 4 of [16]: the RM-values of rigid meromorphic cocycles are conjectured to

satisfy some kind Shimura reciprocity law which restricts their behaviour under complex conjugation. This leads to the fact that these values should have norm 1, and thus their minimal polynomial is reciprocal. A similar kind of Shimura reciprocity law should also hold for any level N, but the statement is a bit trickier to write down as the values depend on a Hecke operator annihilating the space of obstructions, and the cocycles constructed use divisors related to more than a single RM-point.

Similarly, the high divisibility of the leading coefficients comes from a conjectural *Gross-Zagier* style factorization, akin to the differences of singular moduli in the classical case. Particularly, for $N = 1$, if τ_1, τ_2 are two RM-points, we can consider $J_{\tau_1}[\tau_2]$ which takes the place of the differences of singular moduli in the p-adic case. A similar kind of factorization is to be expected of $J_{\tau_1}[\tau_2]$ as for the classical case (See [16] Section 4.4 for more details). Again, for general levels, we encounter the same roadblocks if we try to write down some precise conjecture: the values constructed in the last examples use more general divisors than a single RM-point τ and their factorization depend heavily on a choice of Hecke operator.

5.4 Future Research

A natural way to generalize results of this thesis would be to give a complete characterization of rigid analytic / meromorphic cocycles for arbitrary orthogonal groups (As defined in [11]) or for other kinds of field than real quadratic ones (As partly done in [22]). Results in that direction could give us a systematic way of computing and obtaining values lying in certain ring class fields, giving us a wider window into Hilbert's 12th problem.

Regarding computations related to explicit class field theory, we have been incredibly specific regarding those in this thesis, only computing values of $J^{\times}_{\Delta,\chi}\{r,s\}(z)$ for Δ of degree 0 and when $N = q$ is a prime. Naturally, one would want to get results for more general divisors as well as more general levels, possibly leading to interesting conjectures associated to p-adic singular moduli and Gross-Zagier style factorization *à la* [16] Chapter 4. Specifically, it would be natural (but probably tricky) to construct a cocycle $J^{(N)}_{\tau}$ of level N associated to an RM-point τ, using the triviality of the module $(H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}_p))^{\oplus 2}/\operatorname{im}(\varphi_s \oplus \varphi_t)$, instead of taking divisors Δ of degree 0. One could also study the effect of the Hecke operator used to annihilate the space of obstructions on the leading coefficient of the polynomial / the algebraic value, which could also lead to some new interesting conjectures. We should add that a common problem that appears in a lot of different contexts is that the heights of the algebraic values are usually really high, which leads to the unrecognizability of many results. General techniques that seem to lower this height would definitely be of use.

A more thorough study of the Eisenstein cocycles of Section 3.5.3 in the multiplicative case and understanding the "trivializing lattice" of combinations of them, as is done for Dedekind-Rademacher cocycle for Γ^p, would also be a natural way to study these objects.

Finally, the theory of rational period functions and transfer operators seems to have deep ties with rigid meromorphic functions as explained in Section 3.3 and more precisely in the comments after Proposition 3.3.3. We have unfortunately barely touched on the apparent links between them, and so a deeper study between these different perspectives could give us a more concrete way to manipulate these objects.

Appendix A

Code

```
#!/usr/bin/env python
# coding: utf-8

# In[1]:


Prec = 600 #p-adic precision
p = 2 #prime number
Level=3
Z = Integers(p^(Prec))
R.<var1> = Z[]
Zi.<i> = Z.extension((var1)^2 +1)
Kp.<s> = Qq(p^2, prec=Prec, print_mode='terse')
Rp.<y> = PowerSeriesRing(Z, default_prec=Prec)
Rpi.<varcomp> = PowerSeriesRing(Zi, default_prec=Prec)
Padic.<t> = PolynomialRing(Kp)
Padic2.<t2>= PowerSeriesRing(Kp, default_prec=Prec)
T = Qp(p, prec = Prec, type = 'capped-rel' , print_mode = 'series')
S.<u> = PolynomialRing(T)
R.<varreal> = RealField(1000)[]
R1 = RealField(1000)
Comp.<varcomp> = ComplexField(1000)[]

#returns the square root of x mod p^Prec if it exists
def Sqrt(y):
    r = sqrt(Mod(y,p))
    r = Mod(lift(r), p^Prec)
```

```
    for j in range(Prec):
        r = Z(r)/Z(2) + Z(y)/Z(2*r)
    return r

#Switches the columns of a 2x2 matrix
def SwitchMat(A):
    a = A[0][0]
    b = A[0][1]
    c = A[1][0]
    d = A[1][1]
    return Matrix([[b,a],[d,c]])

#returns the inverse of a complex number with mod p^Prec-coefficients
def InverseCompl(tau):
    real = tau[0]
    imag = tau[1]
    res = (real - i*imag)/(real^2 + imag^2)
    return res

#returns the inverse of a power series in the same Tate algebra
def Inverse(G):
    a = Rp(G(y = 0))
    b = Rp((G - a) * a^(-1))
    res = Rp(0)
    for j in range(Prec):
        res += (-b)^j
    res *= a^(-1)
    return res

#returns the cocycle J^n
def CocyclePower(J, n):
    Jfin = []
    for j in range(p+1):
        Jfin += [Rp(1)]
    for j in range(p+1):
        Jfin[j] = J[j]^n
    return Jfin

#multiply two lists of same length term by term
def MultiplyList(F,G):
    H = []
```

```
    for j in range(len(F)):
        H.append(F[j]*G[j])
    return H

def InverseList(F):
    return [F[1],F[0]]

#concatenate a list with itself n times
def Concat(F,n):
    G=[]
    for j in range(n):
        G+=F
    return G

#gives the ''linear combination'' of two lists, where the first entry is the
#''positive part'' and the second entry is the ''negative one''
def LinComb(F,n,G,m,Level):
    fin = []
    for q in range(0,Level):
        n1 = n
        m1 = m
        if (F[q][2]*n1 + G[q][2]*m1 == 0):
            print('Linear combination gives 0')
        else:
            print('Linear combination does not give 0')
        if (n<0):
            F[q] = InverseList(F)[q]
            n1 = -n1
        if (m<0):
            G[q] = InverseList(G[q])
            m1 = -m1
        fin += [[Concat(F[q][0],n1)+Concat(G[q][0],m1), Concat(F[q][1],n1)+Concat(G[q][1],m1)]]
    return fin


#divide two lists of same length term by term
def DivideList(F,G):
    H = []
    Ginv = []
    for j in range(p+1):
        Ginv += [Rp(1)]
```

```
    for j in range(p+1):
        Ginv[j] = Inverse(G[j])
    for j in range(len(F)):
        H.append(F[j]*Ginv[j])
    return H

#divide a cocycle with p+1 entries by a common constant
def DivideByConstant(J,tau):
    H=[]
    for j in range(p+1):
        H.append(J[j]/tau)
    return H


#Normalises the cocycle so that the constant term is 1.
def NormaliseCocycle(J):
    for j in range(p+1):
        J[j] /= J[j][0]
    return J

#returns the inverse of the cocycle J
def InverseCocycle(J):
    Jinv = []
    for j in range(p+1):
        Jinv += [Rp(1)]
    for j in range(p+1):
        Jinv[j] = Inverse(J[j])
    return Jinv

#returns the root mod p of a degree 2 polynomial f
def Rootmodp (f):
    root = p
    for j in range(p):
        if mod(f[0]*j^2 + f[1]*j + f[2],p) == 0:
            root = j
    return root

#gives a list of all quadratic forms (in matrix form) of discriminant D intersecting (0, 1/t) for
#this list is split into two depending if the intersection is positive or negative
def LLnew(D,N):
    resglob = []
```

```
    sD = floor(sqrt(D))
    for q in range(0,N):
        resp = []
        resm = []
        for b in range (-sD-500, sD+500):
            ac = D - b^2
            divs = [-x for x in divisors(ac)]+ divisors(ac)
            for a in divs:
                c = -ac/a
                if ((a % p == 0) and (b%p ==0) and (c%p ==0)):
                    mm = []
                else:
                    r1 = (-b + sqrt(D))/(a)
                    r2 = (-b - sqrt(D))/(a)
                    if (q==0):
                        if((r1>0) and (r2<0)):
                            mm = [-b, -c, a, b]
                            mmi = [b,c,-a, -b]
                        else:
                            mm = []
                    else:
                        if(((r2 < 0) or (r2 > 1/q)) and (0<r1<(1/q))):
                            mm = [-b, -c, a, b]
                            mmi = [b,c,-a, -b]
                        else:
                            mm = []
                if((kronecker(a,N) == 1) and (mm != [])):
                    resp += [mm]
                if ((kronecker(a,N)==-1) and (mm != [])):
                    resm += [mmi]
        resglob += [[resp, resm, len(resp) - len(resm)]]
    return resglob


#returns a list of matrices in Sl2(Z) that sends the fraction a/c to 0
def Bezout_Factonew(a,c):
    ListeFract = [(a,c)]
    Liste_Mat = []
    while(a!= 0 and c != 0):
        d,u,v = xgcd(a,c)
        a = -v
```

```
        c = u
        ListeFract.append((a,c))
    if (c == 0):
        if (a == 1):
            ListeFract.append((0,1))
        if (a == -1):
            ListeFract.append((0,-1))
    for j in range(len(ListeFract)-1):
        Matrice = Matrix([[ListeFract[j][0],ListeFract[j+1][0]],[ListeFract[j][1],ListeFract[j+1]
])
        Liste_Mat.append(Matrice)
    return Liste_Mat

#returns the rigid period function of level 0 associated to the discriminant D
def Level0(D0):
    Num = 1
    Den = 1
    for f in D0[0]:
        a = ZZ(f[0])
        b = ZZ(f[1])
        c = ZZ(f[2])
        d = ZZ(f[3])
        Num *= c*(varreal)^2 + (d-a)*(varreal) - b
    for f in D0[1]:
        a = ZZ(f[0])
        b = ZZ(f[1])
        c = ZZ(f[2])
        d = ZZ(f[3])
        Den *= c*(varreal)^2+ (d-a)*(varreal) - b

    return [Num, Den]


#returns |Level| vectors of p+1 power series corresponding to the rigid meromorphic cocycle of di
#they correspond to m{0,1/t} for t between 0 and Level-1.
def Level1(Div,Level):
    vecfin = []
    for q in range (0,Level):
        D = Div[q]
        vec = []
        for j in range(p+1):
```

```
            vec += [Rp(1)]
        for f in D[0]:
            G = Fct(f)
            vec[G[1]] *= G[0]
        for f in D[1]:
            G = Fct(f)
            vec[G[1]] *= Inverse(G[0])
        for j in range(p+1):
            vec[j] /= vec[j][0]
        vecfin += [vec]

    return vecfin

#returns the rational function attached to a binary quadratic form with variable in the correspon
def Fct(F):
    a = ZZ(F[0])
    b = ZZ(F[1])
    c = ZZ(F[2])
    d = ZZ(F[3])
    j = Rootmodp([c,d-a,-b])
    if(j == p):
        res = Rp(Rp(c)*y^2 + Rp(d-a)*y - Rp(b))
    else:
        res = Rp((Rp(c)*(Rp(j) + 1/y)^2 + Rp(d-a)*(Rp(j) + 1/y) - Rp(b) )*y^2)
    return [res,j]

#returns the affinoid corresponding to A*j where a is a 2x2 matrix and j in an affinoid.
def NewAffinoid(A,j):
    stop = 0
    a = A[0][0]
    b = A[0][1]
    c = A[1][0]
    d = A[1][1]
    if(j == p):
        Num = Mod(a,p)
        Den = Mod(c,p)
    else:
        Num= Mod(a*j+b, p)
        Den = Mod(c*j+d, p)
    if(Num == Mod(0,p) and Den == Mod(0,p)):
        stop = 1
```

```
        jj=1
    if(stop != 1):
        if(Den == Mod(0,p)):
            jj = p
        else:
            jj = lift(Num/Den)
    return jj

#applies the matrix A^(-1) to a power series in a given affinoid and changes variables depending
def Apply(A,H):
    stop = 0
    a = A[0][0]
    b = A[0][1]
    c = A[1][0]
    d = A[1][1]
    G = H[0]
    j = H[1]
    if(j == p):
        Num = Mod(a,p)
        Den = Mod(c,p)
    else:
        Num = Mod(a*j+b, p)
        Den = Mod(c*j+d, p)
    if(Num == Mod(0,p) and Den == Mod(0,p)):
        stop = 1
    a = Rp(a)
    b = Rp(b)
    c = Rp(c)
    d = Rp(d)
    if(stop == 1):
        F = Rp(1)
        jj = 0
    if(stop != 1):
        if(Den == Mod(0,p)):
            jj = p
        else:
            jj = lift(Num/Den)
        if(j == p and jj == p):
            substx = Rp((d*y-b)/Rp(a) * sum(Rp((c/a)^k)*y^k for k in range(0,Prec+1)))
            F = G(y = substx)
        elif(j != p and jj == p):
```

```
            r = Rp((c*j + d)/(a*j + b))
            substx = Rp(-(a-c*y)*Rp((a*j+b)^(-1)) * sum(r^k*y^k for k in range(0,Prec+1)))
            F = G(y = substx)

        elif(j == p and jj != p):
            r = Rp((-c*jj +a)/c)
            substx = Rp(-(d + (d*jj-b)*y)/Rp(c) * sum(r^k*y^k for k in range(0,Prec+1)))
            F = G(y = substx)

        elif(j != p and jj != p):
            r =  Rp(-(jj - (a*j+b)/(c*j+d)))
            substx = (-c+ (a-c*jj)*y)*Rp((c*j+d)^(-1)) * sum(r^k*y^k for k in range(0,Prec+1))
            F = G(y = substx)
    return [F,jj]

#Gives the vector [F_(A*a){0, infty}(A^(-1)*z)] in their respective variable t_(A*a)
def RMCApply(A,F):
    vec =[]
    for j in range(p+1):
        vec += [Rp(1)]
    for j in range(p+1):
        G = Apply(A,[F[j],j])
        jj = G[1]
        vec[jj] *= G[0]
    return vec

# Gives the value of J(a/b,\infty)
def TranslateRMCnew(Frac, F, Finv,Level):
    vec =[]
    for j in range(p+1):
        vec += [Rp(1)]
    Matrix = Bezout_Factonew(Frac[0],Frac[1])
    for Mat in Matrix:
        a = Mat[0][0]
        b = Mat[0][1]
        c =  Mat[1][0]
        d = Mat[1][1]
        if (Mod(c,Level)==0):
            q = 0
            vec = MultiplyList(RMCApply(Mat, Finv), vec)
        else:
```

```
            q = lift(Mod(c.inverse_mod(Level) * d,Level))
            if (q==0):
                Mat = SwitchMat(Mat)
                vec = MultiplyList(RMCApply(Mat, F[q]), vec)
            else:
                MatApp = [[a,b-(q*a)],[c,d -(q*c)]]
                MatApp = SwitchMat(MatApp)
                vec = MultiplyList(RMCApply(MatApp, F[q]), vec)

    vec = MultiplyList(vec, F[0])
    return vec

#returns the Level k+1 constribution of the RMC given Level k
def NextGen(J,Level):
    fin = []
    Jinv =[]
    for j in range(p+1):
        Jinv += [Rp(1)]
    for j in range(p+1):
        Jinv[j] = Inverse(J[0][j])
    for q in range(Level):
        if (q==0):
            fin += [Next0(J,Jinv,Level)]
        else:
            fin += [Nextnew(J,Jinv,Level,q)]
    return fin

#returns the Level k+1 of J^k(0,infty)
def Next0(J,Jinv,Level):
    vec =[]
    for j in range(p+1):
        vec += [Rp(1)]
    for a in range(p):
        Fa = TranslateRMCnew([-a,-p], J, Jinv,Level)
        Fa = RMCApply([[p,-a],[0,1]], Fa)
        vec = MultiplyList(Fa, vec)

    vec = MultiplyList(vec, RMCApply(([[1,0],[0,p]]), J[0]))
    for j in range(p+1):
        vec[j] /= vec[j][0]
    return vec
```

```
#returns the Level k+1 of J^k(0,1/q)
def Nextnew(J,Jinv,Level,q):
    vec = []
    for j in range(p+1):
        vec += [Rp(1)]
    for a in range(p):
        c = ((1+a*q)/(q*p)).numerator()
        d = ((1+a*q)/(q*p)).denominator()
        FaL = TranslateRMCnew([-a,-p], J, Jinv,Level)
        FaR = TranslateRMCnew([c,d], J, Jinv,Level)
        Fa = MultiplyList(FaL,InverseCocycle(FaR))
        Fa = RMCApply([[p,-a],[0,1]], Fa)
        vec = MultiplyList(Fa, vec)

    c = (p/q).numerator()
    d = (p/q).denominator()
    FinfL = TranslateRMCnew([0,1], J, Jinv,Level)
    FinfR = TranslateRMCnew([c,d], J, Jinv,Level)
    Finf = MultiplyList(FinfL,InverseCocycle(FinfR))
    Finf = RMCApply([[1,0],[0,p]], Finf)
    vec = MultiplyList(vec, Finf)
    for j in range(p+1):
        vec[j] /= vec[j][0]
    return vec

#multiplies all the Level k contribution of the RMC up to the Prec
#the list if split in two: the even contribution and the odd one
def LevelMnew(D,Level):
    J=Level1(D,Level)
    Sodd = J
    Seven = []
    S = []
    for j in range(p+1):
        S += [Rp(1)]
    for j in range(Level):
        Seven += [S]
    i = 0
    for j in range(Prec):
        J = NextGen(J,Level)
```

```
        for q in range(Level):
            if (Mod(i,2)==0):
                Seven[q] = MultiplyList(J[q],Seven[q])
            else:
                Sodd[q] = MultiplyList(J[q],Sodd[q])
        i = i+1
        print("Level ", j, " done")

    return [Seven,Sodd]

#returns G(A*z) where G is of level 0.
def Apply0(A,G):
    return G((A[0][0]*(varreal) + A[0][1])/(A[1][0]*(varreal) + A[1][1]))

#same as TranslateRMC but for Level 0
def TranslateRMC0(Frac, J):
    JFin = 1
    Matrix = Bezout_Facto(Frac[0],Frac[1])
    for Mat in Matrix:
        JFin /= Apply0(InverseMat(Mat),J)
    JFin *= J
    return JFin

#same as MSEval but for Level 0
def EvalLevel0(J , A):
    J = J[0]/J[1]
    tau = FixedPointnew(A)
    f = R(algdepQp2(tau,2)[0]).roots()[0][0]
    Rig = TranslateRMC0([A[0][0],A[1][0]], J)
    return (Rig)(f)

#same as Eval but for Qp2
def Evalnew(J, tau):
    res = Kp(1)
    for j in range(p+1):
        if(j < p):
            comp = 1/(tau-Kp(j))
        if(j==p):
            comp = tau
        Fval = Padic2(J[j])(t2 = comp)
        res *= Fval
```

```
    return res

#returns the value J[tau] where J is a RMC and tau is the p-adic fixed point of the matrix A
def MSEval(J , A, Level):
    tau = FixedPointnew(A)
    print(tau)
    Jinv =[]
    for j in range(p+1):
        Jinv += [Rp(1)]
    for j in range(p+1):
        Jinv[j] = Inverse(J[0][j])
    print(A[0][0],A[1][0])
    Rig = TranslateRMCnew([ZZ(A[0][0]),ZZ(A[1][0])], J, Jinv,Level)

    res = Evalnew(Rig, tau)
    return res

#returns the value J[tau] where J is a RMC and tau has A as its automorph
def MSEvalp2(J , A, Level, tau):
    Jinv =[]
    for j in range(p+1):
        Jinv += [Rp(1)]
    for j in range(p+1):
        Jinv[j] = Inverse(J[0][j])
    print(A[0][0],A[1][0])
    Rig = TranslateRMCnew([ZZ(A[0][0]),ZZ(A[1][0])], J, Jinv,Level)

    res = Evalnew(Rig, tau)
    return res

#Mobius transformation applied to tau
def Mobius(A, tau):
    res = (A[0][0]*tau + A[0][1])/(A[1][0]*tau + A[1][1])
    return res

#returns the inverse of a matrix in Sl2(Z)
def InverseMat(A):
    a = A[0][0]
    b = A[0][1]
    c = A[1][0]
    d = A[1][1]
```

```
    return [[d, -b],[-c,a]]

#returns the p-adic fixed point of A in Qp2
def FixedPointnew(A):
    a = A[0][0]
    b = A[0][1]
    c = A[1][0]
    d = A[1][1]
    ad = a-d
    if ( (ad % p == 0) and (b % p == 0) and (c%p == 0)  ):
        ad = ad/p
        b = b/p
        c = c/p
    Delta = (ad)^2 +4*b*c
    if (Mod(Delta,p).is_square()):
        print("Root not in Hp")
    I = (t^2+1).roots()[0][0]
    delta = Kp(Sqrt(-Delta))*I
    res = (Kp(ad) + delta)/Kp(2*c)
    return res

#evaluated a polynomial f at tau
def Evaluate(f, tau):
    res = 0
    print(f.coefficients())
    for j in f.coefficients():
        res += ZZ(j[0])*tau^(ZZ(j[1]))
    return res

#evaluated a polynomial f at tau
def Evaluatenew(f, tau):
    res = 0
    l =0
    for j in f.coefficients():
        res += ZZ(j)*tau^l
        l+=1
    return res

def coeffs(a):
    coeffs = [0,0]
    p = a.parent().prime()
```

```
    for i in range(a.precision_absolute()):
        if a.expansion()[i] == []:
            pass
        else:

            coeffs[0] += a.expansion()[i][0]*(p**(i+a.ordp()))
            if len(a.expansion()[i])==1:
                pass
            else:
                coeffs[1] += a.expansion()[i][1]*(p**(i+a.ordp()))

    return coeffs

#returns the height of a polynomial f
def PolyHeight(f):
    h = 0
    for coeff in f.coefficients():
        if abs(coeff)> h:
            h = abs(coeff)
    return h

#divides every coefficient of f by the common gcd.
def ReducePoly(f):
    vec = []
    for j in f.coefficients():
        vec.append(j[0])
    g = gcd(vec)
    if (g != 0):
        return f/gcd(vec)
    else:
        return f

#finds a polynomial relation in Qp2 satisfied by x
def algdepQp2(a,deg):

    ZZ = Integers()
    QQ = Rationals()
    RR = RealField(500)
    PolZ.<x> = PolynomialRing(ZZ)

    M = matrix(QQ,deg+3)
```

```
    p = a.parent().prime()
    m = a.precision_absolute()
    N = p**(floor(5*m/7))
    for j in range(0,deg+1):

        M[j,j] = 1
        M[j,deg+1] = coeffs(a^j)[0]
        M[j,deg+2] = coeffs(a^j)[1]
    M[deg+1,deg+1] = N
    M[deg+2,deg+2] = N
    Y = M.LLL()
    X  = Y[1,deg+2]*Y[2]-Y[2,deg+2]*Y[1]
    P1 = X[deg+1]-X[0]
    for j in range(1,deg+1):
        P1 -= floor(X[j])*x**j
    return (ReducePoly(P1), list(P1.factor()))

#same as algdepQp2 but with a slightly different algorithm. It is sometimes a bit more precise.
def algdepQp2new(a,deg):

    ZZ = Integers()
    QQ = Rationals()
    RR = RealField(500)
    PolZ.<x> = PolynomialRing(ZZ)

    p = a.parent().prime()
    m = a.precision_absolute()
    N = p**(floor(5*m/7))

    M = matrix(QQ,deg+3)
    for j in range(0,deg+1):
        M[j,j] = 1
        M[j,deg+1] = coeffs((a)^(j))[0]
        M[j,deg+2] = coeffs((a)^(j))[1]
    M[deg+1,deg+1] = N
    M[deg+2,deg+2] = N
    Y = M.LLL()

    P1 = floor(Y[1][deg+1] - Y[1][0])
    for j in range(2,deg+2):
        P1 = P1 - floor(Y[1][j-1])*x^(j-1)
```

```
    Fac1 = list(P1.factor())
    P1   = Fac1[len(Fac1)-1][0]

    return (P1,Fac1)

#finds a polynomial relation in Qp2 satisfied by tau * (eps_tau)^n
def algdepeps(D, tau, deg, depth=120):
    K.<var2> = QuadraticField(D)
    UK = UnitGroup(K)
    feps = (UK.fundamental_units()[0])
    eps = Padic(feps.minpoly()).roots()[0][0]
    zeta = Padic(cyclotomic_polynomial(p^2-1)).roots()[0][0]


    m = tau.precision_absolute()
    PolZ.<x> = PolynomialRing(ZZ)
    CurrDeg = deg
    CurrP = 10^200 * x^deg
    CurrHt = 10^200
    Currk = -depth
    Currl = 0
    for k in range(-depth, depth):
        for l in range(0,p^2):
            a = tau * zeta^l * eps^k
            P1 = algdepQp2new(a,deg)[0]
            if Evaluatenew(P1, a).valuation(p) > floor(5*m/7):
                if ((P1.degree() < CurrDeg) or (PolyHeight(P1) < CurrHt)):
                    CurrDeg = P1.degree()
                    CurrP = P1
                    CurrHt = PolyHeight(P1)
                    Currk = k
                    Currl = l
    a = tau * zeta^Currl * eps^Currk
    return (CurrDeg, CurrHt,Currl, Currk,a,CurrP)

#evaluates the cocycle J (level 0 and 1+) at some tau related to the discriminant D2
def EvalLevel0and1(D1, n1, D2, n2):
    J0 = Level0(D1,n1)
    Jp = LevelM(D1,n1)
    A,tau = Stabilizer(D2, n2)
    S0 = EvalLevel0(J0,D2,n2)
```

```
    res0 = Padic(algdep(S0,2)).roots()[0][0]
    resp = MSEval2(Jp, A)
    print(res0,resp)
    return res0*resp


#returns T_l(J) where T_l is the Hecke operator associated to l
def HeckeGen(J,Level,l):
    fin = []
    Jinv =[]
    for j in range(p+1):
        Jinv += [Rp(1)]
    for j in range(p+1):
        Jinv[j] = Inverse(J[0][j])
    for q in range(Level):
        if (q==0):
            fin += [Hecke0(J,Jinv,Level,l)]
        else:
            fin += [HeckeN(J,Jinv,Level,q,l)]
    return fin

#Applies T_l to J(0, infty)
def Hecke0(J,Jinv,Level,l):
    vec =[]
    for j in range(p+1):
        vec += [Rp(1)]
    for a in range(l):
        Fa = TranslateRMCnew([-a,-l], J, Jinv,Level)
        Fa = RMCApply([[1,-a],[0,1]], Fa)
        vec = MultiplyList(Fa, vec)

    vec = MultiplyList(vec, RMCApply(([[1,0],[0,1]]), J[0]))
    return vec


#Applies T_l to J(0, 1/q)
def HeckeN(J,Jinv,Level,q,l):
    vec = []
    for j in range(p+1):
        vec += [Rp(1)]
    for a in range(l):
```

```
        c = ((1+a*q)/(q*l)).numerator()
        d = ((1+a*q)/(q*l)).denominator()
        FaL = TranslateRMCnew([-a,-l], J, Jinv,Level)
        FaR = TranslateRMCnew([c,d], J, Jinv,Level)
        Fa = MultiplyList(FaL,InverseCocycle(FaR))
        Fa = RMCApply([[1,-a],[0,1]], Fa)
        vec = MultiplyList(Fa, vec)

    FinfL = TranslateRMCnew([0,1], J, Jinv,Level)
    FinfR = TranslateRMCnew([l,q], J, Jinv,Level)
    Finf = MultiplyList(FinfL,InverseCocycle(FinfR))
    Finf = RMCApply([[1,0],[0,1]], Finf)
    vec = MultiplyList(vec, Finf)
    return vec


# In[ ]:


#Example of computation

#Divisor associated to Delta_13 - 2* Delta_5. It should be of degree 0.
Div = LinComb(LLnew(13*p^2,Level),1,LLnew(5*p^2,Level),-2,Level)

#Computes the cocycle associated to the divisor Div and of the given level.
J = LevelMnew(Div,Level)

#root is the RM-point we're evaluating the cocycle at and A is its automorph in \Gamma_0(N)
root = (7*t^2-3*t-7).roots()[0][0]
A = [[26,21],[21,17]]

#Evaluates the cocycle J at the RM-point root (even and odd part separatly)
tau_even = MSEvalp2(J[0],A,Level, root)
tau_odd = MSEvalp2(J[1],A,Level, root)

#Computes T_3(J_odd) and its value at root
JHecke_3 = HeckeGen(J[1],Level,3)
tau3_odd = MSEvalp2(JHecke_3,A,Level,root)

#recognizes the minimal polynomial of the obtained values
algdepQp2(tau3_odd * tau_odd ,8)
```

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