Explicit computations of Hilbert modular forms on $\mathbb{Q}(\sqrt{5})$

by

Lassina Dembélé

Department of Mathematics and Statistics
McGill University, Montreal

September 2002

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for PhD. degree

Contents

Acknowledgements.

I am grateful to my supervisor Prof. H. Darmon who has introduced me to the very beautiful subject of Number Theory. His great patience and wonderful personality have always been a source of encouragement for me. Not only is he a great mathematician but he is also an exceptional as a person.

I thank all my friends G. Alpargu and her husband A. Bourget, T. Jackson, M.–H. Nicole, I. Longhi and C. E. M'lan with whom I shared a lot while studying mathematics at McGill. On a more personal level, I would like to thank my parents from whom I have been away for all these years. It has not always been easy but their moral support has always helped inspire me. I will also like to thank my friends, T. A. Baldé, J. Grah, J.–M. K. D. N'dri, R. Ali Kazmi and P. Reddappagari, for their inavaluable support during my stay at McGill and in Montreal. A special thanks to R. Friedman for all the efforts she put in making sense of my distorted english.

I thank the department of mathematics and statistics of McGill university for their collaboration and for their financial support. A special thanks goes to Prof. N. Elkies who kindly helped me with the Pari search of all elliptic curves in this thesis.

Abstract. We compute the eigenvalues of the first few Hecke operators $T_{\mathfrak{p}}$ ($\mathrm{N}(\mathfrak{p}) \leq 100$) for all Hilbert modular forms on $\mathbb{Q}(\sqrt{5})$ of parallel weight 2 and level of norm less than 500. This is done by exploiting the Jacquet–Langlands correspondence which allows us to transfer such computations to forms on the (totally definite) Hamilton quaternion algebra over $\mathbb{Q}(\sqrt{5})$. We use these computations to test numerically a conjecture of Oda on periods of Hilbert modular forms.

Résumé. Nous calculons les valeurs propres des opérateurs de Hecke $T_{\mathfrak{p}}$ ($\mathrm{N}(\mathfrak{p}) \leq 100$) pour toutes les formes modulaires de Hilbert primitives de poids (2, 2) et de niveau de norme plus petite que 500 sur $\mathbb{Q}(\sqrt{5})$. Pour ce faire, nous exploitons la correspondence de Jacquet–Langlands qui permet de se ramener au calcul de formes automorphes sur l'algèbre de quaternions de Hamilton sur $\mathbb{Q}(\sqrt{5})$. À l'aide de deux examples, nous illustrons la consistance de nos calculs avec la conjecture de Oda sur les périodes de formes modulaire de Hilbert.

Introduction.

Thanks to the works of Wiles [**57**], Wiles and Taylor [**53**] and Breuil, Conrad, Diamond and Taylor [**7**], it is now known that the Taniyama–Shimura–Weil conjecture over $\mathbb{Q}$ is true; i.e., that every elliptic curve $E/\mathbb{Q}$ is modular. As a by-product, this put an end to the mathematical saga of Fermat's Last Theorem that Ribet [**43**] had already proved to be a consequence of the TSW–conjecture. Before the modularity of all rational elliptic curves was establised, the main numerical results supportive of this conjecture were given by the Antwerp tables [**5**], later extended by Cremona [**14**].

Even with the substantial progress accomplished by Wiles in [**57**] and Skinner–Wiles [**50**], the hope for a proof of the full TSW–conjecture for elliptic curves over general number fields still seems far away. Meanwhile, motivated by Wiles [**57**] and related works, Darmon [**15**] has proposed a strategy to tackle the generalized Fermat Last Theorem. This strategy requires a very good understanding of Hilbert modular forms over the real subfields of certain cyclotomic fields and consists of a series of conjectures. So numerical investigations in order to gather more data supportive of Shimura–Tanyama–Weil conjecture over general number fields still has a role to play as it did in the rational case.

The main ingredient in Cremona [**14**] is the notion of *modular symbols* that gives a very concrete description of the first Homology group $\mathrm{H}_1(X_0(N), \mathbb{Z})$ of the modular curve of level N amenable to computation. There have been some attempts to generalized this notion to higher dimension but the theory is still less elaborate, and is often limited to establishing non–vanishing of L–functions by exhibiting specific modular symbols (see [**1, 2, 3, 35, 40**]). So to get numerical results about the TSW–conjecture over arbitrary number fields, it is preferable to resort to different computational methods. However, let us mention the work of Oda [**40**] who has closely investigated this conjecture in the spirit of Deligne [**19**] in the case of Hilbert modular surfaces. This leads to the so–called Oda's conjecture. To state it, let F be a totally real field of degree $g = [F : \mathbb{Q}]$ and $\mathfrak{c}$ an integral ideal in F. We consider the Hilbert modular variety $X_0(\mathfrak{c})$ and let $\mathbb{H}^g_{sp}(X_0(\mathfrak{c}), \mathbb{Q})$ be its middle cuspidal cohomology. This carries a homogeneous Hodge structure on which the Hecke algebra acts. Let $\mathbf{f}$ be a newform and $K_\mathbf{f}$ the number field generated by its Fourier coefficients.

Conjecture (Oda [**24**, Chap. VI. sec. 1.]**):** *There is a piece* $\mathrm{H}^g(M_\mathbf{f}, \mathbb{Q})$ *of* $\mathbb{H}^g_{sp}(X_0(\mathfrak{c}), \mathbb{Q})$, *an abelian variety* $A_\mathbf{f}$ *of dimension* $[K_\mathbf{f} : \mathbb{Q}]$ *defined over* F *with good reduction outside of* $\mathfrak{c}$, *a homomorphism* $K_\mathbf{f} \to \mathrm{End}(A_\mathbf{f}) \otimes \mathbb{Q}$, *and an isomorphism*

$$\bigotimes_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} \mathrm{H}^1(A^\sigma_\mathbf{f}, \mathbb{C}) \to \mathrm{H}^g(M_\mathbf{f}, \mathbb{C})$$

of $K_\mathbf{f}$*–rational Hodge structures (*$A^\sigma_\mathbf{f}$ *is the abelian variety obtained from* $A_\mathbf{f}$ *by the action of* σ*).*

The piece $\mathrm{H}^g(M_\mathbf{f}, \mathbb{Q})$ is felt to be the Hodge realization of the conjectural motive associated to $\mathbf{f}$.

When $g = 2$, Oda [**40**] showed that $\mathrm{H}^2(M_\mathbf{f}, \mathbb{Q})$ is indeed the tensor product of two $K_\mathbf{f}$–rational Hodge structures of weight 1 coming from the isogeny classes of abelian varities $A^1_{\mathbf{f}}$ and $A^2_{\mathbf{f}}$. Unfortunately not much is known about the field of definition of those varieties.

The goal of this thesis was two fold: first discuss an alternative way of computing Hilbert modular newforms of weight 2 on certain quadratic fields and second, see if the forms obtained could provide one with some numerical evidence on the Oda conjecture. Our main focus has been the quadratic field $F = \mathbb{Q}(\sqrt{5})$ though there seems to be no major obstacle to extending those computational methods to any real quadratic field of class number one. While we have more or less succeeded with the first goal, the second one should rather be seen more as a work in progress, and the few results we could obtain merely illustrate the mystery of the Oda conjecture.

To explain briefly our method, let $F = \mathbb{Q}(\sqrt{5})$ and $\mathfrak{c}$ be an integral ideal. Let $\mathbf{f}$ be a newform of weight 2 and level $\mathfrak{c}$ (see Chapter I, sec. 3 for the precise definition). Then, the Hamilton quaternion algebra B over F has class number one. We consider its maximal order R consisting of the icosians; we fix an Eichler order $\mathrm{R}_{\mathfrak{c}}$ of level $\mathfrak{c}$ in R. By the Jacquet–Langlands correspondence, $\mathbf{f}$ can be identified with a newform on $\mathrm{R}^{\times}_{\mathfrak{c},\mathbb{A}}\backslash\mathrm{B}^{\times}_{\mathbb{A}}/\mathrm{B}^{\times}$. However, $\mathrm{R}^{\times}_{\mathfrak{c},\mathbb{A}}\backslash\mathrm{B}^{\times}_{\mathbb{A}}/\mathrm{B}^{\times}$ is just the complex analytic space of a zero–dimensional variety. Thus it consists of a finite discrete space. The action of Hecke operators on the complex valued functions on this space is the basic ingredient in the computations. We could compute all the eigenvalues $a_{\mathfrak{p}}$ of the Hecke operators $T_{\mathfrak{p}}$ for all prime $\mathfrak{p}$ of norm $\mathrm{N}(\mathfrak{p}) \leq 10000$ for any level of norm $\mathrm{N}(\mathfrak{c}) \leq 500$, but only the first few (with $\mathrm{N}(\mathfrak{p}) \leq 100$) have been listed. It is then easy to compute the periods of such a newform when the plurigenus p_g of the modular surface $X_0(\mathfrak{c})$ is equal to 1. We have done so for the two smallest levels; namely, when $\mathfrak{c}$ is one of the prime living above 31 or 41 in F. By direct investigations, we obtain the minimal model of the corresponding elliptic curves.

In the first chapter, we recall the main facts about the Jacquet–Langlands correspondence, then explain in the last section how this is used to perform our computations. In the second chapter, we use the fact that Oda [**40**] had already establised that integrating a newform along some specific cycles, the corresponding periods are just twists of special values of L–functions, to compute the period moduli of the abelian varieties associated to the newforms of level one of the primes above 31 or 41.

CHAPTER 1

Computing Hilbert modular forms.

In this chapter, we explain how to compute all Hilbert modular forms on $F = \mathbb{Q}(\sqrt{5})$ of level an integral ideal $\mathfrak{c}$ and parallel weight 2. Let $\mathbf{f}$ be a newform of level $\mathfrak{c}$ and weight 2 on $\mathbf{GL}_2$. By the Jacquet-Langlands correspondence, it is associated to an eigenform on the unit group in the Hamilton quaternion algebra over $\mathbb{Q}(\sqrt{5})$. The space of modular forms one then gets by making this transfer is simple, hence more suitable for computational purposes. Before delving into those computations, let us recall some facts about Hilbert modular forms, Hecke operators and the Jacquet-Langlands correspondence. We start with a brief review of quaternion algebras.

1. Quaternion algebras: a brief review.

The main references for this section are Vignéras [**54**] and Reiner [**42**].

Definition 1.1. *A* quaternion algebra *over F is a central simple F–algebra of rank 4 over F.*

Let B be a quaternion algebra over F. By the non-commutativity of B, every element $\alpha \in B$ generates a quadratic F–subalgebra $L = F[\alpha]$ of B. The characteristic polynomial of α is of the form $x^2 - \mathrm{tr}(\alpha)x +$

$\mathrm{n}(\alpha)$, where $\mathrm{tr}(\alpha)$ and $\mathrm{n}(\alpha)$ denote respectively the *reduced trace* and *norm* of α.

Let τ be the non-trivial automorphism of L. The algebra B viewed as a module over L, under left multiplication is free of rank 2; hence if $\beta \in B - L$, one has $B = L \oplus L\beta$. We can always find a couple (L, a) such that $\beta^2 = a$ (see Reiner [**42**, Chap. VII, sec. 29.]). Then, the multiplication in B is completely determined by the rule $\beta\alpha = \alpha^\tau\beta$.

EXAMPLE 1.2. 1) $F = \mathbb{Q}$ and $B = M_2(F)$ the 2×2 matrices. Here, we may take $L = F \times F$ and $a = 1$
2) $F = \mathbb{Q}$ and $B = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k$, where $i^2 = -1$, $j^2 = -1$ and $k = ij = -ji$, the Hamilton quaternion algebra. Here, we may take $L = \mathbb{Q} + \mathbb{Q}i$, $\beta = j$ and $a = -1$.
3) In general, by the structure theorem of Wedderburn for simple algebras, a quaternion algebra B is either a *division* algebra or *isomorphic* to the matrix algebra $M_2(F)$. If $\mathrm{char}(F) \neq 2$, we can write B in the form $B = F \oplus Fi \oplus Fj \oplus Fk$, where $i^2 = a$, $j^2 = b$, $a, b \in F^\times$ and $k = ij = -ji$. We denote this algebra by $B = \left(\frac{a,b}{F}\right)$. In $B = \left(\frac{a,b}{F}\right)$, the conjugate of an element $q = x + yi + zj + wk$ is given by $\bar{q} = x - yi - zj - wk$, and the reduced trace and norm respectively by $\mathrm{tr}(q) = 2x$ and $\mathrm{n}(q) = x^2 + ay^2 + bz^2 - abw^2$.

DEFINITION 1.3. *An order R in B is a* finite *$\mathcal{O}_F$–subalgebra of B such that $B = F \cdot R$.*

EXAMPLE 1.4. Let I be a *lattice* in B (i.e. a finitely generated $\mathcal{O}_F$–module which contains an F–basis of B) and put

$$\mathcal{O}_l(I) = \{x \in B |\, xI \subset I\},$$

$$\mathcal{O}_r(I) = \{x \in B |\, Ix \subset I\}.$$

One easily sees that these are $\mathcal{O}_F$-algebras contained in B. In fact, they are orders in B. Indeed, let $e_1, \ldots, e_4$ be an F-basis of B contained in I. For any $x, y \in B$, $xy = a_1e_1 + \cdots + a_4e_4$, for some $a_i \in F$. So killing denominators of the a_i's, there exists $a \in \mathcal{O}_F$ such that $axy \in I$. Fixing x and letting y run over the set $\{e_1, \ldots, e_4\}$, we see that there exists $a \in \mathcal{O}_F$ such that $ax \in \mathcal{O}_l(I)$. Hence $B = F \cdot \mathcal{O}_l(I)$. So we only need see that $\mathcal{O}_l(I)$ is a *finite* $\mathcal{O}_F$-algebra. Choose an element $a \in \mathcal{O}_F \cap I - \{0\}$. One sees that

$$\mathcal{O}_l(I) \subset a^{-1}I.$$

Since $a^{-1}I$ is a finitely generated $\mathcal{O}_F$–module, so is $\mathcal{O}_l(I)$. A similar argument shows that $\mathcal{O}_r(I)$ is an order in B.

DEFINITION 1.5. *An* ideal *in B is a lattice $I \subset B$. We call $\mathcal{O}_l(I)$ (resp. $\mathcal{O}_r(I)$) the* left *(resp.* right*) order of I. We say that I is*
(i) two-sided *if $\mathcal{O}_l(I) = \mathcal{O}_r(I)$;*
(ii) normal *if $\mathcal{O}_l(I)$ and $\mathcal{O}_r(I)$ are maximal orders in B;*
(iii) integral *if I is contained in both $\mathcal{O}_l(I)$ and $\mathcal{O}_r(I)$;*
(iv) principal *if I is of the form $I = \mathcal{O}_l(I)a = a\mathcal{O}_r(I)$.*
We define the inverse *of I to be*

$$I^{-1} = \{x \in B | IxI \subset I\}.$$

The norm $\mathrm{n}(I)$ *of I is the fractional ideal generated by the norms of all elements in I.*
When R is an order in B, we say that I is a left *(resp.* right*) R–*ideal*, if R is contained in $\mathcal{O}_l(I)$ (resp. $\mathcal{O}_r(I)$).*

By Zorn's lemma, *maximal* orders always exist. We say R is an *Eichler order* if it is the intersection of two maximal orders. We will need the following result of Hijikata later.

PROPOSITION 1.6. [**54**, Chap. II, Lemma 2.4] *Let F be a local field, π a uniformizer of F and R an order in $M_2(F)$. Then the following are equivalent.*
(i) R is an Eichler order.
(ii) There exists a unique *pair of maximal orders R_1 and R_2 such that $R = R_1 \cap R_2$.*
(iii) There is a unique *integer $n \in \mathbb{N}$ such that R is conjugate to the order*

$$R_n = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \pi^n \mathcal{O}_F & \mathcal{O}_F \end{pmatrix}.$$

(iv) R contains a subring conjugate to $\begin{pmatrix} \mathcal{O}_F & 0 \\ 0 & \mathcal{O}_F \end{pmatrix}$.

DEFINITION 1.7. *Let F be a local field and R an Eichler order in $M_2(F)$. The integer n in Proposition 1.6 (iii) is called the* level *of R.*

DEFINITION 1.8. *Let R be an order. The* different $\mathfrak{D}(R)$ *of R is the inverse of the dual of R with respect to the trace form. The* (reduced) discriminant *of R is the ideal* $\mathrm{disc}(R) = \mathrm{n}(\mathfrak{D}(R))$ *in F.*

PROPOSITION 1.9. *Let I (resp. R) be an ideal (resp. an order) in B. Then, we have the following.*
a) The dual $I^\vee$ of I is a two–sided *ideal in B.*
b) $\mathfrak{D}(R)$ is an integral two–sided *ideal in B.*
c) If R is a free *$\mathcal{O}_F$–module, then* $\mathrm{disc}(R)^2 = (\det(A))$, *where* $A = (\mathrm{tr}(e_i e_j))$ *for some $\mathcal{O}_F$–basis of R.*

d) If R' is an order contained in R, then $\mathrm{disc}(R') = \mathrm{disc}(R)$ *implies that $R' = R$.*

PROOF. Vignéras [**54**, Chap. IV, Lemme 4.7. and Corollaire 4.8.] $\square$

DEFINITION 1.10. *Let R be an order in B and I, J two left (resp. right) R–ideals. We say that I and J are* right *(resp.* left*)* equivalent *if $J = Ix$ (resp. $J = xI$) for some non-zero $x \in B$.*

PROPOSITION 1.11. *Let R be an order in B. The number of left equivalence classes is* finite *and* equal *to the number of right equivalence classes. All maximal orders have the same class number.*

DEFINITION 1.12. *Let R be an order in B. The* class number *of R is the number of left equivalence classes. The* class number *of B is the class number of any maximal order in B.*

DEFINITION 1.13. *Let R and R' be two orders in B. We say that R and R' have the* same type *if they are conjugate by an inner automorphism of B. The number of equivalence classes of* maximal orders *in B under this equivalence relation is called the* type number *of B.*

Now, let F be a global field and fix a quaternion algebra B over F. Let I be a lattice in B. Then, for any prime $\mathfrak{p}$ of F, the completion $I_{\mathfrak{p}} = I \otimes \mathcal{O}_{\mathfrak{p}}$ of I at $\mathfrak{p}$ is a (free) lattice in $B_{\mathfrak{p}}$; therefore it is an ideal in $B_{\mathfrak{p}}$. The completions $(I_{\mathfrak{p}})_{\mathfrak{p}}$ determine an $\hat{\mathcal{O}}_F$–module in $\hat{B}$, where $\hat{\mathcal{O}}_F$ (resp. $\hat{B}$) is the finite part of $\mathbb{A}$ (resp. of $B_{\mathbb{A}}$).

PROPOSITION 1.14. [**54**, Chap. V, Proposition 5.1.] *Let B be a quaternion algebra over F and fix a lattice L in B. Then, the map*

$$I \mapsto (I_{\mathfrak{p}})_{\mathfrak{p}}$$

is a bijection between the set of all ideals in B and the set of lattices

$$\{(I_{\mathfrak{p}})_{\mathfrak{p}} |\, I_{\mathfrak{p}} = L_{\mathfrak{p}} \quad \text{for almost all}\, \mathfrak{p} \,\text{in}\, F\}.$$

This Proposition and the very simple structure theorem [**54**, Theorems 1.1. & 1.3.] of quaternion algebras over local field is the thrust of the global-adelic language. For example, sending $(x_{\mathfrak{p}})_{\mathfrak{p}}$ to the ideal determined by the local ideals $I_{\mathfrak{p}} = R_{\mathfrak{p}} x_{\mathfrak{p}}$, one gets that the quotient $R_{\mathbb{A}}^{\times} \backslash B_{\mathbb{A}}^{\times}$ parametrizes the set of left R–ideals which are locally principal. Thus, $R_{\mathbb{A}}^{\times} \backslash B_{\mathbb{A}}^{\times} / B^{\times}$ parametrizes all left equivalence classes of all R–ideals which are locally principal.

DEFINITION 1.15. *We say that B* ramifies *at $\mathfrak{p}$ if $B_{\mathfrak{p}}$ is a division ring. We denote by* $\mathrm{Ram}(B)$ *the set of places at which B ramifies.*

The following theorem gives a classification of quaternion algebras over a global field. A proof can be found in Vignéras [**54**, Chap. III.].

THEOREM 1.16. [**54**, Chap. III, Theorem 3.1.] *Let B be quaternion algebra over a global field F. Then* $\mathrm{Ram}(B)$ *is a* finite *set of* even *cardinality. Conversely, for every* finite *set S of places of* even *cardinality, there is a unique quaternion algebra B (up to isomorphism) such that* $\mathrm{Ram}(B) = S$.

The *reduced* discriminant of B is defined to be the integral ideal $\mathrm{disc}(B) \subset \mathcal{O}_F$ determined by all the primes $\mathfrak{p} \in \mathrm{Ram}(B)$. This is a well-defined ideal as $\mathrm{Ram}(B)$ is finite by the above theorem.

Let R be an Eichler order. Then, for any prime $\mathfrak{p} \in \mathrm{Ram}(B)$, $R_\mathfrak{p}$ is the unique maximal order in $B_\mathfrak{p}$; while if $\mathfrak{p} \notin \mathrm{Ram}(B)$, by Hijikata [**54**, Lemma 2. 4], $\mathrm{disc}(R_\mathfrak{p}) = (\pi_\mathfrak{p}^{n_\mathfrak{p}})$, where $n_\mathfrak{p}$ is the level of $R_\mathfrak{p}$ and $\pi_\mathfrak{p}$ a uniformizer of $\mathcal{O}_\mathfrak{p}$.

DEFINITION 1.17. *Let R be an Eichler order. The* level $\mathfrak{N}(R)$ *of R is the unique integral ideal determined by the local discriminant* $\mathrm{disc}(R_\mathfrak{p})$*, for* $\mathfrak{p} \notin \mathrm{Ram}(B)$.

By the local nature of the discriminant, we see that

$$\mathrm{disc}(R) = \mathrm{disc}(B)\mathfrak{N}(R).$$

The class number formula for orders in quaternion algebras can be found in several places. We refer to Vignéras [**54**, Chap. III, IV & V] and references therein. The following result will be enough for our purpose.

PROPOSITION 1.18. [**55**] *Let F be a number field of degree g and B a totally definite quaternion algebra over F (i.e.,* $\mathrm{Ram}(B)$ *contains all the archimedean places of F). Let R be a maximal order in B and h its class number. We fix a set of representatives of all left ideal classes of R, say $\{I_1, \ldots, I_h\}$. For each i, let R_i be the right order of I_i. We let $R_i^\times$ (resp. R_i^1) be the group of units (resp. elements of norm 1) in*

R_i. We denote by w_i the cardinality of R_i^1. Then,

$$\zeta_F(-1)\prod_{\mathfrak{p}|\mathrm{disc}(B)}(1-\mathrm{N}(\mathfrak{p})) = 2^g\sum_{i=1}^{h}\frac{[R_i^\times : \mathcal{O}_F^\times R_i^1]}{w_i}.$$

PROOF. See Vignéras [**55**]. □

EXAMPLE 1.19. Let $F = \mathbb{Q}(\sqrt{5})$ and B the Hamilton quaternion algebra (i.e. the unique quaternion algebra, which ramifies only at both infinite places). Put

$$R = \mathbb{Z}[\omega][e_1, e_2, e_3, e_4],$$

with

$$\begin{aligned} e_1 &= \frac{1}{2}(1-\omega i+\bar{\omega}j),\\ e_2 &= \frac{1}{2}(-\omega i+j+\bar{\omega}k),\\ e_3 &= \frac{1}{2}(\bar{\omega}i-\omega j+k),\\ e_4 &= \frac{1}{2}(i+\bar{\omega}j-\omega k), \end{aligned}$$

and $\omega = (1+\sqrt{5})/2$. Then $\mathrm{disc}(R) = 1$, thus R is a maximal order: this is the icosian ring (see Conway and Sloane [**13**, Chap. VIII, sec. 2.1.]). R^1 is the icosian group and $R^\times$ is the semi–direct product of R^1 with $\mathcal{O}_F^\times = \mathbb{Z}[\omega]^\times$. Keeping the same notations as in the above proposition, we see that by the structure theorem of *finite* multiplicative subgroups of the Hamilton quaternion algebra $\mathbb{H}$ over the real [**54**, Chap. I, Théorème 3.7.], the only possibilities for w_i are 24, 48 and 120. Again from [**54**, Chap. I, Théorème 3.7. and Exercice 3.1.], the only possibilities for $[R_i^\times : \mathcal{O}_F^\times R_i^1]$ are 1, 2, 4. Since $\zeta_{\mathbb{Q}(\sqrt{5})}(-1) = 1/30$, we see that we must necessarily have $h = 1$, $[R_1^\times : \mathcal{O}_F^\times R_1^1] = 1$ and $w_1 = 120$. That is, B has class number 1.

2. Classical Hilbert modular forms.

The main reference for this section is van der Geer [**24**].

Through out the rest of this chapter, F is a totally real number field of degree g. We let $a \to a^{\tau_i}$, $i = 1, \ldots, g$, be all the real embeddings of F. Also, we let $\mathcal{O}_F$ be the ring of integers of F, δ its different and $\mathbb{A}$ its adèle ring. We say that an element $a \in F$ is totally positive if $a^{\tau_i} > 0$, for all $i = 1, \ldots, g$. We denote by F^+ (resp. $\mathcal{O}_F^+$), the set of all totally positive elements in F (resp. $\mathcal{O}_F$). Through out, $\mathfrak{b}$ (resp. $\mathfrak{c}$) will be a fractional (resp. integral) ideal of F.

We consider the reductive group $G = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{GL}_2$ obtained by restriction of scalars à la Weil, so that for any $\mathbb{Q}$-algebra A, $G_A = \mathbf{GL}_2(A \otimes_{\mathbb{Q}} F)$. We let

$$\begin{aligned}\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b}) \quad = \quad & \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\, d \in \mathcal{O}_F,\, b \in \mathfrak{b}^{-1},\, c \in \mathfrak{b}, \right. \\ & \left. \qquad\qquad\qquad ad - bc \in \mathcal{O}_F^{\times +} \right\}.\end{aligned}$$

This is the subgroup of all automorphisms of the projective rank 2 $\mathcal{O}_F$-module $\mathcal{O}_F \oplus \mathfrak{b}$ having totally positive determinant. We put $\Gamma(\mathfrak{b}, \mathfrak{c})$ to be the following subgroup of $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$.

$$\Gamma(\mathfrak{b},\, \mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b}) : a,\, d \in \mathcal{O}_F,\, b \in \mathfrak{b}^{-1},\, c \in \mathfrak{c}\mathfrak{b} \right\}.$$

$\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$ can be embedded into $\mathrm{GL}_2^+(\mathbb{R})^g$ by making use of the g embeddings $\tau_i : F \hookrightarrow \mathbb{R}$. We recall that $\mathrm{GL}_2^+(\mathbb{R})^g$ acts on the g-fold product $\mathcal{H}^g$ of the upper half-plane by extension of the well-known action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}$ by Mobius transformations:

$$\begin{array}{rcl} \mathrm{GL}_2^+(\mathbb{R}) \times \mathcal{H} & \to & \mathcal{H} \\ (\gamma,\, z) & \mapsto & \dfrac{az+b}{cz+d}, \end{array}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This action of $\mathrm{GL}^+(\mathbb{R})^g$ factors through $\mathrm{PGL}^+(\mathbb{R})^g$ and the resulting action (up to a semi-direct factor given by $\mathfrak{S}_g$, the group of g-permutations), gives all the biholomorphic transformations of $\mathcal{H}^g$. For any two ideals $\mathfrak{b}_1$ and $\mathfrak{b}_2$, $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b}_1)$ and $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b}_2)$ are conjugate (by an inner automorphism) in $M_2(F)$ if and only if $\mathfrak{b}_1$ and $\mathfrak{b}_2$ belong to the same genus (i.e., there exists some totally positive $\lambda \in F$ and an ideal $\mathfrak{b}$ such that $\mathfrak{b}_1 = \lambda \mathfrak{b}^2 \mathfrak{b}_2$).

The domain $\mathcal{H}^g$ is contained in $\mathbf{P}^1(\mathbb{C})^g$, and the g embeddings $\tau_i : F \hookrightarrow \mathbb{R}$ induce an embedding

$$\mathbf{P}^1(F) \hookrightarrow \mathbf{P}^1(\mathbb{R})^g \subset \mathbf{P}^1(\mathbb{C})^g.$$

$\mathrm{GL}_2^+(\mathbb{R})^g$ acts on $\mathbf{P}^1(\mathbb{R})^g$ by fractional linear transformations, and so does $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$ on $\mathbf{P}^1(F) \subset \mathbf{P}^1(\mathbb{R})^g$.

Let us fix Γ a subgroup of $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$ of *finite* index. The Γ-equivalent classes of $\mathbf{P}^1(F)$ are called the *cusps* of Γ. The class of $(1\,:\,0)$ is called the cusp ∞. The isotropy group of the cusp ∞ in $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$ is

$$\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})_\infty = \left\{ \begin{pmatrix} \varepsilon_1 & \mu \\ 0 & \varepsilon_2 \end{pmatrix} : \varepsilon_1, \varepsilon_2 \in \mathcal{O}_F^\times,\ \varepsilon_1\varepsilon_2 \in \mathcal{O}_F^{\times +},\ \mu \in \mathfrak{b}^{-1} \right\}.$$

It determines the transformation group

$$\mathrm{PGL}^+(\mathcal{O}_F \oplus \mathfrak{b})_\infty = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} : \varepsilon \in \mathcal{O}_F^{\times +},\, \mu \in \mathfrak{b}^{-1} \right\}.$$

Hence the transformation group corresponding to the isotropy group $\Gamma_\infty = \Gamma \cap \mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})_\infty$ of ∞ in Γ is of the form

$$\hat{\Gamma}_\infty = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} : \varepsilon \in V,\, \mu \in M \right\},$$

where V is a subgroup of $\mathcal{O}_F^{\times +}$, while M is a $\mathbb{Z}$-submodule of $\mathfrak{b}^{-1}$. In fact, since Γ has *finite* index in $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$, M is a sublattice of $\mathfrak{b}^{-1}$, and the Dirichlet unit theorem implies that V is a subgroup of $\mathcal{O}_F^{\times +}$ of rank $g-1$. We call (M, V) the *type* of the cusp ∞.

Let $\mathbf{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g$ be a vector with integer coefficients, and $f : \mathcal{H}^g \to \mathbb{C}$ a meromorphic function. Take $\gamma = (\gamma_1, \ldots, \gamma_g) \in \mathrm{GL}_2^+(\mathbb{R})^g$, with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and put

$$(f||_{\mathbf{k}}\gamma)(z) := \left(\prod_{i=1}^{g} \det(\gamma_i)^{k_i/2}(c_i z_i + d_i)^{-k_i} \right) f(\gamma z),$$

for $z = (z_1, \ldots, z_g) \in \mathcal{H}^g$. This defines an action of $\mathrm{GL}_2^+(\mathbb{R})^g$ on the set of meromorphic functions on $\mathcal{H}^g$.

Definition 1.20. *A* Hilbert modular form *of weight* $\mathbf{k}$ *and level* Γ *is a holomorphic function* $f : \mathcal{H}^g \to \mathbb{C}$ *such that*

$$f\|_{\mathbf{k}}\gamma = f, \quad \text{for all}\, \gamma \in \Gamma.$$

Let $f : \mathcal{H}^g \to \mathbb{C}$ be a Hilbert modular form. Since it is Γ-invariant, we have in particular

$$f(z+\mu) = f(z), \quad \text{for all } z \in \mathcal{H}^g,\ \mu \in M.$$

Therefore, it admits a Fourier expansion of the form

$$f(z) = \sum_{\nu \in M^\vee} a_\nu e^{2\pi i \mathrm{Tr}(\nu z)},$$

where $\mathrm{Tr}(\nu z) = \sum_{i=1}^{g} \nu^{\tau_i} z_i$, and $M^\vee$ is the dual lattice of M with respect to the trace form. When $g > 1$, every Hilbert modular form is automatically holomorphic at cusps as the next lemma shows.

Lemma 1.21 (Koecher's principle). *Assume that $g > 1$. Then, f is* holomorphic *at the cusp ∞ (hence at all cusps) in the following sense:*

$$a_\nu \neq 0 \Rightarrow \nu = 0 \text{ or } \nu \gg 0.$$

Proof. Let (M, V) be the type of the cusp ∞, and take $\varepsilon \in V$. Then, $\gamma(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma$. Hence, $f \|_{\mathbf{k}} \gamma(\varepsilon) = f$. Equating the q–expansion of both members of this equality, it follows that

$$a_{\varepsilon\nu} = \mathrm{N}(\varepsilon)^{\mathbf{k}/2} a_\nu, \quad \text{for all } \nu \in M^\vee,$$

where we use the notation $\mathrm{N}(\varepsilon) = \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g}$. Now, let us assume that there is a non–zero $\nu_0 \in M^\vee$ not totally positive such that $a_{\nu_0} \neq 0$. Without lost of generality, we can assume that $\nu_0^{\tau_1} < 0$. By the Dirichlet units theorem, we can find $\varepsilon \in V$ such that

$$\varepsilon^{\tau_1} > 1 \quad \text{and} \quad \varepsilon^{\tau_i} < 1,\ i = 2, \ldots, g.$$

We now consider the subseries of $f(z) = \sum_{\nu\in M^\vee} a_\nu e^{2\pi i \mathrm{Tr}(\nu z)}$ index by the set $\{\nu_0\varepsilon^m,\ m\in\mathbb{N}\}$, in which we put $z=(i,\ldots,i)$. Then

$$a_{\nu_0\varepsilon^m} e^{-2\pi \mathrm{Tr}(\nu_0\varepsilon^m)} = \mathrm{N}(\varepsilon)^{m\mathbf{k}/2} a_{\nu_0} e^{-2\pi\mathrm{Tr}(\nu_0\varepsilon^m)}.$$

But, as $m\to\infty$, $e^{-2\pi\mathrm{Tr}(\nu_0\varepsilon^m)} \sim e^{-2\pi\nu_0^{\tau_1}(\varepsilon^{\tau_1})^m}$, and the exponential growth ensures that $\mathrm{N}(\varepsilon)^{m\mathbf{k}/2} a_{\nu_0} e^{-2\pi\mathrm{Tr}(\nu_0\varepsilon^m)} \to \infty$. Therefore the series does not converge, which is a contradiction. So we must have $a_{\nu_0}=0$. $\square$

DEFINITION 1.22. *We say that f is a* cusp *form if the constant term a_0 in the Fourier expansion is equal to 0 for any $f\|_{\mathbf{k}}\gamma$, $\gamma\in\mathrm{GL}_2^+(F)$ (i.e., if f vanishes at all cusps).*

We will denote by $M_{\mathbf{k}}(\Gamma)$ (resp. $S_{\mathbf{k}}(\Gamma)$) the space of Hilbert modular forms (resp. cusp forms) of weight $\mathbf{k}$ and level Γ.

COROLLARY 1.23. $S_{\mathbf{k}}(\Gamma) = M_{\mathbf{k}}(\Gamma)$ *unless* $k_1=\cdots=k_g$.

PROOF. Let assume that there is $f\in M_{\mathbf{k}}(\Gamma)$ not a cusp form. Then at some cusp σ, the q–expansion must give $a_0\neq 0$. From

$$a_0 = \mathrm{N}(\varepsilon)^{\mathbf{k}/2} a_0, \quad \text{for all } \varepsilon\in V,$$

it follows that we must have $\mathrm{N}(\varepsilon)^{\mathbf{k}/2}=1$ for all $\varepsilon\in V$. But this is possible only if we have $k_1=\cdots=k_g$. $\square$

PROPOSITION 1.24. *(i)* $M_{\mathbf{k}}(\Gamma)=0$ *unless* $k_i\geq 0$ *for all* $i=1,\ldots,g$. *(ii)* $M_0(\Gamma)=\mathbb{C}$ *and* $S_0(\Gamma)=0$.

PROOF. van der Geer [**24**, Chap. I. sec. 6.] $\square$

EXAMPLE 1.25. *Eiseinstein series.* Let $\mathcal{B}$ be an ideal class and $\mathfrak{b} \in \mathcal{B}$. Let $k \geq 2$ be even. Put

$$G_{k,\mathcal{B}}(z) = \mathbf{N}(\mathfrak{b})^k \sum_{(c,d)\in\mathbf{P}^1(\mathfrak{ab}\times\mathfrak{b})} \mathbf{N}(cz+d)^{-k},$$

where $\mathbf{P}^1(\mathfrak{ab}\times\mathfrak{b}) = \{(c,\, d) \in \mathfrak{ab}\times\mathfrak{b} | (c,\, d) \neq (0,\, 0)\}/\mathcal{O}_F^\times$. This series does not depend on $\mathfrak{b} \in \mathcal{B}$, and it can be shown to be modular form of weight $\mathbf{k} = (k,\, \cdots,\, k)$ with respect to $\mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{a})$. As in the one–dimensional case, one can find the expansion at ∞ by making use of Poisson summation (see van der Geer [**24**, Chap. 1, sec. 6]). We call $G_{k,\mathcal{B}}$ the *Eiseinstein series* of weight k and class $\mathcal{B}$ with repect to $\mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{a})$.

The Petersson inner product on $S_{\mathbf{k}}(\Gamma)$ is given by

$$\langle f,\, g\rangle = \mu(\Gamma\backslash\mathcal{H}^g)^{-1}\int_{\Gamma\backslash\mathcal{H}^g} f(z)\overline{g(z)}y^{\mathbf{k}}d\mu(z),$$

where $d\mu(z) = \prod_{j=1}^g y_j^{-2}dx_jdy_j$, with $z_j = x_j + y_ji$.

3. Hilbert modular forms in the adelic language.

In this section, we fix a quaternion algebra B over F and let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{B}^\times)$ be the reductive algebraic group obtained by restriction of scalars. For any F-algebra A, $G(A)$ is the group of all A-valued points of G. In particular, $G(\mathbb{A})$ is the adelization of G, and G_∞ and G_f respectively, its infinite and finite parts. The subgroup of all totally positive elements of G_∞ will be denoted G_∞^+. We fix a maximal compact subgroup K_∞ of G_∞^+ and let K_f be a compact open subgroup of G_f. Put $K = K_\infty \times K_f$. Let $\mathfrak{g}$ be the Lie algebra of G_∞. Then we denote by

$\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$, by $U(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, and by $Z(\mathfrak{g}_{\mathbb{C}})$ the center of $U(\mathfrak{g}_{\mathbb{C}})$.

Let $\Phi : G(\mathbb{A}) \to \mathbb{C}$ be a continuous function. We say that Φ is *smooth* if it is C^∞ as a function on G_∞ and *locally constant* on G_f. We say that it is *K–finite to the right* if the complex space $W_\Phi = \mathrm{Span}(\{\Phi_k,\, k \in K\})$, where $\Phi_k : G(\mathbb{A}) \to \mathbb{C}$ is the function sending g to $\Phi_k(g) = \Phi(gk)$, is *finite* dimensional. This means that the representation of K we obtain by right translation of Φ is finite dimensional. The elements in $U(\mathfrak{g}_{\mathbb{C}})$ can be thought of as differential operators or as distributions (see [**23**]). As such, they act on Φ in a natural way. So we can define $Z(\mathfrak{g}_{\mathbb{C}})$–finiteness similarly to K–finiteness. Finally, we say that Φ is *slowy increasing* if there is a continuous representation $\rho : G(\mathbb{A}) \to \mathrm{GL}_n(\mathbb{C})$ such that, for any $g \in G(\mathbb{A})$,

$$|\Phi(g)| \leq C\|\rho(g)\|_2,$$

where $\|A\|_2 = \mathrm{Tr}(AA^t)^{1/2}$ is the Schur norm.

Definition 1.26. *An* automorphic form *on $G(\mathbb{A})$ of level K_f is a* smooth *function $\Phi : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:*
(i) $\Phi(\gamma g z) = \Phi(g)$, for all $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$ and $z \in Z_{\mathbb{A}}$;
(ii) Φ is right K–finite and $Z(\mathfrak{g}_{\mathbb{C}})$–finite;
(iii) Φ is slowly increasing.
If Φ satisfies the further condition that

$$\int_{N(\mathbb{A})/N(F)} \Phi(ng)dn = 0, \quad \textit{for all } g \in G(\mathbb{A}),$$

where N is the unipotent radical of G, we say that Φ is cuspidal *(or a* cusp form*).*

When $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{GL}_2)$, we let $K_\infty = \mathrm{SO}(2)^g$. Then, since K_∞ is abelian, the *finite* dimensionality of the representation W_Φ implies that it is induced by some character χ of $(\mathbb{R}/\mathbb{Z})^g$. If $\chi(\theta) = \exp(2\pi i \mathrm{Tr}(\mathbf{k}\theta)$ for $\theta \in \mathbb{R}^g$ and $\mathbf{k} \in \mathbb{Z}^g$, then

$$\Phi(gr(\theta)) = \chi(\theta)\Phi(g), \text{ for all } g \in G(\mathbb{A}),$$

$$r(\theta) = \left(\begin{pmatrix} \cos(2\pi\theta_i) & -\sin(2\pi\theta_i) \\ \sin(2\pi\theta_i) & \cos(2\pi\theta_i) \end{pmatrix} \right).$$

In this setting, the center of the universal enveloping algebra is generated by the Casimir operators Δ_i corresponding to each archimedean place. Therefore the $Z(\mathfrak{g}_\mathbb{C})$–finiteness of Φ is equivalent to Φ being an eigenvector of each Δ_i. Recall that for the Lie group $\mathrm{GL}_2^+(\mathbb{R})$, the Casimir operator Δ is given by

$$-4\Delta = H^2 + 2RL + 2LR,$$

where

$$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

DEFINITION 1.27. *A* Hilbert modular form *of level K_f and weight $\mathbf{k} \in \mathbb{Z}^g$ is an automorphic form $\Phi : G(\mathbb{A}) \to \mathbb{C}$ such that*

(i) $\Phi(gk) = \Phi(g)$, for all $k \in K_f$ ($k_\infty = 1$);

(ii) $\Phi(gr(\theta)) = \chi(\theta)\Phi(g)$, for all $g \in G(\mathbb{A})$, $r(\theta) \in K_\infty$, where the character χ is defined as above;

(iii) $\Delta_i\Phi = -\dfrac{k_i}{2}\left(\dfrac{k_i}{2} - 1\right)\Phi$, for all $i = 1, \ldots, g$.

We still assume that $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{GL}_2)$, and let $Y(\mathfrak{c})$ and $K_0(\mathfrak{c})$ be the following semigroups.

$$K_0(\mathfrak{c}) = \prod_{\mathfrak{p}} K_{\mathfrak{p}}, \qquad Y(\mathfrak{c}) = G(\mathbb{A}) \cap \left(G_\infty^+ \times \prod_{\mathfrak{p}} Y_{\mathfrak{p}} \right),$$

where the product is taken over all finite primes of F, and

$$Y_{\mathfrak{p}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}}) : b,\, d \in \mathcal{O}_{\mathfrak{p}},\, a-1,\, c \in \mathfrak{c}_{\mathfrak{p}} \right\},$$
$$K_{\mathfrak{p}} = \{ x \in Y_{\mathfrak{p}} : \det(x) \in \mathcal{O}_{\mathfrak{p}}^\times \}.$$

We simply call a Hilbert modular form of level $K_0(\mathfrak{c})$ and weight $\mathbf{k}$ a Hilbert modular form of level $\mathfrak{c}$ and weight $\mathbf{k}$. We will denote by $M_{\mathbf{k}}(\mathfrak{c})$ (resp. $S_{\mathbf{k}}(\mathfrak{c})$) the space of Hilbert modular forms (resp. cusp forms) of weight $\mathbf{k}$ and level $\mathfrak{c}$. The connection of the spaces $M_{\mathbf{k}}(\mathfrak{c})$ (resp. $S_{\mathbf{k}}(\mathfrak{c})$) with the classical Hilbert modular forms is obtained as follows: let $\mathfrak{c}_\lambda$, $\lambda = 1, \ldots, h^+$, be representatives of the narrow ideal classes of F. For each $\lambda = 1, \ldots, h^+$, take $x_\lambda \in G_{\mathbb{A}}$, so that $t_\lambda = \det(x_\lambda)$ generates the ideal $\mathfrak{c}_\lambda$. Then, by the strong approximation theorem,

$$G(\mathbb{A}) = \coprod_{\lambda=1}^{h^+} G(\mathbb{Q}) x_\lambda (G_\infty^+ \times K_0(\mathfrak{c})),$$

and we see that

$$\Gamma_\lambda = \Gamma(\mathfrak{c}_\lambda,\, \mathfrak{c}) = x_\lambda (G_\infty^+ \times K_0(\mathfrak{c})) x_\lambda^{-1} \cap G(\mathbb{Q}).$$

To each h^+–tuple $(f_1, \ldots, f_{h^+}) \in \oplus_{\lambda=1}^{h^+} S_{\mathbf{k}}(\Gamma_\lambda)$, we associate the $\mathbb{C}$-valued function $\mathbf{f}$ on $G(\mathbb{A})$ defined by

$$\mathbf{f}(\gamma x_\lambda w) = (f_\lambda \|_{\mathbf{k}} w_\infty)(\mathbf{i}), \quad \gamma \in G(\mathbb{Q}) \text{ and } w \in G_\infty^+ \times K_0(\mathfrak{c}),$$

where $\mathbf{i} = (i, \ldots, i)$. The function $\mathbf{f}$ uniquely determines the h^+–tuple $(f_1, \ldots, f_{h^+})$.

PROPOSITION 1.28. [**25**, Chap. 3] *The map*

$$\bigoplus_{\lambda=1}^{h^+} S_{\mathbf{k}}(\Gamma_\lambda) \to S_{\mathbf{k}}(\mathfrak{c})$$
$$(f_1, \dots, f_{h^+}) \mapsto \mathbf{f}$$

is an isomorphism of complex vector spaces.

In our general setting, we will denote by $\mathcal{A}^{\mathbf{k}}(G, K_f)$ (resp. $\mathcal{A}_0^{\mathbf{k}}(G, K_f)$) the space of all automorphic (resp. cusp) forms. They satisfy the following results, which we refer to [**30**, Chap. 4] and [**6**] for proofs.

THEOREM 1.29. *The space $\mathcal{A}^{\mathbf{k}}(G, K_f)$ is* finite *dimensional and every element in $\mathcal{A}_0^{\mathbf{k}}(G, K_f)$ is square-integrable.*

Again, let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{GL}_2)$ and $\Phi \in \mathcal{A}_0(G, K_f)$ be a cusp form. We recall that the standard additive character τ_0 of $A_{\mathbb{Q}}$ is given by $\tau_0 = \prod_{\mathfrak{p}} \tau_{\mathfrak{p}}$, where

$$\tau_\infty(x_\infty) = e^{2\pi i x_\infty},$$
$$\tau_{\mathfrak{p}}(x) = e^{-2\pi i y}, \quad \text{where } x = y + z,\ y \in \mathbb{Z}[1/p],\ z \in \mathbb{Z}_p,$$

at each prime p. The standard additive character is obtained by composing τ_0 with the trace map $\mathbb{A} = \mathbb{A}_F \to \mathbb{A}_{\mathbb{Q}}$. Then, any character of $\mathbb{A}/F$ is of the form $\tau_\xi : x \mapsto \tau(\xi x)$, $\xi \in F$. For almost all $g \in G(\mathbb{A})$,

$$x \mapsto \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$$

is a square-integrable function on $\mathbb{A}/F$. Hence, it admits the Fourier expansion:

$$\Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\xi \in F} W_\xi(g) \tau(\xi x),$$

where

$$W_\xi(g) = \int_{\mathbb{A}/F} \bar{\tau}(\xi u)\Phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) du.$$

Definition 1.30. *We call the function* $W_\xi : g \mapsto W_\xi(g)$ *the* ξ^{th}*-Fourier coefficient of* Φ.

Take $\mathbf{f} \in S_{\mathbf{k}}(\mathfrak{c})$ and let $(f_1, \dots, f_{h^+})$ be the corresponding h^+–tuple of classical Hilbert modular forms. Each component f_λ has a q-expansion

$$f_\lambda(z) = \sum a_\lambda(\xi) e^{2\pi i \mathrm{Tr}(\xi z)} \qquad (\xi = 0 \text{ or } 0 \ll \xi \in i(t_\lambda)),$$

where $i(t)$ is the ideal corresponding to the idèle t. Since $f_\lambda(\varepsilon z)\varepsilon^{\mathbf{k}/2} = f_\lambda(z)$ for $\varepsilon \in \mathcal{O}_F^{\times +}$, we see that $a_\lambda(\xi)\xi^{-\mathbf{k}/2}$, for $\xi \neq 0$, depends only on the ideal $i(\xi)$. Now, by definition of the t_λ, every non-zero integral ideal of F is of the form $i(\xi t_\lambda^{-1})$ for some unique λ and some totally positive $\xi \in i(t_\lambda)$. We define $c(\mathfrak{a}, \mathbf{f})$ by

$$c(\mathfrak{a}, \mathbf{f}) = \begin{cases} a_\lambda(\xi) t_\lambda^{-\mathbf{k}/2} & \text{if } \mathfrak{a} = i(\xi t_\lambda^{-1}) \text{ is integral} \\ 0 & \text{if not.} \end{cases}$$

By mimicking the proof of Lemma 3. 6 in [**25**, Gelbart], we get that the q-expansion of the cusp form $\mathbf{f}$ is given by

$$W_\xi\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\right) = \begin{cases} c(\mathfrak{a}, \mathbf{f}) e^{-2\pi \mathrm{Tr}(\xi y_\infty)} & \text{if } \mathfrak{a} = i(\xi) i(t_\lambda)^{-1} \text{ is integral} \\ 0 & \text{otherwise,} \end{cases}$$

$\xi \gg 0, y_\infty \gg 0$.

The Dirichlet L–series associated to $\mathbf{f}$ is

$$L(\mathbf{f}, s) = \sum_{\mathfrak{a}} \frac{c(\mathfrak{a}, \mathbf{f})}{N(\mathfrak{a})^s}.$$

It converges for $\mathrm{Re}(s)$ large enough.

The Petersson inner product on $S_{\mathbf{k}}(\mathfrak{c})$ is then given by

$$\langle \mathbf{f},\, \mathbf{g}\rangle = \sum_{\lambda=1}^{h^+} \langle f_\lambda,\, g_\lambda\rangle.$$

4. Hecke operators.

We refer the reader to Shimura [**48**, Chap. 3] and [**47**, sec. 2], also to Miyake [**38**, Chap. 2, sec. 2], which contain a very nice description of the local Hecke algebras and their relation to the global one. Though the discussion in Miyake [**38**] is over $\mathbb{Q}_p$, it still applies to any p-adic field. In our discussion, we assume that all modular forms have parallel weight, i.e. $\mathbf{k} = (k, \ldots, k)$. (See [**48**] for more details on the non-parallel case). The Hecke algebra, defined by the group $K_0 = K_0(\mathfrak{c})$ and the semi-group $Y = Y(\mathfrak{c})$ consists of formal finite sums $\sum_{y\in Y} c_y K_0 y K_0$, with $y \in Y$ and $c_y \in \mathbb{C}$. The multiplication law is defined as in [**48**, Chap. 3, sec. 1]. It acts on $M_k(\mathfrak{c})$ as follows: for $y \in Y$ with $K_0 y K_0 = \bigcup_j K_0 y_j$, where $(y_j)_\infty = 1$,

$$(\mathbf{f}|K_0 y K_0)(x) = \sum_j \mathbf{f}(x y_j), \qquad x \in G(\mathbb{A}).$$

This is independent of the choice of $y_j's$. For each integral ideal $\mathfrak{a}$ of F, we define the (normalized) Hecke operator $T_{\mathfrak{c}}(\mathfrak{a})$ on $\mathbf{f} \in M_k(\mathfrak{c})$ by

$$T_{\mathfrak{c}}(\mathfrak{a})\mathbf{f} = N(\mathfrak{a})^{k/2-1} \sum_{\det(i(y))=\mathfrak{a}} (\mathbf{f}|K_0 y K_0),$$

where the sum is taken over all distinct double cosets $K_0 y K_0$ for which $\det(i(y)) = \mathfrak{a}$. Also, for eah integral ideal $\mathfrak{a}$ of K prime to $\mathfrak{c}$, we define the operator $S_{\mathfrak{c}}(\mathfrak{a})$

$$S_{\mathfrak{c}}(\mathfrak{a})\mathbf{f} = \mathbf{f}|K_0 a K_0,$$

where a is any element of $\mathbb{A}^\times$ such that $\mathfrak{a} = i(a)$. For convenience, we set $S_\mathfrak{c}(\mathfrak{a})$ to be zero when $\mathfrak{a}$ is not prime to $\mathfrak{c}$. As for modular forms over $\mathbb{Q}$, $R(K_0, Y)$ is generated by the operators $T_\mathfrak{c}(\mathfrak{p})$ and $S_\mathfrak{c}(\mathfrak{p})$ for all prime ideal $\mathfrak{p}$.

By definition of the multiplication in $R(K_0, Y)$ and as in the one dimensional case (see Theorem 3.24 [**48**, Chap.3, sec. 3]), we have

$$T_\mathfrak{c}(\mathfrak{m})T_\mathfrak{c}(\mathfrak{n}) = \sum_{\mathfrak{m}+\mathfrak{n}\subset\mathfrak{a}} N(\mathfrak{a})^{k-1}S_\mathfrak{c}(\mathfrak{a})T_\mathfrak{c}(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n});$$

we also have the formal Euler product (see [**47**] and [**48**, Chap. 3, sec.3])

$$\sum_{\mathfrak{a}} T_\mathfrak{c}(\mathfrak{a})N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \left(1 - T_\mathfrak{c}(\mathfrak{p})N(\mathfrak{p}^{-s}) + S_\mathfrak{c}(\mathfrak{p})N(\mathfrak{p})^{k-1-2s}\right)^{-1}.$$

The Hecke operators act on the Fourier coefficients of $\mathbf{f}$ by

$$c(\mathfrak{m}, T_\mathfrak{c}(\mathfrak{a})\mathbf{f}) = \sum_{\mathfrak{m}+\mathfrak{n}\subset\mathfrak{a}} N(\mathfrak{a})^{k-1}c(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, S_\mathfrak{c}(\mathfrak{a})\mathbf{f}).$$

Since, the Hecke operators commute and are self-adjoint with respect to the Peterson inner product, they have a common basis of eigenvectors. If $\mathbf{f}$ is an eigenform, the corresponding eigenvalues are the $c(\mathfrak{a}, \mathbf{f})$. We say that $\mathbf{f}$ is normalized if $c(\mathcal{O}_F, \mathbf{f}) = 1$. When $\mathbf{f}$ is a normalized eigenform, its Dirichlet L–series has a formal Euler product

$$L(\mathbf{f}, s) = \prod_{\mathfrak{p}} (1 - c(\mathfrak{p}, \mathbf{f})N(\mathfrak{p})^{-s} + N(\mathfrak{p})^{k-1-2s})^{-1}.$$

PROPOSITION 1.31. *The eigenvalues of $T_\mathfrak{c}(\mathfrak{a})$ are algebraic integers.*

PROOF. See [**47**, Prop. 2.2]. ☐

5. Automorphic representations.

We refer the reader to Jacquet–Langlands [**32**, Chap. I & II], Godement [**26**] or Gelbart [**25**, Chap. II & III], for the definition of admissible representations and the local and global Hecke theory; there is also a very nice survey of representations of p–adic groups by Cartier in [**9**]. We let $\mathcal{H}_{\mathfrak{p}}$ be the local Hecke algebra at the prime $\mathfrak{p}$ and $\mathcal{H}_{\mathbb{A}}$ the global one (which is obtained as the restricted tensor product of the local ones; see Flath [**23**]).

Definition 1.32. *Let Φ be in $\mathcal{A}^{\mathbf{k}}(G,\, K_f)$. We say that W_Φ is an* automorphic *representation if it is irreducible. If further, $\Phi \in \mathcal{A}_0^{\mathbf{k}}(G,\, K_f)$, we say that W_Φ is* cuspidal.

Remark 1.33. We only consider automorphic repesentations associated with the trivial character. By Theorem 4.5 [**6**], such a representation is admissible. Also, since we are only interested in *unitary* admissible representations, we will not distinguished between such representations and their corresponding $\mathcal{H}_{\mathbb{A}}$–modules; indeed, the functor that sends the isomorphism class of a unitary admissible $G(\mathbb{A})$–module to the one of the corresponding $\mathcal{H}_{\mathbb{A}}$–module is *full* and *faithful*.

A natural question is the understanding of the map $\Phi \mapsto W_\Phi$. This is very well–understood in the case of $\mathbf{GL}_2$. We will restrict ourselves to this case in the rest of this section and refer to [**25**, Chap. 5, sec. A] and [**32**, Chap. 2, sec. 10], for more on that subject and for the following theorem which make use of the uniqueness of the Whittaker model.

THEOREM 1.34 (Multiplicity one). *The representation R_0 decomposes into a countable direct (Hilbert) sum of cuspidal representations:*

$$R_0 = \bigoplus_{\alpha} W_\alpha.$$

In this decomposition, each W_α has multiplicity one *(i.e., up to isomorphism, W_α appears only once). Moreover, for each α, β, if there is a finite set of places S such that for each $\mathfrak{p} \notin S$, $W_{\alpha,\mathfrak{p}} \cong W_{\beta,\mathfrak{p}}$, then $W_\alpha = W_\beta$.*

Take $\Phi \in \mathcal{A}_0^{\mathbf{k}}(G, K_f)$ and assume that the corresponding representation π_Φ is cuspidal. Then, by Theorem 4 in Flath [**23**], it factorizes as $\pi_\Phi = \otimes \pi_\mathfrak{p}$. The characterization of cuspidal representations is then purely local and we need the following few defintions to start.

DEFINITION 1.35. *Let π be an* infinite *dimensional irreducible admissible representation of $G_\mathfrak{p} = \mathrm{GL}_2(F_\mathfrak{p})$ at a finite prime $\mathfrak{p}$ on the vector space V. We let $\pi_\mathfrak{p}$ be a uniformizer at $\mathfrak{p}$, and ω_π the central character of π. The* conductor $\mathfrak{c}_\pi$ *of π is the largest ideal $\mathfrak{c}$ of $\mathcal{O}_\mathfrak{p}$ such that $V^{K_1(\mathfrak{c})} \neq 0$, where*

$$K_1(\mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) |\ c,\, d-1 \in \mathfrak{c} \right\}.$$

We say that π is unramified *(or* class 1 *or* special*) if $\mathfrak{c}_\pi = \mathcal{O}_\mathfrak{p}$.*

Here is the list of possible conductors for a p-adic field (see Gelbart [**25**, Remark 4.25]):

- If $\pi = \pi(\chi_1, \chi_2)$ is in the principal series, then $\mathfrak{c}_\pi = \mathfrak{f}_1\mathfrak{f}_2$, where $\mathfrak{f}_i$, $i = 1, 2$ is the conductor of χ_i.

- If $\pi = \pi(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ is special, then $\mathfrak{c}_\pi = \mathfrak{f}^2 \cap \pi\mathcal{O}_\mathfrak{p}$, where $\mathfrak{f}$ is the conductor of χ.
- If π is supercuspidal, then $\mathfrak{c}_\pi = \pi^n\mathcal{O}_\mathfrak{p}$, for some $n \geq 2$.

Note that $\mathfrak{c}_\pi$ is divisible by the conductor of ω_π and we have the following theorem.

THEOREM 1.36. ([**10**, Theorem 1]) *For $\mathfrak{c} = \mathfrak{c}_\pi$, the space $V^{K_1(\mathfrak{c})}$ is one-dimensional and*

$$v \in V^{K_1(\mathfrak{c})} \Leftrightarrow \pi(g)v = \omega_\pi(d)v, \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{c}),$$

where

$$K_0(\mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) | c \in \mathfrak{c} \right\}.$$

DEFINITION 1.37. *The vector $v \in V$,* unique up to constant, *invariant under $K_1(\mathfrak{c})$, with $\mathfrak{c} = \mathfrak{c}_\pi$, is called the* new vector *of π.*

Piecing these together, we get the notion of a new form.

DEFINITION 1.38. *Let W be a cuspidal representation of weight $\mathbf{k}$. Then, it is an irreducible admissible representation of G_f. Therefore, it factors as $\pi = \otimes\pi_\mathfrak{p}$. For each finite prime $\mathfrak{p}$, let $v_\mathfrak{p}$ be a new vector of $\pi_\mathfrak{p}$. We assume that, for almost all $\mathfrak{p}$, $v_\mathfrak{p}$ is equal to the vector we have used to define the restricted tensor product. We say that $v = \otimes v_\mathfrak{p}$, defined up to a constant, is the* new vector *of W. Let Φ be in $\mathcal{A}_0^\mathbf{k}(G, K_f)$, and* $W_\Phi = \mathrm{Span}(\Phi \cdot G_f)$. *We say that Φ is a* new form *if W_Φ is irreducible and Φ is a new vector for W_Φ. The* conductor *of Φ is then the ideal $\mathfrak{c} = \prod_\mathfrak{p} \pi_\mathfrak{p}^{f_\mathfrak{p}}$, $f_\mathfrak{p}$ being the conductor of $W_{\Phi,\mathfrak{p}}$.*

REMARK 1.39. By Theorem 2 in Flath [**23**] and the discussion following it, almost every $W_{\mathfrak{p}}$ is unramified; therefore the new vector is well-defined up to a constant. Further we see that, by the multiplicity one theorem, $W_\Phi^{K_1(\mathfrak{c})}$ must be one-dimensional; which means that Φ is an eigenform.

We denote by $\mathcal{A}_0^{\mathbf{k},\,new}(G,\,K_f)$ the subspace of new forms in $\mathcal{A}_0^{\mathbf{k}}(G,\,K_f)$.

THEOREM 1.40. [**16**, Deligne, Scholie 2.4.2] *The map $\Phi \mapsto W_\Phi$, which associates to each cusp form the irreducible G_f-module it generates, is an injection of $\mathcal{A}_0^{\mathbf{k},\,new}(G,\,K_f)$ into the set of isomorphism classes of cuspidal representations.*

When $K_f = K_0 = K_0(\mathfrak{c})$, the holomorphic newforms are just $S_{\mathbf{k}}^{new}(\mathfrak{c}) = S_{\mathbf{k}}(\mathfrak{c}) \cap \mathcal{A}_0^{\mathbf{k},\,new}(G,\,K_0)$. For a more classical definition of newforms, see Shimura [**47**, sec. 2, Prop. 3].

6. The Jacquet-Langlands correspondence.

The last step in building the bridge between the new forms on the algebraic group $G' = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{B}^\times$ and the ones on $G = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{GL}_2$ is making explicit the correspondence between automorphic representations on $G'_{\mathbb{A}}$ and on $G_{\mathbb{A}}$. Let $\mathrm{Ram}(\mathrm{B})$ be the set of places at which B is ramified. We take a cusp form π' on $\mathrm{B}_{\mathbb{A}}^\times$, and let $\pi' = \otimes\pi'_{\mathfrak{p}}$ be its factorization. If $\mathfrak{p} \notin \mathrm{Ram}(\mathrm{B})$, we fix an isomorphism $\mathrm{B}_{\mathfrak{p}} = \mathrm{M}_2(F_{\mathfrak{p}})$ such that the maximal compact open subgroup $K'_{\mathfrak{p}}$ we have fixed in defining the Hecke algebra of B goes to $K_{\mathfrak{p}} = \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$. We can carry the representation $\pi'_{\mathfrak{p}}$ to a representation $\pi_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ via this isomorphism, and by construction, those two representations will be isomorphic as Hecke modules. Although our choice of isomorphism has been arbitrary, it will not affect the construction we will be making (see [**32**]). It remains now to take care of the local representations at places in Ram(B). This is done by making use of the Weil representations (see Jacquet-Langlands [**32**] and Gelbart [**25**], for more details). By taking the restricted tensor product of the local representations we just obtained, we get an admissible irreducible representation $\pi = JL(\pi')$. That $\pi = JL(\pi')$ is automorphic is a consequence of the celebrated Jacquet-Langlands correspondence.

THEOREM 1.41. *The map* $\pi' \mapsto \pi = JL(\pi')$ *is an injection of the irreducible constituents of* $\mathcal{A}^{\mathbf{k}}(G', K')$ *into* $\mathcal{A}_0^{\mathbf{k},new}(G, K)$. *Its image consists of the representations* π *with* $\pi_{\mathfrak{p}}$ *square-integrable for* $\mathfrak{p} \in \mathrm{Ram}(\mathrm{B})$.

PROOF. The proof of this result relies essentially on the trace formula and we refer to [**25**, Chap. 9 and 10] for a detailed treatment. □

7. Applications.

We would like to compute the coefficients $a_{\mathfrak{p}}$ ($\mathrm{N}(\mathfrak{p}) \leq 100$) of all the (normalized) eigenvectors of the $T_{\mathfrak{p}}$ acting on $S_2(\mathfrak{c})$ with $\mathrm{N}(\mathfrak{c}) \leq 500$ on $F = \mathbb{Q}(\sqrt{5})$. To this end, let us consider the (unique, up to isomorphism) totally definite quaternion algebra B over F, which is unramified at all finite places. The algebra B can be identified with the standard Hamilton quaternion algebra, since 2 is inert in F:

$$\mathrm{B} = \{x + yi + zj + wk, \quad x,\, y,\, z,\, w \in F\}\,.$$

As we have seen in Example 1.19, the class number of B, hence the type number is one. Every maximal order in B is then conjugate to the icosian ring

$$\mathrm{R} = \mathbb{Z}[\omega][e_1,\, e_2,\, e_3,\, e_4],$$

whose group of units $\mathrm{R}^\times$ is isomorphic to the semi-direct product of $\mathbb{Z}$ with the binary icosahedral group of order 120. (Cf. [**13**, Chap. 8, sec. 2.1.]) For any prime $\mathfrak{p}$, we fix an isomorphism $\mathrm{B}_\mathfrak{p} = M_2(F_\mathfrak{p})$ so that $\mathrm{R}_\mathfrak{p} = M_2(\mathcal{O}_{F,\,\mathfrak{p}})$. Now, let $\mathrm{R}_\mathfrak{c}$ be an Eichler order of level $\mathfrak{c}$ in R, and write

$$\hat{\mathrm{R}}_\mathfrak{c} = \mathrm{R}_\mathfrak{c} \otimes \hat{\mathcal{O}}_F, \qquad \hat{\mathrm{R}} = \mathrm{R} \otimes \hat{\mathcal{O}}_F \qquad \text{and} \qquad \hat{\mathrm{B}} = \mathrm{B} \otimes \hat{\mathcal{O}}_F.$$

We recall that $\hat{\mathrm{R}}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times$ parametrizes the set of left ideal classes of R. Thus, since B has class number one,

$$\hat{\mathrm{R}}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times = \{\hat{\mathrm{R}}^\times \mathrm{B}^\times\}, \quad \text{and} \quad \hat{\mathrm{B}}^\times / \mathrm{B}^\times = \hat{\mathrm{R}}^\times / \mathrm{R}^\times.$$

Hence

$$\begin{aligned} \hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times &= \hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{R}}^\times / \mathrm{R}^\times = \left(\prod_{\mathfrak{q} \mid \mathfrak{c}} \mathrm{R}_{\mathfrak{c},\,\mathfrak{q}}^\times \backslash \mathrm{R}_\mathfrak{q}^\times \right) / \mathrm{R}^\times \\ &= \left(\prod_{\mathfrak{q} \mid \mathfrak{c}} \mathbf{P}^1(\mathcal{O}_\mathfrak{q} / \mathfrak{q}^{e_\mathfrak{q}}) \right) / \mathrm{R}^\times, \end{aligned}$$

where $\mathfrak{c} = \prod_{\mathfrak{q} \mid \mathfrak{c}} \mathfrak{q}^{e_\mathfrak{q}}$ and

$$\mathbf{P}^1(A) = \left\{ (a,\, b) \in A^2 : \ \alpha a + \beta b = 1 \text{ for some } (\alpha,\, \beta) \in A^2 \right\} / A^\times,$$

for any ring A. This is then a finite set, and $L^2(\hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times)$ is the vector space of $\mathbb{C}$-valued maps defined on it. By the Jacquet-Langlands correspondence, $S_2(\mathfrak{c})$ is isomorphic as a Hecke module to the complex space

$$L^2(\hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times),$$

on which the Hecke operators act in the standard way. To carry out our computations, we will explain a bit more how this action goes. This will lead us to the theory of Brandt matrices in the case of algebraic modular forms (as in Gross [**28**] and Khuri–Makdisi [**33**]). Indeed, in the spirit of Gross [**28**], $L^2(\hat{\mathrm{R}}_{\mathfrak{c}}^{\times}\backslash\hat{\mathrm{B}}^{\times}/\mathrm{B}^{\times})$ can be identified with the space of algebraic modular forms on $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{B}^{\times}/F^{\times})$ of level the compact open subgroup with $K = \hat{\mathrm{R}}_{\mathfrak{c}}^{\times}$ with values in the trivial representation $V = \mathbb{C}$. To make this more precise as a statement, let us recall some definitions and facts. We assume for a while that G is a connected reductive group defined over $\mathbb{Q}$ such that $G(\mathbb{R})$ is *compact*. We let V be a (finite) dimensional complex representation of $G(\mathbb{R})$ and fix a compact open subgroup K of $G_f = G(\mathbb{A}_f)$.

Definition 1.42. [**33**] *The space of* algebraic modular forms *of* level K *and* weight V *on* G *is* $M(V, K) := \mathrm{Hom}_{G(\mathbb{R})}(V, L^2(K\backslash G_{\mathbb{A}}/G(\mathbb{Q})))$.

By Proposition 8.3 in [**28**] (see also [**33**, Lemma 2.2]), $M(V, K)$ can be identified with the complex space of all functions $f : K\backslash G_{\mathbb{A}}/G(\mathbb{Q}) \to V^*$ such that $f(g_\infty g) = g_\infty^{-1} f(g)$ for all $g_\infty \in G(\mathbb{R})$. This is in fact an isomorphism of Hecke modules.

Proposition 1.43. *Let $(g_i)_{1\le i\le k}$ be a complete set of representatives of $K\backslash G_f/G(\mathbb{Q})$. Define $\Gamma_i = G(\mathbb{Q})\cap g_i^{-1}Kg_i$. Then Γ_i is a finite group, as it embeds discretely into the compact group $G(\mathbb{R})$. Let V^{Γ_i} be the subspace of all Γ_i-invariant vectors in V. Then the map*

$$\begin{array}{rcl} M(V, K) & \to & \bigoplus_{i=1}^{k} V^{\Gamma_i} \\ f & \mapsto & (f(g_i))_{1\le i\le k} \end{array}$$

is an isomorphism of complex spaces.

PROOF. The proof essentially uses the fact that each element in $M(V, K)$ viewed as a functions on the double coset space $K\backslash G_{\mathbb{A}}/G(\mathbb{Q})$ is completely determined by its values on the g_i; see [**28**, Proposition 4.5] or [**33**, Lemma 2.3]. □

By letting $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{B}^\times/F^\times)$ and K the image of $\hat{\mathrm{R}}_{\mathfrak{c}}^\times$ by the projection $\hat{B}^\times \to G_f$, and $V = \mathbb{C}$ the trivial representation, we see that $L^2(\hat{\mathrm{R}}_{\mathfrak{c}}^\times\backslash\hat{\mathrm{B}}^\times/\mathrm{B}^\times)$ is nothing else but the space of all algebraic modular forms on G of level K and trivial weight as we claimed before. The definition of Brandt matrices for the action of Hecke operators on such forms has been given by Khuri–Makdisi in the case when V is non-trivial. In what follows, we will content ourselves with recalling that definition only for algebraic modular forms of weight the trivial representation and refer to Khuri–Makdisi [**33**, sec. 3] for the general case. So, let $(g_i)_{1\le i\le k}$ be a complete set of representatives for the classes in $\hat{\mathrm{R}}_{\mathfrak{c}}^\times\backslash\hat{\mathrm{B}}^\times/\mathrm{B}^\times$, and recall that we have an isomorphism of complex spaces:

$$\begin{array}{rcl} L^2(\hat{\mathrm{R}}_{\mathfrak{c}}^\times\backslash\hat{\mathrm{B}}^\times/\mathrm{B}^\times) & \to & \mathbb{C}^k \\ \mathbf{f} & \mapsto & (\mathbf{f}(g_i))_{1\le i\le k}. \end{array}$$

Also recall that in this setting, the definition of Hecke operators is as follows: take $t \in G_f$ and write the double coset as a disjoint union $KtK = \cup_{i=1}^s Kt_i$. Then,

$$\mathbf{f}\|KtK(g) = \sum_{i=1}^s \mathbf{f}(t_i g), \qquad \text{for } g \in G_{\mathbb{A}}.$$

Viewing every element in $L^2(\hat{\mathrm{R}}_{\mathfrak{c}}^\times\backslash\hat{\mathrm{B}}^\times/\mathrm{B}^\times)$ as a column vector via the isomorphism above, we see that the action of t on $L^2(\hat{\mathrm{R}}_{\mathfrak{c}}^\times\backslash\hat{\mathrm{B}}^\times/\mathrm{B}^\times)$ is then given by a matrix $\mathcal{B}_t$, called the *Brandt matrix* of t. We see that it is independent on both sets of coset representatives $(g_i)_{1\le i\le k}$ and

$(t_j)_{1\leq j\leq s}$. For a prime $\mathfrak{p} \in \mathcal{O}_F$, we will simply denote by $\mathcal{B}_\mathfrak{p} = (b_{ij})$ the Brandt matrix of $T_\mathfrak{p}$. When B is unramified at $\mathfrak{p}$ there is a well-known alternative description of the $\mathcal{B}_\mathfrak{p} = (b_{ij})$. Indeed, fix an isomorphism $\mathrm{B}_\mathfrak{p} = \mathrm{M}_2(F_\mathfrak{p})$ such that $\mathrm{R}_\mathfrak{p} = \mathrm{M}_2(\mathcal{O}_\mathfrak{p})$, and let $t_\mathfrak{p} \in \mathrm{R}_\mathfrak{p}$ correspond to $\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}$, where $\pi_\mathfrak{p}$ is a uniformizer of $\mathcal{O}_\mathfrak{p}$. We can view $t_\mathfrak{p}$ as an element of $\hat{\mathrm{B}}^\times$ by letting all its other entries be equal to the identity matrix in the corresponding $\mathrm{B}_\mathfrak{q}$'s. Then, the following proposition is standard.

THEOREM 1.44. [**37**, Pizer] *Let $\mathfrak{p}$ be a prime at which* B *splits such that $(\mathfrak{p}, \mathfrak{c}) = 1$, and let $I_1, \ldots, I_k$ be a complete set of representatives of the ideal classes in $\hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times$ so that I_i corresponds to g_i. Then b_{ij} is the number of subideals $L_\nu \subset I_i$ satisfying the following conditions:*
(i) $I_i/L_\nu \cong (\mathcal{O}_\mathfrak{p}/\pi_\mathfrak{p}\mathcal{O}_\mathfrak{p})^2$,
(ii) L_ν is in the same ideal class as I_j; so that $L_\nu = \gamma_\nu I_j$ for some $\gamma_\nu \in \mathrm{B}^\times$.

The computation of the Brandt matrices exploits this theorem and the duality between the action of Hecke operators on $L^2(\hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times)$ and on the free group on $X = \hat{\mathrm{R}}_\mathfrak{c}^\times \backslash \hat{\mathrm{B}}^\times / \mathrm{B}^\times$. Indeed, let us use the improper notation that $\mathrm{Div}(X)$ equals the free abelian group on X. We define the map

$$\begin{aligned} S_2(\mathfrak{c}) \times X &\to \mathbb{C} \\ (\mathbf{f}, x) &\mapsto \langle \mathbf{f}, x\rangle = \mathbf{f}(x), \quad x \in X, \end{aligned}$$

that we extend linearly in the second factor to $\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{C}$. One easily sees that this is a perfect pairing and by definition of Hecke operators,

$$\langle \mathbf{f}\|T,\, D\rangle = \langle \mathbf{f},\, D\|T\rangle,$$

for all $\mathbf{f} \in S_2(\mathfrak{c})$, $D \in \mathrm{Div}(X)$ and $T \in R(K_0(\mathfrak{c}), Y)$. Further, the action of Hecke on $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ preserves the lattice $\mathrm{Div}(X)$ by definition. Therefore, it is enough to compute the action of Hecke on $\mathrm{Div}(X)$. The resulting matrices are then the tranposes of the Brandt matrices. The action of Hecke on $\mathrm{Div}(X)$ is described in several places. We refer to Gross [**28**, sec. 4] and Bertolini and Darmon [**4**, sec. 1.5] where this is done in the setting of quaternion algebras over $\mathbb{Q}$. If B splits at the prime $\mathfrak{p}$ such that $(\mathfrak{p},\, \mathfrak{c}) = 1$, then

$$(\mathrm{R}_{\mathfrak{c}} \otimes \mathcal{O}_{\mathfrak{p}})^{\times}\backslash \mathrm{B}_{\mathfrak{p}}^{\times}/F_{\mathfrak{p}}^{\times} \cong \mathbf{PGL}_2(\mathcal{O}_{\mathfrak{p}})\backslash\mathbf{PGL}_2(F_{\mathfrak{p}})$$

is the Bruhat-Tits tree of $\mathbf{PGL}_2(F_{\mathfrak{p}})$ whose vertices correspond to similarity classes of $\mathcal{O}_{\mathfrak{p}}$-lattices in $F_{\mathfrak{p}}^2$. This is a homogeneous tree of degree $\mathrm{N}(\mathfrak{p})+1$. Let $g \in \mathbf{PGL}_2(\mathcal{O}_{\mathfrak{p}})\backslash\mathbf{PGL}_2(F_{\mathfrak{p}})$ corresponds to the lattice L. The neighbours of g which determine all sublattices of L of index $\mathfrak{p}$, are given by

$$g_i = \begin{pmatrix} 1 & i \\ 0 & \pi_{\mathfrak{p}} \end{pmatrix} g,\ i \in \mathbb{F}_{\mathfrak{p}}, \quad g_{\infty} = \begin{pmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} g.$$

To get the action of $T_{\mathfrak{p}}$ on $\mathrm{Div}(X)$, we then take $\mathrm{N}(\mathfrak{p})+1$ global elements γ_i, $i \in \mathbb{F}_{\mathfrak{p}}$ and γ_{∞} in R such that $g_i = \gamma_i g$, $i \in \mathbb{F}_{\mathfrak{p}}$ and $g_{\infty} = \gamma_{\infty} g$. Then, by Theorem 1.44,

$$T_{\mathfrak{p}}.x = \sum_{i\in\mathbb{F}_{\mathfrak{p}}} \gamma_i.x + g_{\infty}x, \qquad \text{for } x \in X.$$

When $\mathfrak{p}|\mathfrak{c}$, the operator $S_{\mathfrak{p}}$ is described in a similar way. Practically, we let the elements γ_i, $i \in \mathbb{F}_{\mathfrak{p}}$ and γ_∞ act on the projective space $\prod_{\mathfrak{q}|\mathfrak{c}} \mathbf{P}^1(\mathcal{O}_{\mathfrak{q}}/\mathfrak{q}^{e_{\mathfrak{q}}})$. We then take the summation over the elements which are not equivalent to $(0, 0)$ under the action of $\mathrm{R}^\times$. To do the computations, we find a fundamental domain S of the action of $\mathrm{R}^\times$ on $\prod_{\mathfrak{q}|\mathfrak{c}} \mathbf{P}^1(\mathcal{O}_{\mathfrak{q}}/\mathfrak{q}^{e_{\mathfrak{q}}})$. We then let R act on S by embedding it diagonally into $\hat{\mathrm{R}}$.

Algorithms.

1) **Generating the icosian group:** We could have used a set of generators for the icosian group and their relations (cf. [**13**, Chap. 8, sec. 2.1] to generate this group. Instead, we have chosen to find all 4–tuples in $\mathbb{Z}[\omega]$ such that the quaternion $q = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$ has reduced norm 1 (see example 1.19).

2) **Finding a fundamental domain:** We first find a set of representatives of the space $\mathbf{P}^1(\mathcal{O}/\mathfrak{c})$. We have chosen to work with the product

$$\mathbf{P}^1(\mathcal{O}/\mathfrak{c}) = \prod_{\mathfrak{p}|\mathfrak{c}} \mathbf{P}^1(\mathcal{O}/\mathfrak{p}^{e_{\mathfrak{p}}}).$$

Then the coset representatives for each local factor $\mathbf{P}^1(\mathcal{O}/\mathfrak{p}^{e_{\mathfrak{p}}})$ are taken to be all couples

$$(1,\, a) \qquad a \in \mathfrak{p}/\mathfrak{p}^{e_{\mathfrak{p}}} \qquad \text{and} \qquad (a,\, 1) \qquad a \in \mathcal{O}/\mathfrak{p}^{e_{\mathfrak{p}}}.$$

We let the icosian group $\mathrm{R}^\times$ acts on this set of coset representatives. This is done via the local isomorphism $\mathrm{R}_{\mathfrak{p}} = \mathrm{M}_2(\mathcal{O}_{\mathfrak{p}})$ which reduces to

$$\mathrm{R} \otimes (\mathcal{O}_F/\mathfrak{p}^{e_{\mathfrak{p}}}) = \mathrm{M}_2(\mathcal{O}_F/\mathfrak{p}^{e_{\mathfrak{p}}}).$$

Note that by Hensel's Lemma, we only need find that reduced isomorphism. This amounts to finding a set of generators of $\mathrm{M}_2(\mathcal{O}_F/\mathfrak{p}^{e_\mathfrak{p}})$ which satisfies the appropriate relations corresponding to the basis we have chosen for $\mathrm{R}_\mathfrak{p}$. This determines a fundamental domain for this action.

3) **Generating the Hecke operators:** Let $\mathfrak{p}$ be a prime in $F = \mathbb{Q}(\sqrt{5})$. To compute $T_\mathfrak{p}$, it is enough to find coset representatives of reduced norm (a uniformizer of) $\mathfrak{p}$ in the form

$$q = x + yi + zj + wk \quad \text{with } x,\, y,\, z,\, w \in \mathbb{Z}[\omega].$$

To this end, we again consider the local isomorphism $\mathrm{R}_\mathfrak{p} = \mathrm{M}_2(\mathcal{O}_\mathfrak{p})$ and reduce it to $\mathrm{R} \otimes \mathbb{F}_\mathfrak{p} = \mathrm{M}_2(\mathbb{F}_\mathfrak{p})$. We then consider the Eichler order in $\mathrm{M}_2(\mathcal{O}_F)$ whose reduction consists of all upper triangular matrices. We use the fact that $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ acts transitively on $\mathbf{P}^1(\mathbb{F}_\mathfrak{p})$ and that the stabilizer of ∞ under this action is the group of invertible elements in our fixed Eichler order. Our algorithm consists in finding global elements $q \in \mathrm{R}$ of reduced norm $\mathfrak{p}$ such that $q'q^{-1} \in \mathrm{R}_\mathfrak{p}^\times$ reduces to a non upper triangular element. This could have been much easier and faster if we were able to prove that R is Euclidean. Which we couldn't do, so we have chosen instead to represents the quadratic form $\mathrm{n}(q) = x^2 + y^2 + z^2 + w^2$; so our algorithm is $O(\mathrm{N}(\mathfrak{p})^2)$ time. If $p \in \mathbb{Z}$ is a prime that splits, we only stored the cosets representatives corresponding to $T_\mathfrak{p}$ since we can get $T_{\bar{\mathfrak{p}}}$ by Galois conjugation.

Conventions:

- Let p be a prime in $\mathbb{Z}$ which splits in $\mathcal{O}_F = \mathbb{Z}[\omega]$. We have a canonical isomorphism $\mathcal{O}_F = \mathbb{Z}[x]/(x^2 - x - 1)$, and

$$\mathcal{O}_F \otimes \mathbb{F}_p = \mathbb{F}_p[x]/(x^2 - x - 1) = \mathbb{F}[x]/(x - \alpha) \times \mathbb{F}[x]/(x - \beta),$$

  where α is the reduction of ω modulo p and β the one of $\bar{\omega}$. We denote by pA the prime which reduces to $(x - \alpha)$, and pB the one which reduces to $(x - \beta)$.
- For quadratic fields, we use the notations

$$\omega_D = \begin{cases} \sqrt{D} & \text{if } D \neq 1 \mod 4 \\ \frac{1+\sqrt{D}}{2} & \text{if } D = 1 \mod 4. \end{cases}$$

- The prime $\mathfrak{p}$ has norm $2 \leq \mathrm{N}(\mathfrak{p}) \leq 100$.
- The ideal of smallest norm above which we could find a modular form has norm 31. The level $\mathfrak{c}$ has norm $31 \leq \mathrm{N}(\mathfrak{c}) \leq 500$.
- The first row of the table gives the levels which have been listed in the increasing order of the norm. For each level $\mathfrak{c}$, the corresponding columns give the list of all modular forms of level $\mathfrak{c}$. The listing is done up to Galois conjugation of the levels.
- We have listed separately all levels $\mathfrak{c}$ which have normalized eigenforms with Fourier coefficients generating a field of dimension greater or equal to 3. In such case, the notation $\mathfrak{c} : P_1, \ldots, P_s$ means that, up to Galois conjugation, we have s newforms $\mathbf{f}_1, \ldots, \mathbf{f}_s$ and that the field generated by the Hecke eigenvalues of $\mathbf{f}_i$ is $\mathbb{Q}(\alpha_i)$, where α_i is a root of the irreducible polynomial P_i which is also an Hecke eigenvalue. We then

give the coefficients of each eigenform in a chosen integral basis. We only specify this integral basis when it does not correspond to the canonical power basis $\{1, \alpha_i, \ldots, \alpha_i^{d_i-1}\}$, where $d_i = \deg P_i$.

	$31A$	2.3	$41A$	$3.\sqrt{5}$	7	$\sqrt{5}.11A$	$61A$	2^3	$71A$
$\mathfrak{p}$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$
2	-3	–	-2	-3	0	-1	$-2\omega_5$	–	-1
3	2	–	-4	–	5	-2	$-3+\omega_5$	2	-2
$\sqrt{5}$	-2	-4	-1	–	-4	–	$-2+3\omega_5$	-2	0
7	2	10	-6	-14	–	14	$-2+4\omega_5$	10	-4
$11A$	4	2	-2	-4	-3	–	$-1+\omega_5$	-4	0
$11B$	-4	2	5	-4	-3	0	$2-4\omega_5$	-4	0
$19A$	-4	0	-1	4	0	8	$2-\omega_5$	4	-4
$19B$	4	0	6	4	0	-4	$-3-3\omega_5$	4	2
$29A$	-2	0	9	-2	5	-6	$-4+5\omega_5$	-2	-6
$29B$	-2	0	2	-2	5	6	$4+2\omega_5$	-2	6
$31A$	–	-8	-10	0	2	-4	$4-5\omega_5$	0	2
$31B$	8	-8	4	0	2	8	$6+2\omega_5$	0	8
$41A$	-6	2	–	10	2	-6	$6-2\omega_5$	2	12
$41B$	-6	2	0	10	2	6	$-10+4\omega_5$	2	6
$59A$	12	-10	-3	-4	-10	0	$4-10\omega_5$	12	-12
$59B$	-4	-10	4	-4	-10	-12	$7\omega_5$	12	6
$61A$	6	2	6	-2	-8	2	–	-10	-4
$61B$	-2	2	-8	-2	-8	-10	0	-10	-10
$71A$	0	12	9	-8	-8	0	$2+3\omega_5$	8	–
$71B$	-8	12	-12	-8	-8	0	$-4\omega_5$	8	6
$79A$	0	0	-11	0	5	8	$-8+2\omega_5$	-16	14
$79B$	16	0	-4	-8	5	-4	$8-4\omega_5$	-16	-4
$89A$	-6	10	-8	-6	0	-18	$2+2\omega_5$	-6	18
$89B$	10	10	-1	-6	0	6	$-11+3\omega_5$	-6	6

	$2 \cdot 19A$			$79A$	$2^2.\sqrt{5}$	3^2	$89A$	$\sqrt{5}.19A$	$3.11A$
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$
2	–	–	–	1	–	−1	−1	−1	1
3	−5	1	1	−2	−2	–	4	−2	–
$\sqrt{5}$	1	−3	−3	−2	–	0	0	–	−2
7	0	−4	−4	−2	−10	14	−4	2	2
$11A$	2	−6	3	0	0	0	0	0	–
$11B$	−3	3	−6	−4	0	0	−6	0	−4
$19A$	–	–	–	8	−4	−4	−4	–	4
$19B$	5	−7	−7	4	−4	−4	2	−4	−4
$29A$	5	3	−6	−2	6	0	6	6	6
$29B$	−10	−6	3	6	6	0	6	−6	−2
$31A$	−3	5	5	−8	−4	8	−4	−4	8
$31B$	7	5	5	0	−4	8	−4	8	−8
$41A$	2	6	6	−2	6	0	0	−6	−6
$41B$	2	6	6	2	6	0	6	−6	2
$59A$	10	6	−12	−4	12	0	0	12	12
$59B$	0	−12	6	4	12	0	12	12	12
$61A$	12	8	8	14	2	2	−4	−10	−2
$61B$	−8	8	8	10	2	2	14	14	−2
$71A$	−8	0	−9	12	−12	0	0	0	8
$71B$	7	−9	0	−16	−12	2	12	12	−8
$79A$	15	−1	−1	–	8	−16	2	−16	8
$79B$	5	−1	−1	8	8	−16	−16	8	16
$89A$	0	0	9	18	−6	0	–	−6	2
$89B$	−15	9	0	−14	−6	0	6	6	−14

	$2\cdot\sqrt{5}^2$		$101A$	$109A$	$2.29A$		$11A^2$	11	
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$
2	$-$	$-$	$-1-\omega_2$	$2\omega_5$	$-$	$-$	$-\omega_3$	0	$1-\omega_{33}$
3	-5	5	$-2-2\omega_2$	$1-3\omega_5$	0	-4	-1	-5	$2+\omega_{33}$
$\sqrt{5}$	$-$	$-$	$1+2\omega_2$	$-2\omega_5$	1	-1	ω_3	1	$-2+\omega_{33}$
7	-10	10	$4-2\omega_2$	$2-5\omega_5$	5	1	ω_3	-10	2
$11A$	-3	-3	$-2\omega_2$	$-2-\omega_5$	2	-2	$-$	$-$	-1
$11B$	-3	-3	$1+\omega_2$	$-4+2\omega_5$	-3	5	$2\omega_3$	$-$	-1
$19A$	5	-5	$-1+5\omega_2$	$4+2\omega_5$	-5	-1	-2	0	-4
$19B$	5	-5	$-3-\omega_2$	$-3\omega_5$	0	-8	$-4\omega_3$	0	-4
$29A$	0	0	$1-2\omega_2$	$-4+7\omega_5$	$-$	$-$	$5\omega_3$	0	$-2-2\omega_{33}$
$29B$	0	0	-4	$5-\omega_5$	-5	9	-3	0	$-2-2\omega_{33}$
$31A$	2	2	-2	$1-3\omega_5$	7	-3	$-2\omega_3$	7	ω_{33}
$31B$	2	2	$-3-3\omega_2$	$-3+3\omega_5$	-8	4	$-2\omega_3$	7	ω_{33}
$41A$	-3	-3	$1-4\omega_2$	$6-\omega_5$	-3	-7	9	-8	$2+2\omega_{33}$
$41B$	-3	-3	8	$-6+6\omega_5$	12	0	$-4\omega_3$	-8	$2+2\omega_{33}$
$59A$	0	0	$2-4\omega_2$	$1-5\omega_5$	0	4	-6	5	$4+\omega_{33}$
$59B$	0	0	$4+2\omega_2$	$-4+8\omega_5$	0	4	-6	5	$4+\omega_{33}$
$61A$	2	2	11	$-6\omega_5$	12	-8	$3\omega_3$	12	$6-2\omega_{33}$
$61B$	2	2	$6+2\omega_2$	$-6\omega_5$	2	6	7	12	$6-2\omega_{33}$
$71A$	12	12	$-6+4\omega_2$	$4-8\omega_5$	7	9	6	-3	$-3\omega_{33}$
$71B$	12	12	$2-6\omega_2$	$-4+4\omega_5$	-8	16	6	-3	$-3\omega_{33}$
$79A$	-10	10	$-2\omega_2$	$5-3\omega_5$	-10	10	$6\omega_3$	-10	$-8+2\omega_{33}$
$79B$	-10	10	$2+4\omega_2$	$8-2\omega_5$	-6	-11	14	-10	$-8+2\omega_{33}$
$89A$	15	-15	$7+6\omega_2$	$3-7\omega_5$	15	-1	$4\omega_3$	15	$2-\omega_{33}$
$89B$	15	-15	$-10+6\omega_2$	$16+\omega_5$	0	-8	$-7\omega_3$	15	$2-\omega_{33}$

	$131A$	$2.31A$	$\sqrt{5}^3$	$139A$	$2^2.3$	$\sqrt{5}.29A$			$149A$
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f)$
2	$-\omega_3$	$-$	0	$1-\omega_2$	$-$	-2	1	2	0
3	$-2+2\omega_3$	-2	0	$-1-\omega_2$	$-$	3	-6	1	$2-4\omega_5$
$\sqrt{5}$	ω_3	0	$-$	$2\omega_2$	2	$-$	$-$	$-$	$2\omega_5$
7	$-2-6\omega_3$	14	0	$6-4\omega_2$	-2	8	2	-4	$2+\omega_5$
$11A$	$2-2\omega_3$	-6	$2-5\omega_5$	$-3-2\omega_2$	-4	-2	4	-6	$-\omega_5$
$11B$	1	0	$-3+5\omega_5$	-2	-4	5	-4	-3	$-2-2\omega_5$
$19A$	-2	2	0	$1+4\omega_2$	0	-8	4	-4	$3-\omega_5$
$19B$	-2	2	0	$3+\omega_2$	0	-1	-4	5	$-5+5\omega_5$
$29A$	$-4+2\omega_3$	0	0	$-3+4\omega_2$	6	$-$	$-$	$-$	$5\omega_5$
$29B$	$-1-4\omega_3$	6	0	$-1-6\omega_2$	6	-5	-2	9	$7+\omega_5$
$31A$	$-3+4\omega_3$	$-$	$-8+5\omega_5$	$7-\omega_2$	4	-10	8	2	$-6+6\omega_5$
$31B$	$4+\omega_3$	-4	$-3-5\omega_5$	$-2+2\omega_2$	4	-3	0	5	$-6\omega_5$
$41A$	$4-\omega_3$	6	$7-5\omega_5$	$-1+5\omega_2$	-10	7	10	-9	$-2-2\omega_5$
$41B$	$-4\omega_3$	-6	$2+5\omega_5$	$-8-2\omega_2$	-10	0	-6	0	$-3+5\omega_5$
$59A$	$-\omega_3$	-12	0	$6+4\omega_2$	-4	-3	-12	-9	$4-8\omega_5$
$59B$	$6-2\omega_3$	0	0	$10-2\omega_2$	-4	4	4	12	$2+\omega_5$
$61A$	$6+2\omega_3$	8	$2-5\omega_5$	$-3+3\omega_2$	2	13	-2	5	$3+3\omega_5$
$61B$	-4	-10	$-3+5\omega_5$	$2-8\omega_2$	2	-8	-2	-4	$-2-7\omega_5$
$71A$	$4+4\omega_3$	-12	$7+5\omega_5$	$-2+6\omega_2$	0	-12	0	12	$-8+10\omega_5$
$71B$	$4+4\omega_3$	0	$12-5\omega_5$	$-2-8\omega_2$	0	2	-16	-6	$6-3\omega_5$
$79A$	$1-8\omega_3$	8	0	$-2+4\omega_2$	12	10	16	2	$2-9\omega_5$
$79B$	$4-2\omega_3$	8	0	$2-2\omega_2$	12	-4	8	-4	$2+\omega_5$
$89A$	$-7-4\omega_3$	6	0	6	10	6	-6	-6	$12-9\omega_5$
$89B$	13	-6	0	$4-4\omega_2$	10	13	10	15	$6-2\omega_5$

	$\sqrt{5}.31A$	$2.41A$	$3.19A$		$2^2.11A$		$179A$		$2.3.\sqrt{5}$
$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f)$
2	1	–	-1	$-1+2\omega_5$	–	–	-1	1	–
3	-2	2	–	–	-2	$2-4\omega_5$	4	0	–
$\sqrt{5}$	–	2	0	$2-2\omega_5$	0	$2\omega_5$	0	0	–
7	-6	-6	-10	$2-4\omega_5$	8	$-8+6\omega_5$	5	-5	2
$11A$	0	-2	0	$4-4\omega_5$	–	–	3	-5	0
$11B$	4	-4	6	$-4+2\omega_5$	6	$-2-2\omega_5$	-3	3	0
$19A$	0	8	–	–	-4	0	8	-4	-4
$19B$	-4	-6	-4	0	2	$-6+2\omega_5$	-1	1	-4
$29A$	-2	-6	-6	$-2+4\omega_5$	-6	$2-4\omega_5$	-3	5	-6
$29B$	-2	8	0	$-6+2\omega_5$	-6	$-2+4\omega_5$	-9	7	-6
$31A$	–	-4	-4	$4-4\omega_5$	2	$-6+6\omega_5$	-10	2	8
$31B$	-4	-8	-4	$4\omega_5$	-4	$4-4\omega_5$	-7	1	8
$41A$	2	–	-6	$-2+8\omega_5$	-6	$-2+8\omega_5$	3	-3	-6
$41B$	2	6	12	$-2-2\omega_5$	0	$8-2\omega_5$	-6	6	-6
$59A$	-12	0	6	$-8+6\omega_5$	12	$-4+8\omega_5$	0	8	0
$59B$	-12	4	-6	$-4-2\omega_5$	0	$-8-4\omega_5$	12	-4	0
$61A$	10	0	14	2	8	$-6\omega_5$	2	-10	-10
$61B$	-2	-8	2	$-10+4\omega_5$	-10	2	-13	-11	-10
$71A$	-4	0	-6	$4+6\omega_5$	0	12	6	10	0
$71B$	16	0	6	$8-2\omega_5$	12	$4-4\omega_5$	0	12	0
$79A$	-8	16	8	$4-8\omega_5$	-10	$-2-6\omega_5$	8	12	8
$79B$	16	-4	-4	$8+4\omega_5$	8	0	8	0	8
$89A$	10	10	6	$2-4\omega_5$	6	$-2+4\omega_5$	-12	4	18
$89B$	18	14	12	$-2-6\omega_5$	6	$6-12\omega_5$	0	4	18

	$191A$		2.7		$199A$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	-3	1	$-$	$-$	0	-4	3
3	2	$2-4\omega_5$	-2	0	0	-2	0
$\sqrt{5}$	2	$-2+2\omega_5$	0	$2-\sqrt{6}$	1	-3	-2
7	-2	$-6+2\omega_5$	$-$	$-$	5	5	2
$11A$	0	$4-2\omega_5$	0	$2\sqrt{6}$	2	0	-4
$11B$	4	0	0	$2\sqrt{6}$	-3	-3	0
$19A$	-4	$-4+4\omega_5$	2	$-4-\sqrt{6}$	0	2	0
$19B$	4	-4	2	$-4-\sqrt{6}$	5	-7	-4
$29A$	-2	$-6+4\omega_5$	-6	$-2+2\sqrt{6}$	0	6	-6
$29B$	-10	$6-4\omega_5$	-6	$-2+2\sqrt{6}$	0	-6	6
$31A$	0	0	-4	$4-2\sqrt{6}$	-8	-4	10
$31B$	-8	$4\omega_5$	-4	$4-2\sqrt{6}$	2	-4	-4
$41A$	2	$-6+4\omega_5$	6	$-6-2\sqrt{6}$	-3	-3	6
$41B$	-10	$2+2\omega_5$	6	$-6-2\sqrt{6}$	-8	12	4
$59A$	0	$4-10\omega_5$	-6	$4+\sqrt{6}$	-5	3	4
$59B$	12	$-4+8\omega_5$	-6	$4+\sqrt{6}$	10	0	-2
$61A$	10	$6-6\omega_5$	8	$6+\sqrt{6}$	2	-10	14
$61B$	14	$-2+8\omega_5$	8	$6+\sqrt{6}$	7	-1	-14
$71A$	12	$8-10\omega_5$	0	$-6-2\sqrt{6}$	2	-12	8
$71B$	0	$4-8\omega_5$	0	$-6-2\sqrt{6}$	-3	-3	-12
$79A$	-16	0	8	$-2+2\sqrt{6}$	10	-10	10
$79B$	-4	$2\omega_5$	8	$-2+2\sqrt{6}$	-10	-10	8
$89A$	-14	$10+4\omega_5$	-6	10	-10	0	14
$89B$	-6	-6	-6	10	0	12	6

	$\sqrt{5}.41A$			$11A.19A$			$11B.19A$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	−1	1	3	−3	−1	1	−2	1	2
3	4	2	−4	−4	4	−2	2	2	−2
$\sqrt{5}$	–	–	–	−4	0	2	4	−2	0
7	−4	10	4	−12	8	6	−12	−6	8
$11A$	6	−4	−2	–	–	–	−5	4	−3
$11B$	0	−4	0	0	0	−4	–	–	–
$19A$	−4	−4	4	–	–	–	–	–	–
$19B$	−4	4	−4	2	2	0	1	4	−7
$29A$	−6	−2	−6	2	−6	6	1	−2	9
$29B$	0	6	−8	−10	6	−2	7	−2	9
$31A$	12	0	10	4	−4	0	−5	−8	5
$31B$	2	0	−6	−8	−4	0	−2	−8	−10
$41A$	–	–	–	0	0	6	−3	−6	3
$41B$	−6	−6	10	0	−12	6	−11	10	−3
$59A$	−12	4	12	4	−12	4	−3	12	−3
$59B$	12	−12	4	−10	6	8	10	4	−6
$61A$	−10	−2	6	6	2	−2	8	−10	−4
$61B$	2	14	2	−4	−4	−6	−2	−2	2
$71A$	−12	8	4	−8	0	0	−13	8	3
$71B$	6	−8	−2	−14	−6	−12	12	0	0
$79A$	8	−8	−16	−8	−16	8	4	−8	8
$79B$	−4	0	−4	16	8	−16	−10	8	2
$89A$	0	18	−8	14	6	18	10	10	6
$89B$	−6	−6	−6	−6	6	10	−14	10	6

	$2.\sqrt{5}.11A$			$3.\sqrt{5}^2$		
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$
2	–	–	–	0	0	3
3	0	-2	4	–	–	–
$\sqrt{5}$	–	–	–	–	–	–
7	-10	2	2	-5	5	14
$11A$	–	–	–	2	2	-4
$11B$	2	0	-6	2	2	-4
$19A$	0	-4	-4	-5	5	-4
$19B$	0	-4	8	-5	5	-4
$29A$	-10	6	-6	10	-10	2
$29B$	0	-6	0	10	-10	2
$31A$	-8	8	-4	-3	-3	0
$31B$	2	-4	-10	-3	-3	0
$41A$	2	-6	-6	-8	-8	10
$41B$	-8	6	0	-8	-8	10
$59A$	0	-12	12	-10	10	4
$59B$	10	0	-6	-10	10	4
$61A$	2	-10	14	7	7	-2
$61B$	2	14	2	7	7	-2
$71A$	12	0	-12	-8	-8	-8
$71B$	-8	0	0	-8	-8	-8
$79A$	0	8	8	0	0	0
$79B$	10	-4	-10	0	0	0
$89A$	10	-6	6	0	0	6
$89B$	-10	-6	-6	0	0	6

	$2 \cdot 59B$		$239A$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	1	-1	-3	1	1
3	$-2\omega_5$	$2\omega_{21}$	-5	-1	2
$\sqrt{5}$	$1-\omega_5$	$1-\omega_{21}$	-3	1	-2
7	-8	$-4\omega_{21}$	6	-6	6
$11A$	$-2+3\omega_5$	$2-\omega_{21}$	1	-5	4
$11B$	$-2+3\omega_5$	$2+\omega_{21}$	-3	3	0
$19A$	$-4\omega_5$	0	7	5	-4
$19B$	$-3+7\omega_5$	$-5+\omega_{21}$	-2	-2	4
$29A$	$5-5\omega_5$	$-5+\omega_{21}$	-8	0	6
$29B$	$-4-\omega_5$	$-3\omega_{21}$	-8	0	6
$31A$	$2-6\omega_5$	$2-2\omega_{21}$	4	-4	-4
$31B$	$-6+4\omega_5$	2	3	1	-8
$41A$	$7-7\omega_5$	$-3+3\omega_{21}$	3	1	-8
$41B$	$8-\omega_5$	$-8+\omega_{21}$	-11	9	-6
$59A$	$-3-\omega_5$	$5+3\omega_{21}$	-7	11	-4
$59B$	1	-1	10	14	-4
$61A$	$-6-2\omega_5$	$2+2\omega_{21}$	-6	-14	-2
$61B$	$5\omega_5$	$-8+\omega_{21}$	-7	5	-10
$71A$	$-1-3\omega_5$	$-3+3\omega_{21}$	14	-6	0
$71B$	14	$2-4\omega_{21}$	-12	-8	16
$79A$	$-2+\omega_5$	$-10+\omega_{21}$	-12	-8	16
$79B$	$2-4\omega_5$	$10-4\omega_{21}$	-1	-3	0
$89A$	$-1+5\omega_5$	$-5+5\omega_{21}$	0	-4	2
$89B$	$4+6\omega_5$	$-6\omega_{21}$	-7	9	-6

	$2.61A$		$\sqrt{5}.7$		2^4	$269A$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	–	–	-4	ω_{17}	–	-3	-1	2
3	1	5	-5	$-1-\omega_{17}$	-2	-1	1	$-2\omega_{13}$
$\sqrt{5}$	0	0	–	1	2	-3	3	0
7	-4	8	–	1	-10	-5	5	$-2+4\omega_{13}$
$11A$	0	-4	-3	$1-\omega_{17}$	-4	0	0	$-2+2\omega_{13}$
$11B$	-3	3	-3	$1-\omega_{17}$	4	-5	-3	0
$19A$	2	-6	2	$-2-2\omega_{17}$	4	-3	5	$-5+\omega_{13}$
$19B$	2	6	2	$-2-2\omega_{17}$	-4	1	-1	$-2+4\omega_{13}$
$29A$	0	-8	3	$-1+3\omega_{17}$	-2	-5	3	$6-3\omega_{13}$
$29B$	-9	9	3	$-1+3\omega_{17}$	-2	-2	-6	$4+2\omega_{13}$
$31A$	5	-3	-4	0	0	6	-10	$4-5\omega_{13}$
$31B$	-4	0	-4	0	0	-10	2	$2-2\omega_{13}$
$41A$	-6	-10	-12	$2\omega_{17}$	-2	12	0	$5-5\omega_{13}$
$41B$	0	0	-12	$2\omega_{17}$	-2	6	6	$-8-\omega_{13}$
$59A$	3	1	0	-4	12	3	3	$6+3\omega_{13}$
$59B$	6	-10	0	-4	-12	-7	-9	$10-4\omega_{13}$
$61A$	–	–	8	$6\omega_{17}$	10	6	2	-4
$61B$	-1	-11	8	$6\omega_{17}$	10	2	-10	$2\omega_{13}$
$71A$	-3	-7	0	8	8	10	-6	$6\omega_{13}$
$71B$	6	-10	0	8	-8	-6	-6	$-7+\omega_{13}$
$79A$	8	8	-1	$-5+\omega_{17}$	16	12	-16	$1+\omega_{13}$
$79B$	-1	-13	-1	$-5+\omega_{17}$	-16	-8	8	$-6-\omega_{13}$
$89A$	12	-4	-12	$4-2\omega_{17}$	-6	-18	6	$7-\omega_{13}$
$89B$	-3	13	-12	$4-2\omega_{17}$	-6	-13	3	$3+3\omega_{13}$

	$5 \cdot 11A$		$3 \cdot 31A$			$2 \cdot 71A$	
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$
2	1	$1-2\omega_5$	-1	1	3	$-$	$-$
3	2	$-2+4\omega_5$	$-$	$-$	$-$	2	$2-2\sqrt{2}$
$\sqrt{5}$	$-$	$-$	2	2	-2	0	$2\sqrt{2}$
7	-14	$2-4\omega_5$	-2	10	-10	-2	$5\sqrt{2}$
$11A$	$-$	$-$	-2	0	-2	-2	$-3\sqrt{2}$
$11B$	0	2	2	-4	2	4	-2
$19A$	-8	$-2+4\omega_5$	0	8	-4	6	$-4+\sqrt{2}$
$19B$	4	0	4	-4	4	2	$-2-2\sqrt{2}$
$29A$	6	$-2+4\omega_5$	0	-10	4	4	$6+\sqrt{2}$
$29B$	-6	$4-8\omega_5$	6	-2	-2	0	$-2-\sqrt{2}$
$31A$	-4	2	$-$	$-$	$-$	-8	$-6+\sqrt{2}$
$31B$	8	-8	-8	4	-4	8	$-2+2\sqrt{2}$
$41A$	-6	2	-12	10	-12	-6	$-7\sqrt{2}$
$41B$	6	-8	0	-6	0	-10	$2+4\sqrt{2}$
$59A$	0	$-2+4\omega_5$	-4	4	12	0	$12+2\sqrt{2}$
$59B$	12	$4-8\omega_5$	10	4	2	2	$6+4\sqrt{2}$
$61A$	2	-8	0	-6	0	8	$-4+8\sqrt{2}$
$61B$	-10	2	-6	14	10	-10	$2\sqrt{2}$
$71A$	0	-8	10	-4	-6	$-$	$-$
$71B$	0	12	8	0	-8	8	$-6+5\sqrt{2}$
$79A$	-8	$4-8\omega_5$	-8	8	0	4	$-6+3\sqrt{2}$
$79B$	4	$2-4\omega_5$	-8	0	-8	12	-4
$89A$	18	$2-4\omega_5$	-6	10	18	-10	$-4+8\sqrt{2}$
$89B$	-6	$-2+4\omega_5$	12	-6	-8	-6	$4-4\sqrt{2}$

	$2^2.19A$		$\sqrt{5}.61A$		$2^3.\sqrt{5}$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	$-$	$-$	1	$1-2\omega_5$	$-$	$-$	$-$
3	4	$2-2\omega_5$	2	2	2	2	-6
$\sqrt{5}$	-2	2	$-$	$-$	$-$	$-$	$-$
7	-6	6	2	$2-4\omega_5$	-6	-6	2
$11A$	0	$-4+4\omega_5$	-4	4	-4	4	4
$11B$	2	$-2\omega_5$	0	$-2+4\omega_5$	4	-4	4
$19A$	$-$	$-$	0	$-6+4\omega_5$	-4	-4	4
$19B$	8	$-4+4\omega_5$	0	$-2+4\omega_5$	-4	-4	4
$29A$	2	$2+4\omega_5$	6	2	-10	6	-2
$29B$	-10	-2	-2	$-2-4\omega_5$	6	-10	-2
$31A$	-2	$-4-2\omega_5$	-8	$4-8\omega_5$	8	0	-8
$31B$	-10	-2	-4	$-6+4\omega_5$	0	8	-8
$41A$	-4	$2-2\omega_5$	2	$2-8\omega_5$	-6	10	-6
$41B$	-10	$-2-4\omega_5$	2	10	10	-6	-6
$59A$	-12	8	4	-8	-12	4	-4
$59B$	10	$-8-2\omega_5$	0	$-6+4\omega_5$	4	-12	-4
$61A$	4	$-2+10\omega_5$	$-$	$-$	-2	-2	-2
$61B$	0	$-10+6\omega_5$	-2	$-6+8\omega_5$	-2	-2	-2
$71A$	-8	$-8+8\omega_5$	0	$4-8\omega_5$	-8	0	0
$71B$	-2	$-4-2\omega_5$	-4	$2-4\omega_5$	0	-8	0
$79A$	14	$4-10\omega_5$	8	$-4+8\omega_5$	-16	0	0
$79B$	0	$8-8\omega_5$	-4	$-2+12\omega_5$	0	-16	0
$89A$	6	$6-4\omega_5$	10	$-10+4\omega_5$	10	10	-6
$89B$	6	$-6+12\omega_5$	-6	$10-8\omega_5$	10	10	-6

	$2 \cdot 79B$					$319A$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_5)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	1	-1	1	1	1	-3	$1-\sqrt{2}$	$2-\sqrt{3}$
3	$3-\omega_{17}$	$1-\sqrt{6}$	2	-3	1	-2	$2-2\sqrt{2}$	$-2+2\sqrt{3}$
$\sqrt{5}$	$-1-\omega_{17}$	1	-2	2	0	-2	$2\sqrt{2}$	0
7	-2	$-8-2\sqrt{6}$	-2	10	2	-10	$-2-6\sqrt{2}$	8
$11A$	$2\omega_{17}$	2	2	0	0	-1	1	-1
$11B$	$2\omega_{17}$	$2\sqrt{6}$	2	0	0	-4	-2	$2\sqrt{3}$
$19A$	$1+\omega_{17}$	$1-\sqrt{6}$	2	-5	-1	-4	$2\sqrt{2}$	$2-2\sqrt{3}$
$19B$	-6	$2-2\sqrt{6}$	-6	0	-4	0	$2\sqrt{2}$	2
$29A$	$-3-\omega_{17}$	$-3+3\sqrt{6}$	-4	-3	-3	-2	$-4+2\sqrt{2}$	$4\sqrt{3}$
$29B$	$2-4\omega_{17}$	$-2+2\sqrt{6}$	-2	-4	0	1	1	-1
$31A$	$4-4\omega_{17}$	$2\sqrt{6}$	0	-2	2	-4	$-4-2\sqrt{2}$	2
$31B$	$1+3\omega_{17}$	$1-4\sqrt{6}$	4	4	2	8	$2-4\sqrt{2}$	$-4+2\sqrt{3}$
$41A$	-10	2	-10	2	-6	-2	$6-2\sqrt{2}$	$-6-2\sqrt{3}$
$41B$	$-3+\omega_{17}$	$1+\sqrt{6}$	-2	-3	-3	6	$-6+2\sqrt{2}$	$-4\sqrt{3}$
$59A$	$3+3\omega_{17}$	$-11+\sqrt{6}$	6	3	3	4	$6\sqrt{2}$	$-6+2\sqrt{3}$
$59B$	$1-\omega_{17}$	$-7+2\sqrt{6}$	0	-2	0	0	4	$6-4\sqrt{3}$
$61A$	$8-4\omega_{17}$	-8	4	8	8	-10	$2\sqrt{2}$	$-4-4\sqrt{3}$
$61B$	$3-\omega_{17}$	$-1-2\sqrt{6}$	2	0	2	6	$-6+4\sqrt{2}$	$2+4\sqrt{3}$
$71A$	$-1-3\omega_{17}$	$5+2\sqrt{6}$	-4	2	0	8	$-8-6\sqrt{2}$	$-6+2\sqrt{3}$
$71B$	$-4+4\omega_{17}$	$6-4\sqrt{6}$	0	-10	-6	12	$4+4\sqrt{2}$	$-6-4\sqrt{3}$
$79A$	$-10+2\omega_{17}$	$2-2\sqrt{6}$	-8	-10	-10	-8	10	$-4-2\sqrt{3}$
$79B$	1	-1	1	1	1	-8	$8\sqrt{2}$	$-4+4\sqrt{3}$
$89A$	-14	10	-14	10	-6	10	$-4-6\sqrt{2}$	$-8\sqrt{3}$
$89B$	$2\omega_{17}$	$-12+2\sqrt{6}$	2	0	0	-14	$-6+4\sqrt{2}$	$6+4\sqrt{3}$

	2.3^2		$2.\sqrt{5}.19A$			$2.3.11A$				
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_4)$	$a_\mathfrak{p}(f_5)$
2	–	–	–	–	–	–	–	–	–	–
3	–	–	2	-2	2	–	–	–	–	–
$\sqrt{5}$	4	0	–	–	–	-2	3	1	1	3
7	10	-4	0	-4	-2	-14	-11	-5	5	11
$11A$	-2	$-3\sqrt{2}$	4	0	-4	–	–	–	–	–
$11B$	-2	$3\sqrt{2}$	2	6	0	4	2	2	2	-6
$19A$	0	2	–	–	–	4	7	-5	-5	-1
$19B$	0	2	-2	2	0	4	-7	-5	5	-1
$29A$	0	$6\sqrt{2}$	4	0	2	-10	0	0	0	0
$29B$	0	$-6\sqrt{2}$	-2	-6	6	6	-7	5	-5	-9
$31A$	-8	-4	0	-4	-8	0	6	2	2	-10
$31B$	-8	-4	4	-4	-8	0	-1	-3	7	5
$41A$	-2	$-3\sqrt{2}$	-12	12	-10	-6	-5	7	-3	9
$41B$	-2	$3\sqrt{2}$	2	-6	10	-6	2	2	2	-6
$59A$	10	$-3\sqrt{2}$	0	0	-4	-12	4	0	0	-12
$59B$	10	$3\sqrt{2}$	-4	-12	-4	4	11	-5	-15	9
$61A$	2	-10	6	-10	6	-2	-12	12	-8	8
$61B$	2	-10	-2	-10	-10	14	-5	-13	7	-1
$71A$	-12	$6\sqrt{2}$	0	0	8	8	5	-3	-13	3
$71B$	-12	$-6\sqrt{2}$	-8	12	0	-8	12	-8	-8	12
$79A$	0	8	-16	8	8	0	-14	10	-10	-10
$79B$	0	8	-6	-10	4	-16	-14	-10	10	14
$89A$	-10	$3\sqrt{2}$	6	-18	2	-6	10	-10	-10	-6
$89B$	-10	$-3\sqrt{2}$	-14	6	10	6	-11	15	-5	9

	$11A \cdot 31B$				
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_5)$
2	1	1	-1	$1-2\omega_5$	1
3	-2	-6	-4	0	$-2\omega_{17}$
$\sqrt{5}$	2	2	-4	$2\omega_5$	2
7	-6	-14	-6	$2-4\omega_5$	$-2-4\omega_{17}$
$11A$	-1	-1	1	1	-1
$11B$	0	4	-6	2	$-2+2\omega_{17}$
$19A$	4	8	0	$-4-2\omega_5$	$2+2\omega_{17}$
$19B$	0	4	6	$-2+4\omega_5$	$-2+2\omega_{17}$
$29A$	-2	-2	2	$2-4\omega_5$	-2
$29B$	-2	-10	6	$-2+4\omega_5$	$2-4\omega_{17}$
$31A$	-4	-8	-6	2	$-2-2\omega_{17}$
$31B$	-1	-1	-1	1	-1
$41A$	2	10	-2	2	$-2+4\omega_{17}$
$41B$	2	2	-6	2	2
$59A$	-4	-4	-12	$4-8\omega_5$	-4
$59B$	4	4	8	$-8-4\omega_5$	4
$61A$	-2	-10	2	2	$2-4\omega_{17}$
$61B$	10	14	-12	-8	$8+2\omega_{17}$
$71A$	8	4	-4	$-8+10\omega_5$	$10-2\omega_{17}$
$71B$	4	-4	10	$2-10\omega_5$	$8-4\omega_{17}$
$79A$	0	0	-4	$-12+4\omega_5$	0
$79B$	0	4	-4	$-8+6\omega_5$	$-2+2\omega_{17}$
$89A$	-6	-10	-10	$2+6\omega_5$	$-4-2\omega_{17}$
$89B$	-6	2	-6	$2-4\omega_5$	$-10+4\omega_{17}$

	$349A$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	$-2-\omega_5$	$1-\sqrt{2}$	ω_{21}
3	$-4-\omega_5$	3	$-\omega_{21}$
$\sqrt{5}$	$-3+2\omega_5$	$2\sqrt{2}$	1
7	$-2-3\omega_5$	$-4+4\sqrt{2}$	$3\omega_{21}$
$11A$	$2-5\omega_5$	-2	$2-\omega_{21}$
$11B$	$-3+2\omega_5$	$1-\sqrt{2}$	-3
$19A$	$-3+6\omega_5$	$-2-2\sqrt{2}$	$5-2\omega_{21}$
$19B$	$-3-2\omega_5$	$-2-2\sqrt{2}$	$5-2\omega_{21}$
$29A$	$-6+3\omega_5$	$-4+2\sqrt{2}$	ω_{21}
$29B$	$1+\omega_5$	$4+4\sqrt{2}$	$-5+\omega_{21}$
$31A$	$2-3\omega_5$	$-2+2\sqrt{2}$	$2+\omega_{21}$
$31B$	$1+3\omega_5$	$2-4\sqrt{2}$	$-3-\omega_{21}$
$41A$	$7-5\omega_5$	$-6+2\sqrt{2}$	$-3+3\omega_{21}$
$41B$	$2+\omega_5$	$8+2\sqrt{2}$	$-8-\omega_{21}$
$59A$	$7-4\omega_5$	4	$-5+4\omega_{21}$
$59B$	$-4\omega_5$	$1+\sqrt{2}$	$-4\omega_{21}$
$61A$	$-13+4\omega_5$	$3-6\sqrt{2}$	7
$61B$	$-8+3\omega_5$	-8	$2+\omega_{21}$
$71A$	-6	$-5+7\sqrt{2}$	2
$71B$	$1+\omega_5$	$-8-6\sqrt{2}$	$-3+5\omega_{21}$
$79A$	-3	$4+2\sqrt{2}$	5
$79B$	$-1+4\omega_5$	$-3-5\sqrt{2}$	$-5+6\omega_{21}$
$89A$	$6-6\omega_5$	$-3+10\sqrt{2}$	$10-2\omega_{21}$
$89B$	$11-\omega_5$	$-12\sqrt{2}$	$5-\omega_{21}$

	$2 \cdot 89B$					$3 \cdot 41B$				
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_5)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_5)$
2	-1	-1	-1	-1	1	-2	1	1	-2	$1+\omega_{17}$
3	-2	0	0	0	$-2-2\sqrt{3}$	-1	-1	1	-1	1
$\sqrt{5}$	-4	3	2	$2-2\omega_5$	$1+\sqrt{3}$	-3	2	-2	-1	$2-\omega_{17}$
7	-10	0	0	0	$-6-2\sqrt{3}$	-10	6	2	-6	$2-4\omega_{17}$
$11A$	-2	2	2	2	$-2\sqrt{3}$	-5	-4	0	5	$-\omega_{17}$
$11B$	-6	-2	0	$4\omega_5$	$2+2\sqrt{3}$	0	4	4	-2	-4
$19A$	-2	-4	-2	$-2+4\omega_5$	-2	-8	0	8	6	-4
$19B$	4	0	0	0	-2	3	-4	4	-1	$-\omega_{17}$
$29A$	8	-3	-4	$-4-2\omega_5$	$3+\sqrt{3}$	-2	2	-2	2	-2
$29B$	-6	4	2	$2-4\omega_5$	$-4-2\sqrt{3}$	-1	6	-10	9	$2+\omega_{17}$
$31A$	0	-1	-2	$-2-2\omega_5$	$5+\sqrt{3}$	6	-8	-4	4	$2\omega_{17}$
$31B$	0	-2	0	$4\omega_5$	$-2-2\sqrt{3}$	6	-4	4	-10	$-2\omega_{17}$
$41A$	-6	2	2	2	$4\sqrt{3}$	-6	6	-6	0	$-6+4\omega_{17}$
$41B$	-6	-2	0	$4\omega_5$	$-4-4\sqrt{3}$	1	-1	1	-1	1
$59A$	8	-14	-12	$-12+4\omega_5$	$4-2\sqrt{3}$	-8	4	-8	4	$4+4\omega_{17}$
$59B$	10	-8	-4	$-4+8\omega_5$	$-4+4\sqrt{3}$	7	0	0	-3	$-5\omega_{17}$
$61A$	-10	10	6	$6-8\omega_5$	$-2+4\sqrt{3}$	6	10	-10	-8	$-2+4\omega_{17}$
$61B$	12	-1	-2	$-2-2\omega_5$	$3+3\sqrt{3}$	8	-6	-10	6	$6-2\omega_{17}$
$71A$	0	14	8	$8-12\omega_5$	$-6-2\sqrt{3}$	-6	0	-8	-12	$-2\omega_{17}$
$71B$	-16	-8	-8	-8	$4-4\sqrt{3}$	-9	-12	12	9	$-4+7\omega_{17}$
$79A$	0	15	10	$10-10\omega_5$	$-1+3\sqrt{3}$	-12	8	-12	-4	$8-4\omega_{17}$
$79B$	-4	-11	-8	$-8+6\omega_5$	$1+7\sqrt{3}$	1	-8	16	-11	$4+\omega_{17}$
$89A$	-6	-10	-10	-10	2	-7	2	10	-1	$6-\omega_{17}$
$89B$	1	-1	-1	-1	1	6	-2	-6	-8	$-6+8\omega_{17}$

	379A		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	ω_5	$-2-\omega_5$	$2-\sqrt{3}$
3	$1-2\omega_5$	$-3+4\omega_5$	$2+\sqrt{3}$
$\sqrt{5}$	$1+\omega_5$	$-3+\omega_5$	-2
7	$-3+3\omega_5$	$-1-\omega_5$	$6+3\sqrt{3}$
11A	$3-2\omega_5$	$-1-2\omega_5$	-2
11B	$-2-\omega_5$	$3\omega_5$	$2+2\sqrt{3}$
19A	1	-7	$2+2\sqrt{3}$
19B	1	$1-4\omega_5$	1
29A	$2-2\omega_5$	$-2+6\omega_5$	-3
29B	$3-6\omega_5$	$-1-4\omega_5$	$-2-2\sqrt{3}$
31A	$-3+7\omega_5$	$-1+3\omega_5$	$4\sqrt{3}$
31B	$-5\omega_5$	$-4+3\omega_5$	$2-4\sqrt{3}$
41A	$-6+2\omega_5$	$-2+6\omega_5$	$2+6\sqrt{3}$
41B	$5-4\omega_5$	$5-4\omega_5$	$-6-2\sqrt{3}$
59A	$-3+5\omega_5$	$1-3\omega_5$	$-6+2\sqrt{3}$
59B	3	$7-6\omega_5$	-8
61A	$-5+2\omega_5$	$3-6\omega_5$	0
61B	$-2+9\omega_5$	$10+\omega_5$	$-8-2\sqrt{3}$
71A	$-6+8\omega_5$	$-2+4\omega_5$	$6-5\sqrt{3}$
71B	$-5+4\omega_5$	$7-4\omega_5$	$1-8\sqrt{3}$
79A	$8+3\omega_5$	$-14+\omega_5$	$-10+4\sqrt{3}$
79B	$9-\omega_5$	$3+7\omega_5$	$-6-3\sqrt{3}$
89A	$-7+8\omega_5$	$5+2\omega_5$	2
89B	$4+2\omega_5$	$-12+6\omega_5$	$7-4\sqrt{3}$

	$2^2.\sqrt{5}^2$		$2.101A$				$3^2.\sqrt{5}$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	–	–	–	–	–	–	-3	2	2
3	2	0	-4	0	0	4	–	–	–
$\sqrt{5}$	–	–	-3	-2	1	2	–	–	–
7	10	0	-6	-4	-10	-4	-14	-4	-4
$11A$	0	$4-4\omega_5$	-6	-1	2	-1	4	$-3\omega_5$	$3-3\omega_5$
$11B$	0	$4\omega_5$	-1	6	-3	-2	4	$-3+3\omega_5$	$3\omega_5$
$19A$	4	0	-1	2	5	-2	4	$-4+6\omega_5$	$2-6\omega_5$
$19B$	4	0	1	4	-5	0	4	$2-6\omega_5$	$-4+6\omega_5$
$29A$	-6	0	1	-4	5	0	2	$6-6\omega_5$	$-6\omega_5$
$29B$	-6	0	0	9	0	3	2	$6\omega_5$	$-6+6\omega_5$
$31A$	-4	$4-4\omega_5$	0	-5	-8	5	0	$-1-3\omega_5$	$-4+3\omega_5$
$31B$	-4	$4\omega_5$	9	-11	7	-1	0	$-4+3\omega_5$	$-1-3\omega_5$
$41A$	6	$-10+4\omega_5$	5	3	-3	-1	-10	$-6-3\omega_5$	$9-3\omega_5$
$41B$	6	$-6-4\omega_5$	0	-2	-8	-6	-10	$-9+3\omega_5$	$6+3\omega_5$
$59A$	-12	0	-8	6	0	14	4	6	-6
$59B$	-12	0	-14	-5	10	-9	4	6	-6
$61A$	2	$-6-4\omega_5$	-7	-6	-3	-10	-2	$2-3\omega_5$	$-1+3\omega_5$
$61B$	2	$-10+4\omega_5$	-12	1	-8	7	-2	$-1+3\omega_5$	$2-3\omega_5$
$71A$	-12	$4\omega_5$	16	-12	12	-4	8	$3+3\omega_5$	$-6+3\omega_5$
$71B$	-12	$4-4\omega_5$	-10	-16	2	-8	8	$6-3\omega_5$	$-3-3\omega_5$
$79A$	-8	0	10	-8	10	-8	0	$8-6\omega_5$	$2+6\omega_5$
$79B$	-8	0	-12	6	0	-10	0	$2+6\omega_5$	$8-6\omega_5$
$89A$	6	0	-7	9	-15	17	6	$6+6\omega_5$	$-12+6\omega_5$
$89B$	6	0	-4	12	0	0	6	$12-6\omega_5$	$-6-6\omega_5$

	$11A.41A$					$2^2.29A$			
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_4)$	$a_\mathfrak{p}(f_5)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_4)$
2	-1	3	-2	1	$-\omega_{41}$	$-$	$-$	$-$	$-$
3	1	1	2	ω_{41}	ω_{41}	4	-4	0	$2\omega_5$
$\sqrt{5}$	3	-1	-3	$-\omega_{41}$	1	0	-4	4	$-2\omega_5$
7	-4	4	8	$-2-2\omega_{41}$	0	2	-2	2	$-6+4\omega_5$
$11A$	$-$	$-$	$-$	1	1	0	-2	-4	$2+2\omega_5$
$11B$	0	0	-5	4	$-3+\omega_{41}$	0	-4	0	0
$19A$	2	-6	-5	$2\omega_{41}$	$5-\omega_{41}$	-4	-4	4	4
$19B$	5	1	-2	$-6+\omega_{41}$	$-\omega_{41}$	2	4	-6	$4-4\omega_5$
$29A$	-6	-6	7	-2	$5-\omega_{41}$	$-$	$-$	$-$	$-$
$29B$	-6	2	-4	-2	$2\omega_{41}$	6	-6	-2	$2+4\omega_5$
$31A$	8	0	-2	0	$2+2\omega_{41}$	-10	0	-2	$4\omega_5$
$31B$	-4	4	-10	0	$2+2\omega_{41}$	8	4	-8	$-4-4\omega_5$
$41A$	$-$	$-$	$-$	1	1	6	2	6	-2
$41B$	0	0	8	$6-2\omega_{41}$	-8	0	0	-12	$-4-2\omega_5$
$59A$	-6	2	-3	$-2\omega_{41}$	$-5+\omega_{41}$	-12	-2	0	$2-6\omega_5$
$59B$	-9	-1	-4	$6-\omega_{41}$	$-10-\omega_{41}$	6	-8	6	$8\omega_5$
$61A$	14	6	-4	$-2+4\omega_{41}$	$-8-2\omega_{41}$	-10	10	6	$-10+4\omega_5$
$61B$	5	-3	-8	$4+\omega_{41}$	$2-\omega_{41}$	8	-12	-4	$-8-2\omega_5$
$71A$	-6	-6	1	$-4-2\omega_{41}$	$7+\omega_{41}$	-12	-6	16	$-2+2\omega_5$
$71B$	12	-12	6	$-4\omega_{41}$	$2-2\omega_{41}$	-6	4	10	-4
$79A$	-10	14	7	$-4-2\omega_{41}$	$5-\omega_{41}$	-4	-8	-4	$4-8\omega_5$
$79B$	-13	11	0	$6+\omega_{41}$	$-10+\omega_{41}$	8	10	4	$-2+2\omega_5$
$89A$	9	-3	0	$-4-\omega_{41}$	$-10+\omega_{41}$	-6	-10	-6	$2+4\omega_5$
$89B$	0	-16	13	$6+2\omega_{41}$	$-5-\omega_{21}$	-18	-2	6	$14-4\omega_5$

	$\sqrt{5}^2.19A$					$2.11A.11B$				
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_4)$	$a_\mathfrak{p}(f_5)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_4)$	$a_\mathfrak{p}(f_5)$
2	-3	1	3	$-1-\sqrt{6}$	$1-\sqrt{6}$	–	–	–	–	–
3	-2	2	2	$1+\sqrt{6}$	$-1+\sqrt{6}$	2	1	1	5	-3
$\sqrt{5}$	–	–	–	–	–	2	0	0	-4	4
7	4	-2	-4	$8-2\sqrt{6}$	$-8-2\sqrt{6}$	14	-1	-1	5	13
$11A$	-2	0	-2	$-2\sqrt{6}$	$2\sqrt{6}$	–	–	–	–	–
$11B$	-6	0	-6	$3+\sqrt{7}$	$3-\sqrt{6}$	–	–	–	–	–
$19A$	–	–	–	–	–	-4	-1	5	5	7
$19B$	-4	4	4	$2+2\sqrt{6}$	$-2+2\sqrt{6}$	-4	5	-1	5	7
$29A$	-4	-6	4	$-3+2\sqrt{6}$	$3+2\sqrt{6}$	10	3	-3	-5	-3
$29B$	10	6	-10	0	0	10	-3	3	-5	-3
$31A$	2	-4	2	2	2	-8	-7	5	7	-5
$31B$	-4	8	-4	$-1+2\sqrt{6}$	$-1-2\sqrt{6}$	-8	5	-7	7	-5
$41A$	-2	-6	-2	$-2\sqrt{6}$	$2\sqrt{6}$	-2	0	0	-8	-8
$41B$	-2	-6	-2	$-2\sqrt{6}$	$2\sqrt{6}$	-2	0	0	-8	-8
$59A$	4	-12	-4	$3-2\sqrt{6}$	$-3-2\sqrt{6}$	12	0	-12	0	-12
$59B$	-12	-12	12	$-9+\sqrt{6}$	$9+\sqrt{6}$	12	-12	0	0	-12
$61A$	-2	-10	-2	$5+3\sqrt{6}$	$5-3\sqrt{6}$	-10	-1	-7	-13	5
$61B$	-6	14	-6	$-7+\sqrt{6}$	$-7-\sqrt{6}$	-10	-7	-1	-13	5
$71A$	12	0	12	-3	-3	0	9	-3	-3	-7
$71B$	-10	12	-10	$-9-\sqrt{6}$	$-9+\sqrt{6}$	0	-3	9	-3	-7
$79A$	-8	16	8	$-1-\sqrt{6}$	$1-\sqrt{6}$	-12	8	-4	0	4
$79B$	0	-8	0	5	-5	-12	-4	8	0	4
$89A$	8	6	-8	$6-4\sqrt{6}$	$-6-4\sqrt{6}$	-18	9	-15	15	7
$89B$	2	-6	-2	$-6+4\sqrt{6}$	$6+4\sqrt{6}$	-18	-15	9	15	7

	499A			
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$
2	$-1+2\omega_5$	-2	3	0
3	$3-\omega_5$	$2-4\omega_5$	$-3+3\omega_5$	0
$\sqrt{5}$	$2-2\omega_5$	$-3+\omega_5$	$2-2\omega_5$	$-\omega_{29}$
7	$6+3\omega_5$	$2-8\omega_5$	$4-11\omega_5$	$2+2\omega_{29}$
$11A$	$-2-2\omega_5$	$-2-\omega_5$	$-2+2\omega_5$	$3+\omega_{29}$
$11B$	$4\omega_5$	$-2+2\omega_5$	-4	2
$19A$	$2+\omega_5$	$-6+2\omega_5$	$4-5\omega_5$	$2+2\omega_{29}$
$19B$	$4-3\omega_5$	$-4+3\omega_5$	$-4+5\omega_5$	$1+\omega_{29}$
$29A$	$-7-\omega_5$	$-3+3\omega_5$	$1-\omega_5$	$6+\omega_{29}$
$29B$	$-4+3\omega_5$	$-3-\omega_5$	$-4+3\omega_5$	$2-3\omega_{29}$
$31A$	$7-5\omega_5$	$-4+7\omega_5$	$3-\omega_5$	$-7+\omega_{29}$
$31B$	$-2-2\omega_5$	$2-3\omega_5$	$6-2\omega_5$	$1-\omega_{29}$
$41A$	$-2+3\omega_5$	0	$-4+\omega_5$	$-6+2\omega_{29}$
$41B$	$4\omega_5$	$5-7\omega_5$	$-8+4\omega_5$	$-4-\omega_{29}$
$59A$	$-2-\omega_5$	$-4+8\omega_5$	$-4+\omega_5$	$6-4\omega_{29}$
$59B$	$3-\omega_5$	$8-4\omega_5$	$3-9\omega_5$	0
$61A$	$3+3\omega_5$	0	$1+3\omega_5$	-8
$61B$	$-4+2\omega_5$	$-2+3\omega_5$	$-4+6\omega_5$	$-1-3\omega_{29}$
$71A$	$-14+2\omega_5$	$7+\omega_5$	$2-2\omega_5$	$6-\omega_{29}$
$71B$	$-4+2\omega_5$	$2+2\omega_5$	$4-6\omega_5$	-8
$79A$	$-4+3\omega_5$	$-8+5\omega_5$	$4+7\omega_5$	$-1-\omega_{29}$
$79B$	$11-7\omega_5$	$-6+4\omega_5$	$5+5\omega_5$	$4+4\omega_{29}$
$89A$	$-7-\omega_5$	$-5+\omega_5$	$5-5\omega_5$	$-6-\omega_{29}$
$89B$	$-2-6\omega_5$	$4-8\omega_5$	$-6+2\omega_5$	$8-2\omega_{29}$

	$3.\sqrt{5}.11A$				$2.\sqrt{5}^3$	
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$	$a_\mathfrak{p}(f_4)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$
2	3	-3	1	$-1+2\omega$	$-$	$-$
3	$-$	$-$	$-$	$-$	0	0
$\sqrt{5}$	$-$	$-$	$-$	$-$	$-$	$-$
7	-2	2	2	$2-4\omega_5$	10	-10
$11A$	$-$	$-$	$-$	$-$	2	2
$11B$	0	-4	4	$4-4\omega_5$	2	2
$19A$	0	-8	-4	0	$-5\omega_5$	$5-5\omega_5$
$19B$	4	-4	-4	0	$-5+5\omega_5$	$5\omega_5$
$29A$	2	6	-2	$2-4\omega_5$	$-5\omega_5$	$5-5\omega_5$
$29B$	-2	-6	-2	$2-4\omega_5$	$-5+5\omega_5$	$5\omega_5$
$31A$	4	-8	0	$4-4\omega_5$	$-5\omega_5$	$-3+5\omega_5$
$31B$	-8	4	0	$4\omega_5$	$-3+5\omega_5$	$2-5\omega_5$
$41A$	2	6	10	$-10+4\omega_5$	$2-5\omega_5$	$-3+5\omega_5$
$41B$	-10	-10	-6	2	$-3+5\omega_5$	$2-5\omega_5$
$59A$	0	4	12	$-12+4\omega_5$	$5+5\omega_5$	$-10+5\omega_5$
$59B$	4	12	-4	$4-8\omega_5$	$10-5\omega_5$	$-5-5\omega_5$
$61A$	-6	-6	-2	2	$2-10\omega_5$	$-8+10\omega_5$
$61B$	-10	-2	-2	$-2+8\omega_5$	$-8+10\omega_5$	$2-10\omega_5$
$71A$	0	8	16	-8	$-8+5\omega_5$	$-3-5\omega_5$
$71B$	-8	-4	0	$-4+12\omega_5$	$-3-5\omega_5$	$-8+5\omega_5$
$79A$	-16	0	-8	$-4+8\omega_5$	$-10\omega_5$	$10-10\omega_5$
$79B$	4	-12	-8	$4-8\omega_5$	$-10+10\omega_5$	$10\omega_5$
$89B$	-10	2	-6	$-2+4\omega_5$	-10	10
$89A$	14	-6	-6	$-2+4\omega_5$	-10	10

	$13 : x^3 + 6x^2$ $+8x - 2$	$151A : x^3 - x^2$ $-5x + 1$	$181A : x^4 - 12x^2$ $+8x - 1$
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_1)$
2	$(3, -2, -1)$	$(0, 1, 0)$	$(0, 1, 0, 0)$
3	$(2, 10, 3)$	$(6, 1, -3)$	$(-3, 0, 1, 0)$
$\sqrt{5}$	$(-4, -2, 0)$	$(-2, -1, 1)$	$(-2, 12, -1, -2)$
7	$(6, -2, 0)$	$(4, -1, 1)$	$(6, -38, 2, 6)$
$11A$	$(0, 1, 0)$	$(-4, -2, 2)$	$(8, -20, 0, 3)$
$11B$	$(0, 1, 0)$	$(5, -1, -2)$	$(8, -33, 2, 5)$
$19A$	$(-2, -1, 0)$	$(-6, 0, 2)$	$(6, -31, 2, 5)$
$19B$	$(-2, -1, 0)$	$(3, 1, -2)$	$(-6, 26, -2, -4)$
$29A$	$(-6, -12, -3)$	$(-12, -2, 4)$	$(-12, 50, -4, -8)$
$29B$	$(-6, -12, -3)$	$(-6, 2, 2)$	$(-6, 2, 2, 0)$
$31A$	$(-4, -7, -2)$	$(-4, 2, 2)$	$(2, 6, -2, -1)$
$31B$	$(-4, -7, -2)$	$(8, 5, -3)$	$(-10, 50, -4, -8)$
$41A$	$(4, -4, -2)$	$(-8, -3, 5)$	$(-7, 28, -1, -4)$
$41B$	$(4, -4, -2)$	$(8, 2, -4)$	$(-10, 26, 0, -4)$
$59A$	$(10, -3, -2)$	$(-16, -3, 7)$	$(-6, -5, 2, 1)$
$59B$	$(10, -3, -2)$	$(4, -4, 0)$	$(22, -60, 2, 9)$
$61A$	$(2, -10, -3)$	$(8, 6, -4)$	$(18, -12, -2, 2)$
$61B$	$(2, -10, -3)$	$(8, 1, -5)$	$(-8, 40, -1, -6)$
$71A$	$(-10, 17, 6)$	$(18, 1, -5)$	$(12, -14, 0, 2)$
$71B$	$(-10, 17, 6)$	$(2, 4, -2)$	$(2, -12, 0, 2)$
$79A$	$(12, 10, 2)$	$(-2, -4, 2)$	$(22, -56, 2, 9)$
$79B$	$(12, 10, 2)$	$(14, 1, -7)$	$(16, -34, 0, 5)$
$89A$	$(2, 16, 4)$	$(6, 2, -6)$	$(-12, 10, 0, -2)$
$89B$	$(2, 16, 4)$	$(-17, -7, 8)$	$(-4, -2, 1, 0)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^3 - x^2 - 5x + 1)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2 + 1)\}.$$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 - 12x^2 + 8x - 1)$ the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2 + 1), 1/2(\alpha^3 + \alpha)\}.$$

	$211A : x^4 - 2x^3$ $-8x^2 + 6x + 7$	$229A : x^3 - 3x^2$ $-x + 1$	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$
2	$(0, 1, 0, 0)$	-3	$(0, 1, 0)$
3	$(-3, 5, 4, -3)$	-1	$(2, 2, -1)$
$\sqrt{5}$	$(3, -2, -2, 1)$	-4	$(1, -4, 1)$
7	$(-1, -8, -2, 3)$	-10	$(0, -10, 4)$
$11A$	$(1, -3, -1, 1)$	-2	$(1, 8, -3)$
$11B$	$(-4, -2, 2, 0)$	0	$(-3, -2, 1)$
$19A$	$(4, 3, -1, -1)$	-5	$(-3, 5, -1)$
$19B$	$(4, -2, -4, 2)$	-7	$(2, -1, 0)$
$29A$	$(-2, -4, -2, 2)$	6	$(-1, 2, -1)$
$29B$	$(-12, 3, 7, -3)$	3	$(2, -4, 1)$
$31A$	$(-7, 5, 6, -3)$	-3	$(3, 7, -3)$
$31B$	$(6, 5, -1, -1)$	8	$(-3, -8, 3)$
$41A$	$(-3, 1, 2, -1)$	2	$(1, -6, 1)$
$41B$	$(14, -10, -12, 6)$	-9	$(-1, 4, 0)$
$59A$	$(-9, 3, 5, -3)$	9	$(-8, -1, 2)$
$59B$	$(-2, 12, 4, -4)$	-10	$(-1, -12, 3)$
$61A$	$(0, 12, 4, -4)$	0	$(4, 14, -6)$
$61B$	$(4, -4, -6, 4)$	-14	$(-8, -6, 4)$
$71A$	$(12, -14, -8, 6)$	-3	$(-6, -19, 6)$
$71B$	$(5, 0, 0, -1)$	16	$(-9, 2, 1)$
$79A$	$(-10, -8, -2, 4)$	6	$(7, 22, -7)$
$79B$	$(-9, 7, 5, -3)$	-12	$(9, -10, 3)$
$89A$	$(0, 2, -2, 0)$	-5	$(5, 14, -4)$
$89B$	$(-5, 3, 7, -3)$	-6	$(7, 4, -3)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 - 2x^3 - 8x^2 + 6x + 7)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2 + 1), 1/4(\alpha^3 + \alpha^2 + \alpha + 1)\}.$$

	$241A : x^5 - 3x^4 - 7x^3$ $+20x^2 - 4x - 8$	$251A : x^3 + x^2$ $-5x + 1$	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$
2	$(0, 1, 0, 0, 0)$	-3	$(0, 1, 0)$
3	$(-8, -4, 7, 3, -3)$	0	$(3, -2, -1)$
$\sqrt{5}$	$(0, -7, 3, 3, -2)$	-3	$(-3, 1, 1)$
7	$(-6, 8, 2, -2, 0)$	0	$(9, -2, -3)$
$11A$	$(14, 28, -18, -14, 10)$	-6	$(5, -2, -1)$
$11B$	$(4, -4, -3, 0, 1)$	-4	$(5, -2, -1)$
$19A$	$(12, 30, -18, -14, 10)$	0	$(-5, 0, 1)$
$19B$	$(-19, -19, 17, 10, -8)$	8	$(3, 0, -1)$
$29A$	$(10, 16, -12, -8, 6)$	-5	$(-9, 2, 2)$
$29B$	$(-4, -26, 12, 12, -8)$	8	$(-5, 2, 1)$
$31A$	$(-8, -16, 13, 9, -7)$	1	$(-7, 2, 2)$
$31B$	$(12, 16, -12, -8, 6)$	-8	$(8, -2, -2)$
$41A$	$(-21, -47, 29, 22, -16)$	-10	$(8, 0, -2)$
$41B$	$(-12, -10, 11, 6, -5)$	0	$(-11, 4, 3)$
$59A$	$(-14, -28, 14, 12, -8)$	-11	$(-1, 3, 1)$
$59B$	$(-26, -54, 32, 26, -18)$	-9	$(-6, 4, 1)$
$61A$	$(-4, 10, 2, -2, 0)$	-7	$(-4, -1, 2)$
$61B$	$(24, 48, -30, -22, 16)$	8	$(0, 4, 0)$
$71A$	$(32, 46, -35, -25, 19)$	4	$(-11, 10, 3)$
$71B$	$(-4, 12, 2, -2, 0)$	3	$(-4, 6, 1)$
$79A$	$(-24, -29, 25, 15, -12)$	-2	$(-12, 0, 2)$
$79B$	$(-16, -18, 17, 12, -9)$	1	$(-14, 3, 4)$
$89A$	$(11, 13, -7, -6, 4)$	8	$(13, -10, -5)$
$89B$	$(34, 24, -30, -16, 14)$	4	$(17, -4, -3)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^5 - 3x^4 - 7x^3 + 20x^2 - 4x - 8)$, the chosen integral basis is

$$\{1, \alpha, \alpha^2, 1/2(\alpha^3 + \alpha^2 + \alpha), 1/4(\alpha^4 + \alpha^3 + \alpha^2)\}.$$

	$3 \cdot 29B : x^3 + x^2 - 5x + 2$		$271A : x^4 - 14x^2 + 8x + 25$	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$
2	0	$(3, -3, -1)$	1	$(0, 1, 0, 0)$
3	-1	$(1, 0, 0)$	-2	$(-5, -5, 1, 2)$
$\sqrt{5}$	$2 - \sqrt{21}$	$(0, 1, 0)$	2	$(1, 2, 0, -1)$
7	$-9 - \sqrt{21}$	$(2, 2, 1)$	2	$(-6, 8, 2, -2)$
$11A$	$2\sqrt{21}$	$(0, 2, 0)$	4	$(-2, -6, 0, 2)$
$11B$	$1 + \sqrt{21}$	$(-8, 4, 3)$	-4	$(11, 4, -2, -2)$
$19A$	$-2 + 2\sqrt{21}$	$(4, -4, -2)$	-8	$(6, 3, -1, -1)$
$19B$	$-4 - \sqrt{21}$	$(2, 3, 0)$	-4	$(-8, -5, 2, 2)$
$29A$	$1 - \sqrt{21}$	$(-6, 0, 1)$	-2	$(5, 0, -2, 0)$
$29B$	-1	$(1, 0, 0)$	-4	$(6, 0, 0, 0)$
$31A$	$5 - 3\sqrt{21}$	$(-4, 0, 1)$	8	$(-4, -5, 0, 2$
$31B$	$-2 + 4\sqrt{21}$	$(4, -2, -2)$	8	$(2, 10, 0, -4)$
$41A$	$5 - 3\sqrt{21}$	$(-6, 2, 1)$	6	$(10, 7, -3, -3)$
$41B$	2	$(10, -8, -4)$	-10	$(-2, -7, 1, 3)$
$59A$	$-3 + 3\sqrt{21}$	$(-4, -2, -1)$	8	$(-9, 1, 1, 0)$
$59B$	$8 - 3\sqrt{21}$	$(-14, 7, 4)$	-4	$(-1, 4, 0, -2)$
$61A$	$-6 - 2\sqrt{21}$	$(18, -6, -4)$	-14	$(9, 5, -1, -2)$
$61B$	$2 - 5\sqrt{21}$	$(12, -5, -4)$	10	$(23, 6, -6, -3)$
$71A$	$6 + \sqrt{21}$	$(18, -11, -6)$	-8	$(10, 6, -2, -2)$
$71B$	$-4 - 4\sqrt{21}$	$(-8, 8, 4)$	0	$(-14, -2, 4, 2)$
$79A$	$8 + 2\sqrt{21}$	$(-12, 10, 4)$	16	$(-8, -12, 2, 4)$
$79B$	$-6 + \sqrt{21}$	$(-10, 3, 2)$	12	$(-5, 6, 2, -1)$
$89A$	$4 + \sqrt{21}$	$(-12, 5, 2)$	2	$(-20, -6, 4, 4)$
$89B$	$-6 - 4\sqrt{21}$	$(26, -16, -8)$	-14	$(2, 6, 0, -2)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 - 14x^2 + 8x + 25)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2 + 1), 1/4(\alpha^3 + \alpha^2 + \alpha + 1)\}.$$

	$281A: x^5 - x^4 - 11x^3 + 8x^2 + 28x - 8$	$17: x^4 - 4x^3 - 2x^2 + 8x + 1$	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$
2	$(0,1,0,0,0)$	-3	$(0,1,0,0)$
3	$(2,2,-2,-2,2)$	-6	$(-1,-3,2,0)$
$\sqrt{5}$	$(2,2,1,0,-1)$	-2	$(6,0,-5,1)$
7	$(7,7,-1,-2,0)$	2	$(-23,7,24,-6)$
$11A$	$(2,-12,-4,2,2)$	0	$(-4,0,4,-1)$
$11B$	$(-4,-2,2,1,-1)$	0	$(-4,0,4,-1)$
$19A$	$(2,-8,-2,2,0)$	-4	$(-6,4,6,-2)$
$19B$	$(-10,11,8,-1,-4)$	-4	$(-6,4,6,-2)$
$29A$	$(3,-1,-5,-2,4)$	6	$(8,-6,-6,2)$
$29B$	$(-4,4,4,1,-3)$	6	$(8,-6,-6,2)$
$31A$	$(5,-1,-5,-2,4)$	4	$(-4,4,-1,0)$
$31B$	$(6,-5,-2,1,0)$	4	$(-4,4,-1,0)$
$41A$	$(6,10,0,-2,0)$	-6	$(10,-4,-11,3)$
$41B$	$(-6,15,6,-3,-2)$	-6	$(10,-4,-11,3)$
$59A$	$(-8,6,4,0,-2)$	-12	$(2,0,0,0)$
$59B$	$(-8,8,8,2,-6)$	-12	$(2,0,0,0)$
$61A$	$(-4,20,6,-4,-2)$	-10	$(8,-6,-3,1)$
$61B$	$(-12,2,6,1,-3)$	-10	$(8,-6,-3,1)$
$71A$	$(10,6,0,0,-2)$	-4	$(-6,6,2,-1)$
$71B$	$(2,8,3,-2,-1)$	-4	$(-6,6,2,-1)$
$79A$	$(6,-22,-10,4,4)$	12	$(22,-2,-17,4)$
$79B$	$(-8,-18,-4,4,2)$	12	$(22,-2,-17,4)$
$89A$	$(4,-2,2,2,-2)$	10	$(1,5,-4,0)$
$89B$	$(-2,-16,-3,4,1)$	10	$(1,5,-4,0)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^5 - x^4 - 11x^3 + 8x^2 + 28x - 8)$, the chosen integral basis is

$$\{1, \alpha, \alpha^2, 1/2(\alpha^3 + \alpha^2 + \alpha), 1/4(\alpha^4 + \alpha^3 - \alpha^2 + 2\alpha)\}.$$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 - 4x^3 - 2x^2 + 8x + 1)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2 + 1), 1/2(\alpha^3 + \alpha)\}.$$

	$\sqrt{5}\cdot 59B : x^3+x^2-7x+1$		$311A : x^4+x^3-7x^2-x+2$	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$
2	$2-\sqrt{3}$	$(0,1,0)$	-2	$(0,1,0,0)$
3	$1+2\sqrt{3}$	$(-3,0,1)$	1	$(-1,-8,1,2)$
$\sqrt{5}$	-1	$(1,0,0)$	-4	$(3,3,-1,-1)$
7	$-1-6\sqrt{3}$	$(-9,4,3)$	5	$(-1,20,-3,-6)$
$11A$	$-3+2\sqrt{3}$	$(-1,0,1)$	-5	$(1,-8,1,2)$
$11B$	$-\sqrt{3}$	$(8,-3,-3)$	-2	$(-2,8,0,-2)$
$19A$	$2+3\sqrt{3}$	$(6,-1,-3)$	-2	$(-3,-3,1,1)$
$19B$	2	$(-6,0,2)$	1	$(-1,-1,0,0)$
$29A$	$-3\sqrt{3}$	$(0,-3,-1)$	4	$(-2,-14,2,4)$
$29B$	$\sqrt{3}$	$(0,-3,-1)$	-5	$(-5,7,0,-2)$
$31A$	$-4+2\sqrt{3}$	$(-4,2,2)$	-4	$(4,-10,0,2)$
$31B$	$-7-2\sqrt{3}$	$(-1,4,1)$	-3	$(5,0,-1,0)$
$41A$	$-6-3\sqrt{3}$	$(2,-1,-1)$	8	$(3,5,-1,-1)$
$41B$	$4\sqrt{3}$	$(-4,4,2)$	-10	$(-4,-6,2,2)$
$59A$	$-4\sqrt{3}$	$(-12,4,4)$	1	$(-1,12,-1,-2)$
$59B$	-1	$(1,0,0)$	14	$(10,4,-4,-2)$
$61A$	$2-3\sqrt{3}$	$(2,3,-1)$	4	$(3,19,-3,-5)$
$61B$	$-4+6\sqrt{3}$	$(8,-2,-4)$	7	$(-5,-2,1,0)$
$71A$	$-2\sqrt{3}$	$(-16,2,6)$	-10	$(11,9,-3,-3)$
$71B$	$12-\sqrt{3}$	$(-4,1,1)$	-13	$(1,19,-2,-4)$
$79A$	$5-4\sqrt{3}$	$(13,-2,-3)$	-14	$(6,14,-4,-4)$
$79B$	14	$(-18,4,6)$	-8	$(-9,5,1,-1)$
$89A$	$-2\sqrt{3}$	$(16,2,-4)$	0	$(4,22,-2,-6)$
$89B$	-12	$(0,0,-2)$	0	$(-16,-4,4,2)$

For $\mathbb{Q}(\alpha)\cong\mathbb{Q}[x]/(x^3+x^2-7x+1)$, the chosen integral basis is

$$\{1,\alpha,1/2(\alpha^2+1)\}.$$

For $\mathbb{Q}(\alpha)\cong\mathbb{Q}[x]/(x^4+x^3-7x^2-x+2)$, the chosen integral basis is

$$\{1,\alpha,\alpha^2,1/2(\alpha^3+\alpha)\}.$$

	$331A : x^6 - 2x^5 - 17x^4 + 32x^3$ $+39x^2 - 86x + 25$	$11A \cdot 31A : x^3 - x^2 - 12x + 16$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	$(0, 1, 0, 0, 0, 0)$	0	0	$(0, 1, 0)$
3	$(15, -19, -8, 12, 4, -11)$	0	0	$(8, -2, -4)$
$\sqrt{5}$	$(-1, 3, 1, -3, -1, 3)$	2	1	$(-1, 0, 1)$
7	$(-30, 43, 13, -23, -7, 20)$	2	0	$(0, 0, -2)$
$11A$	$(-4, 13, 1, -8, -2, 7)$	1	1	$(-1, 0, 0)$
$11B$	$(-16, 13, 11, -7, -3, 6)$	0	2	$(-2, 0, 2)$
$19A$	$(21, -21, -12, 14, 5, -13)$	-2	0	$(4, -2, -2)$
$19B$	$(14, -26, -6, 14, 4, -12)$	-1	0	$(0, -1, -1)$
$29A$	$(-32, 34, 17, -20, -7, 18)$	-1	0	$(0, 1, -1)$
$29B$	$(6, -6, -6, 4, 2, -4)$	-4	-5	$(7, -2, -1)$
$31A$	$(10, 2, -6, -2, 0, 2)$	1	1	$(-1, 0, 0)$
$31B$	$(-30, 34, 17, -20, -7, 18)$	1	2	$(-18, 5, 7)$
$41A$	$(2, 22, -6, -12, -2, 10)$	-2	-3	$(9, 0, -1)$
$41B$	$(7, -13, -4, 6, 2, -5)$	4	2	$(14, -2, -6)$
$59A$	$(20, -30, -12, 18, 6, -16)$	-11	-10	$(-6, 3, 5)$
$59B$	$(-4, 0, -2, -2, 0, 2)$	5	0	$(-20, 3, 7)$
$61A$	$(22, -30, -12, 18, 6, -16)$	5	2	$(-10, -1, 3)$
$61B$	$(6, -11, -1, 5, 1, -4)$	5	2	$(14, -3, -7)$
$71A$	$(63, -79, -32, 46, 15, -41)$	-2	2	$(-22, 6, 8)$
$71B$	$(-44, 36, 26, -20, -8, 18])$	-2	2	$(-2, 2, 0)$
$79A$	$(-51, 57, 27, -33, -11, 29)$	6	5	$(-21, 6, 9)$
$79B$	$(10, 4, -10, -4, 0, 4)$	-4	-5	$(-1, 0, -1)$
$89A$	$(9, -41, 0, 22, 5, -19)$	-4	0	$(12, -2, -8)$
$89B$	$(-34, 17, 19, -8, -4, 7)$	-6	-10	$(-6, 4, 4)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^6 - 2x^5 - 17x^4 + 32x^3 + 39x^2 - 86x + 25)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2 + 1), 1/2(\alpha^3 + \alpha), 1/4(\alpha^4 - 1), 1/28(\alpha^5 + 2\alpha^4 - 2\alpha^3 - 4\alpha^2 + 9\alpha + 6)\}.$$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^3 - x^2 - 12x + 16)$, the chosen integral basis is

$$\{1, \alpha, 1/4(\alpha^2 - \alpha)\}.$$

	$\sqrt{5}\cdot 71B : x^3+x^2-11x+1$		$359A : x^3-x^2-5x+1$		
$\mathfrak{p}$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_1)$	$a_\mathfrak{p}(f_2)$	$a_\mathfrak{p}(f_3)$
2	$1-\sqrt{2}$	$(0,1,0)$	2	-2	$(0,1,0)$
3	$1+3\sqrt{2}$	$(-3,0,1)$	1	-3	$(0,-2,0)$
$\sqrt{5}$	1	$(1,0,0)$	-3	-3	$(2,0,0)$
7	$-2-6\sqrt{2}$	$(0,-2,0)$	-4	8	$(7,2,-3)$
$11A$	$-3-2\sqrt{2}$	$(6,-1,-1)$	0	2	$(0,0,0)$
$11B$	$-1+2\sqrt{2}$	$(6,-1,-1)$	6	-4	$(1,0,-1)$
$19A$	$1-3\sqrt{2}$	$(-5,0,1)$	-4	-4	$(9,1,-4)$
$19B$	6	$(8,-2,-2)$	2	-2	$(-9,-2,5)$
$29A$	$-1+\sqrt{2}$	$(-7,0,1)$	3	-1	$(7,3,-2)$
$29B$	$-4+2\sqrt{2}$	$(8,0,-2)$	6	4	$(0,-2,0)$
$31A$	$6\sqrt{2}$	$(-2,2,0)$	-4	0	$(11,0,-5)$
$31B$	$-2+2\sqrt{2}$	$(-6,0,2)$	-7	1	$(-11,-3,4)$
$41A$	$-8-2\sqrt{2}$	$(-8,0,2)$	-6	0	$(-11,0,3)$
$41B$	$-3-6\sqrt{2}$	$(0,-1,-1)$	-6	-4	$(-11,0,5)$
$59A$	$3+5\sqrt{2}$	$(-1,-2,-1)$	6	-6	$(-10,-2,4)$
$59B$	$7-\sqrt{2}$	$(-1,0,1)$	9	-15	$(0,2,2)$
$61A$	$-2-2\sqrt{2}$	$(4,2,0)$	11	-5	$(-4,0,-2)$
$61B$	-1	$(-2,3,1)$	-10	12	$(13,6,-7)$
$71A$	$-6-2\sqrt{2}$	$(-10,4,2)$	0	6	$(18,6,-8)$
$71B$	1	$(-1,0,0)$	-3	9	$(-8,0,4)$
$79A$	-4	$(-16,4,4)$	-10	-12	$(13,6,-3)$
$79B$	10	$(-16,2,2)$	-4	2	$(-14,0,6)$
$89A$	$2-8\sqrt{2}$	$(-6,0,0)$	-3	-15	$(5,1,-2)$
$89B$	$-6+4\sqrt{2}$	$(18,-4,-4)$	-12	-14	$(8,0,-2)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^3+x^2-11x+1)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2+1)\}.$$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^3-x^2-5x+1)$, the chosen integral basis is

$$\{1, \alpha, 1/2(\alpha^2+1)\}.$$

	$19A^2 : x^4 - 10x^2 + 5$			$19 : x^3 - 2x^2 - 5x + 4$			
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$
2	-2	2	$(0,1,0,0)$	3	-4	0	$(0,1,0)$
3	-1	1	$(0,-8,0,3)$	6	-2	0	$(2,-2,0)$
$\sqrt{5}$	-3	-3	$(3,0,-1,0)$	-2	3	1	$(3,2,-1)$
7	2	2	$(-4,0,1,0)$	10	-13	-5	$(11,2,-1)$
$11A$	-3	3	$(0,4,0,-2)$	-4	3	$2-\omega_{15}$	$(-3,-2,1)$
$11B$	-3	3	$(0,4,0,-2)$	-4	3	$2+\omega_{15}$	$(-3,-2,1)$
$19A$	-1	-1	$(-6,0,4,0)$	-1	1	-1	$(1,0,0)$
$19B$	0	0	$(0,0,0,0)$	-1	1	-1	$(1,0,0)$
$29A$	6	-6	$(0,3,0,-1)$	6	6	$2\omega_{15}$	$(-6,0,0)$
$29B$	3	3	$(2,0,-3,0)$	6	6	$-2\omega_{15}$	$(-6,0,0)$
$31A$	-7	-7	$(0,0,2,0)$	-4	-4	$2+2\omega_{15}$	$(-8,-2,2)$
$31B$	-5	5	$(0,-12,0,4)$	-4	-4	$2-2\omega_{15}$	$(-8,-2,2)$
$41A$	-3	-3	$(8,0,-5,0)$	-10	-6	2	$(10,2,-2)$
$41B$	3	-3	$(0,7,0,-3)$	-10	-6	2	$(10,2,-2)$
$59A$	12	-12	$(0,6,0,-2)$	0	-6	$-2\omega_{15}$	$(-2,-2,0)$
$59B$	-3	-3	$(-18,0,12,0)$	0	-6	$2\omega_{15}$	$(-2,-2,0)$
$61A$	-14	14	$(0,-16,0,7)$	2	-1	$2+\omega_{15}$	$(9,4,-3)$
$61B$	-5	5	$(0,11,0,-3)$	2	-1	$2-\omega_{15}$	$(9,4,-3)$
$71A$	9	-9	$(0,10,0,-4)$	4	6	$-8-2\omega_{15}$	$(10,-2,0)$
$71B$	0	0	$(0,10,0,-4)$	4	6	$-8+2\omega_{15}$	$(10,-2,0)$
$79A$	5	5	$(10,0,-8,0)$	-4	8	$-10-2\omega_{15}$	$(12,2,-2)$
$79B$	4	-4	$(-2,0,-2,0)$	-4	8	$-10+2\omega_{15}$	$(12,2,-2)$
$89A$	9	-9	$(0,10,0,-2)$	2	12	$-10+2\omega_{15}$	$(-16,4,2)$
$89B$	-18	-18	$(0,-18,0,7)$	2	12	$-10-2\omega_{15}$	$(-16,4,2)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 - 10x^2 + 5)$, the chosen integral basis is

$$\{1, \alpha, 1/4(\alpha^2 + 1), 1/4(\alpha^3 + \alpha)\}.$$

	$389A : x^4 - 3x^3 - 3x^2 + 10x - 4$		$\sqrt{5} \cdot 79A : x^3 + x^2 - 9x - 1$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	$-2 - \omega_5$	$(0, 1, 0, 0)$	1	-3	$(-1, 2, 0)$
3	$-3\omega_5$	$(2, -1, 0, 0)$	4	-2	$(2, -2, -2)$
$\sqrt{5}$	$-3 + 2\omega_5$	$(-4, 5, 3, -2)$	-1	-1	$(1, 0, 0)$
7	$1 + 2\omega_5$	$(-22, 20, 12, -7)$	4	-6	$(2, -6, -2)$
$11A$	-3	$(6, -7, -3, 2)$	-2	-4	$(4, -2, -2)$
$11B$	$-3 + \omega_5$	$(4, -4, -3, 2)$	2	-2	$(0, 0, 2)$
$19A$	$-3 + 3\omega_5$	$(10, -4, -5, 2)$	0	2	$(0, -2, 0)$
$19B$	$-3\omega_5$	$(2, -7, -2, 2)$	0	-8	$(-4, 0, 4)$
$29A$	$-4 + 7\omega_5$	$(16, -19, -8, 6)$	6	-2	$(-6, 4, 4)$
$29B$	$-5 - 2\omega_5$	$(-5, 1, 3, -1)$	-4	0	$(2, -4, -2)$
$31A$	$-2 + 3\omega_5$	$(0, 0, 1, -1)$	0	4	$(4, 0, -4)$
$31B$	$3 - 6\omega_5$	$(0, 1, -1, 0)$	-4	0	$(4, 0, -4)$
$41A$	$9 - \omega_5$	$(-6, 2, 1, 0)$	-6	12	$(-2, 6, 0)$
$41B$	$-8 + 4\omega_5$	$(-14, 20, 8, -6)$	2	-10	$(2, -4, 0)$
$59A$	$-3 + 3\omega_5$	$(6, 1, -4, 1)$	12	0	$(-4, -4, 4)$
$59B$	$3 - 2\omega_5$	$(-17, 23, 9, -7)$	4	-8	$(-4, 4, 0)$
$61A$	$10 - 3\omega_5$	$(1, 3, 0, -1)$	-14	-10	$(-2, 4, 0)$
$61B$	$2 - 9\omega_5$	$(-16, 9, 8, -4)$	-10	12	$(2, -6, 0)$
$71A$	$12 - 4\omega_5$	$(-11, 16, 6, -5)$	-6	0	$(-8, 6, 6)$
$71B$	$-1 + 2\omega_5$	$(4, -5, -3, 2)$	4	0	$(-4, 8, 0)$
$79A$	$2 - 3\omega_5$	$(12, -3, -6, 2)$	-1	1	$(1, 0, 0)$
$79B$	$-9 + 8\omega_5$	$(-2, 1, 1, 0)$	-16	-8	$(0, 0, -4)$
$89A$	$-5 - 8\omega_5$	$(4, 6, 4, -3)$	0	12	$(-10, 4, 6)$
$89B$	$-15 + 3\omega_5$	$(18, -10, -7, 4)$	0	12	$(-10, 4, 6)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 - 3x^3 - 3x^2 + 10x - 4)$, the chosen integral basis is

$$\{1, \alpha, \alpha^2, 1/2(\alpha^3 + \alpha^2 + \alpha)\}.$$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[]/(x^3 + x^2 - 9x - 1)$, the chosen integral basis is

$$\{1, 1/2(\alpha + 1)1/4(\alpha^2 - 1)\}.$$

	$401A: x^6-15x^4+$ $12x^3+39x^2-24x-29$		$409A: x^4-x^3-8x^2+4x+8$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	-1	$(0,1,0,0,0,0)$	-3	$3-\omega_5$	$(0,1,0,0)$
3	-4	$(-5,-16,-7,12,0,-2)$	-4	3	$(0,-3,-1,1)$
$\sqrt{5}$	-3	$(-5,-12,-3,7,0,-1)$	-1	$-2+2\omega_5$	$(0,3,2,-2)$
7	5	$(24,37,5,-27,2,5)$	2	$-2+8\omega_5$	$(6,8,0,-4)$
$11A$	-4	$(23,35,7,-24,1,4)$	-1	$-4+2\omega_5$	$(-2,-5,0,2)$
$11B$	-3	$(1,11,9,-8,-1,1)$	-1	$1-2\omega_5$	$(6,2,-3,0)$
$19A$	3	$(-27,-24,3,16,-2,-3)$	0	$-3+2\omega_5$	$(2,-8,-4,5)$
$19B$	4	$(37,38,-1,-24)$	-4	$2-6\omega_5$	$(0,-2,2,0)$
$29A$	0	$(-5,2,1,0,0,0)$	-4	-6	$(8,4,0,-2)$
$29B$	0	$(-61,-72,-1,46,-4,-8)$	-3	$-2+4\omega_5$	$(-1,-2,0,0)$
$31A$	10	$(-15,-24,-5,14,0,-2)$	-1	$-3+2\omega_5$	$(-2,-3,-2,3)$
$31B$	-1	$(10,22,10,-18,0,3)$	-1	2	$(-2,-5,-4,4)$
$41A$	-4	$(39,56,7,-36,2,6)$	8	$7-4\omega_5$	$(-10,1,3,-2)$
$41B$	-10	$(34,24,-8,-12,2,2)$	-6	$-1+4\omega_5$	$(-10,0,6,-3)$
$59A$	-5	$(19,11,-5,-6,1,1)$	0	$-7+2\omega_5$	$([8,5,-1,-1)$
$59B$	-2	$(-22,-22,-4,14,0,-2)$	12	$-8\omega_5$	$(6,10,0,-4)$
$61A$	-1	$(-19,-41,-16,31,0,-5)$	6	2	$(3,5,2,-4)$
$61B$	-6	$(-71,-78,1,48,-4,-8)$	5	$-4+6\omega_5$	$(-6,9,4,-4)$
$71A$	0	$(24,8,-14,0,2,0)$	-14	$-12+4\omega_5$	$(5,3,-2,0)$
$71B$	-15	$(-47,-62,-9,42,-2,-7)$	15	$3-2\omega_5$	$(4,6,-3,-1)$
$79A$	0	$(-3,19,13,-14,-1,2)$	-11	$-8+12\omega_5$	$(7,2,-4,0)$
$79B$	-16	$(27,42,1,-24,2,4)$	1	$2+4\omega_5$	$(-6,3,2,0)$
$89A$	3	$(19,36,7,-27,2,5)$	6	$13-8\omega_5$	$(-14,3,11,-6)$
$89B$	15	$(2,-17,-15,11,2,-1)$	1	$7+4\omega_5$	$(6,-7,-8,7)$

The integral basis for the field $\mathbb{Q}(\alpha)$, where α is a root of $x^6-15x^4+12x^3+39x^2-24x-29$ is

$$\{1,\alpha,\alpha^2,\frac{1}{2}(\alpha^3+\alpha^2+\alpha+1),\frac{1}{2}(\alpha^4+1),\frac{1}{4}(\alpha^5+\alpha^4-\alpha-1)\}.$$

The integral basis for the field $\mathbb{Q}(\alpha)$, where α is a root of $x^4-x^3-8x^2+4x+8$, is

$$\{1,\alpha,\frac{1}{2}(\alpha^2+\alpha),\frac{1}{4}(\alpha^3+\alpha^2+2\alpha)\}.$$

	$419A : x^4 - 6x^2 + 2x + 5$			$421A : x^8 - 2x^7 - 23x^6 + 46x^5 +$ $148x^4 - 312x^3 - 176x^2 + 576x - 256$
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_1)$
2	-1	-3	$(0,1,0,0)$	$(0,1,0,0,0,0,0,0)$
3	-4	-4	$(-1,0,1,0)$	$(6,-16,-1,14,0,-7,-1,2)$
$\sqrt{5}$	-4	0	$(3,0,-1,0)$	$(13,-19,-7,16,3,-9,-2,3)$
7	6	6	$(10,8,-4,-2)$	$(46,-73,-29,65,14,-35,-8,11)$
$11A$	-1	-3	$(4,7,-2,-2)$	$(-39,66,28,-61,-14,34,8,-11)$
$11B$	3	-5	$(-3,-8,2,2)$	$(12,-5,-13,5,8,-3,-2,1)$
$19A$	-6	2	$(2,-2,0,0)$	$(-36,42,24,-30,-12,14,4,-4)$
$19B$	-6	2	$(-4,-5,2,1)$	$(8,-18,-10,16,7,-9,-3,3)$
$29A$	-4	0	$(-4,-3,2,1)$	$(62,-78,-48,70,24,-38,-10,12)$
$29B$	6	-10	$(12,1,-4,-1)$	$(40,-73,-24,62,10,-34,-7,11)$
$31A$	10	2	$(2,10,-2,-2)$	$(-72,106,50,-92,-24,50,12,-16)$
$31B$	-10	10	$(1,0,-1,0)$	$(-80,134,56,-122,-28,68,16,-22)$
$41A$	4	0	$(-17,-14,7,4)$	$(-56,96,42,-90,-22,50,12,-16)$
$41B$	2	-2	$(-12,2,4,0)$	$(117,-179,-79,162,38,-90,-21,29)$
$59A$	-12	-12	$(13,14,-5,-4)$	$(-62,76,48,-64,-24,36,10,-12)$
$59B$	-5	-13	$(-6,4,1,0)$	$(-91,136,64,-121,-32,66,16,-21)$
$61A$	10	-10	$(4,6,-4,-2)$	$(52,-74,-38,72,18,-42,-10,14)$
$61B$	-1	13	$(2,1,-2,0)$	$(138,-204,-94,178,45,-97,-23,31)$
$71A$	-14	6	$(-1,-4,1,2)$	$(58,-70,-46,56,25,-29,-9,9)$
$71B$	-14	6	$(-10,2,4,0)$	$(-22,28,14,-30,-6,18,4,-6)$
$79A$	6	10	$(8,-1,-2,-1)$	$(63,-109,-55,102,30,-58,-15,19)$
$79B$	-5	-15	$(6,-14,0,3)$	$(-156,230,109,-198,-56,107,27,-34)$
$89A$	-10	-14	$(-4,3,2,-1)$	$(-32,60,18,-50,-10,26,6,-8)$
$89B$	15	7	$(1,20,-2,-4)$	$(64,-83,-47,75,22,-41,-10,13)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^8 - 2x^7 - 23x^6 + 46x^5 + 148x^4 - 312x^3 - 176x^2 + 576x - 256)$, the chosen integral basis is

$$\{1, \alpha, \alpha^2, 1/2(\alpha^3 + \alpha), 1/4(\alpha^4 + \alpha^2), 1/8(\alpha^5 + \alpha^3 + 4\alpha),$$

$$1/16(\alpha^6 + \alpha^4 - 4\alpha^2), 1/64(\alpha^7 + 2\alpha^6 + \alpha^5 + 2\alpha^4 + 12\alpha^3 + 4\alpha^2 + 16\alpha).$$

	$431A : x^3+3x^2-9x-23$					$2\cdot 109A : x^3-x^2-20x-25,$ x^3-x^2-8x-5	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_5)$	$a_{\mathfrak{p}}(f_1):\alpha_1$	$a_{\mathfrak{p}}(f_2):\alpha_2$
2	1	2	-1	-2	$(0,1,0)$	$(-1,0,0)$	$(1,0,0)$
3	2	-5	-2	-1	$(2,0,0)$	$(0,1,0)$	$(0,1,0)$
$\sqrt{5}$	2	0	3	-2	$(-6,0,1)$	$(-2,-1,1)$	$(6,2,-1)$
7	-10	-4	5	-4	$(-5,1,1)$	$(4,1,-1)$	$(0,2,-1)$
$11A$	0	6	0	-6	$(1,-1,0)$	$(3,0,-1)$	$(-3,-1,1)$
$11B$	-4	0	6	-2	$(12,0,-2)$	$(10,2,-4)$	$(2,-2,0)$
$19A$	4	-1	-4	-5	$(8,-2,-2)$	$(-14,-2,6)$	$(-10,-4,2)$
$19B$	-4	5	-1	5	$(-12,-1,2)$	$(-6,-1,2)$	$(-10,-5,2)$
$29A$	-2	9	-3	5	$(7,0,-2)$	$(10,2,-6)$	$(-10,-4,2)$
$29B$	-2	-9	-3	-1	$(5,0,0)$	$(-6,0,2)$	$(10,6,-2)$
$31A$	0	-4	5	0	$(-9,0,2)$	$(-11,-2,5)$	$(-3,-3,1)$
$31B$	4	2	-1	4	$(15,-1,-3)$	$(20,3,-7)$	$(-8,-2,1)$
$41A$	-6	9	-6	-3	$(-12,2,4)$	$(4,2,-2)$	$(12,4,-2)$
$41B$	2	-9	12	-5	$(-14,0,2)$	$(13,2,-7)$	$(17,9,-3)$
$59A$	8	-12	-3	4	$(-3,3,1)$	$(-4,1,0)$	$(0,-1,0)$
$59B$	4	-12	6	2	$(-14,2,2)$	$(8,-2,-2)$	$(0,4,-2)$
$61A$	-6	8	-1	-12	$(17,1,-3)$	$(17,0,-5)$	$(-13,3,1)$
$61B$	2	-4	2	2	$(7,-3,-2)$	$(-16,-3,6)$	$(-8,-7,2)$
$71A$	8	3	-12	-5	$(12,-2,-2)$	$(-14,-4,4)$	$(-18,-8,4)$
$71B$	8	-6	-12	-2	$(8,0,0)$	$(-32,-4,14)$	$(12,2,-2)$
$79A$	8	-4	2	14	$(-8,2,0)$	$(-24,-4,8)$	$(20,4,-4)$
$79B$	0	-1	-10	3	$(14,2,-2)$	$(-3,2,-1)$	$(5,3,-1)$
$89A$	10	15	-15	-1	$(10,-3,-4)$	$(6,3,-6)$	$(10,1,-2)$
$89B$	2	-6	-12	-8	$(-16,0,4)$	$(22,5,-11)$	$(-10,-4,1)$

The integral basis for the field $\mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3+3x^2-9x-23$ is

$$\{1, \alpha, 1/2(\alpha^2+1)\}.$$

The integral basis for the field $\mathbb{Q}(\alpha_1)$, where α_1 is a root of the polynomial $x^3-x^2-20x-25$, is

$$\{1, \alpha_1, 1/5(\alpha_1^2-\alpha_1)\}.$$

	$439A : x^3 - x^2 - 9x + 5$			$3 \cdot 7 : x^3 - x^2 - 13x + 5$				
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_5)$
2	$2 - \omega_5$	$-2 - \omega_5$	$(0, 1, 0)$	3	-3	1	1	$(-1, 2, 0)$
3	$2 + \omega_5$	$-4 + \omega_5$	$(-4, -1, 1)$	-1	1	1	1	$(-1, 0, 0)$
$\sqrt{5}$	$-2 + \omega_5$	$-2 + \omega_5$	$(2, 0, 0)$	2	-2	2	2	$(2, -2, 0)$
7	$2 - 5\omega_5$	$2 + 7\omega_5$	$(0, -2, 0)$	-1	1	1	1	$(-1, 0, 0)$
$11A$	1	$1 - 4\omega_5$	$(-1, -1, 0)$	-6	4	-4	4	$(-3, 0, 0)$
$11B$	1	$-3 + 4\omega_5$	$(6, 0, -2)$	-6	4	4	-4	$(-3, 0, 0)$
$19A$	$-1 + 4\omega_5$	$-3 - 2\omega_5$	$(4, 0, 0)$	6	4	-8	0	$(2, 0, -2)$
$19B$	$2 - 6\omega_5$	$-2 - 2\omega_5$	$(6, -1, -1)$	6	4	0	-8	$(2, 0, -2)$
$29A$	$1 + 5\omega_5$	$1 - \omega_5$	$(4, 0, -2)$	2	-2	6	-10	$(2, -4, 0)$
$29B$	-3	$-1 - 2\omega_5$	$(4, -2, 0)$	2	-2	-10	6	$(2, -4, 0)$
$31A$	-1	$-1 - 4\omega_5$	$(-6, 0, 2)$	-10	0	-4	4	$(2, 4, -2)$
$31B$	$2 + \omega_5$	$5\omega_5$	$(-10, 0, 2)$	-10	0	4	-4	$(2, 4, -2)$
$41A$	$-3 - 4\omega_5$	$-1 - 2\omega_5$	$(2, 1, -1)$	2	2	6	6	$(-6, -2, 4)$
$41B$	$-3 + 7\omega_5$	$9 - 5\omega_5$	$(-4, 2, 0)$	2	2	6	6	$(-6, -2, 4)$
$59A$	$7 - 3\omega_5$	$-1 + 7\omega_5$	$(-10, 0, 2)$	8	12	12	-4	$(-16, 4, 4)$
$59B$	$-7 + 7\omega_5$	$-7 + 7\omega_5$	$(-6, 0, 2)$	8	12	-4	12	$(-16, 4, 4)$
$61A$	$1 - 2\omega_5$	$7 - 4\omega_5$	$(-2, 1, 1)$	-2	-2	-10	6	$(6, 0, -4)$
$61B$	$3 + 6\omega_5$	$-11 + 4\omega_5$	$(-12, 1, 1)$	-2	-2	6	-10	$(6, 0, -4)$
$71A$	$-11 - 2\omega_5$	$7 - 4\omega_5$	$(-14, 2, 4)$	10	0	0	-8	$(-8, 0, 0)$
$71B$	-5	$-11 + 6\omega_5$	$(0, 1, 1)$	10	0	-8	0	$(-8, 0, 0)$
$79A$	$-9 + 7\omega_5$	$-5 + \omega_5$	$(16, 3, -3)$	-4	-16	8	-8	$(4, 0, -4)$
$79B$	$2 - 4\omega_5$	$-6 + 12\omega_5$	$(3, 3, 0)$	-4	-16	-8	8	$(4, 0, -4)$
$89A$	$-2 - 5\omega_5$	$-6 - 5\omega_5$	$(17, 1, -6)$	-6	-14	6	6	$(-6, 2, 0)$
$89B$	$-5 - 6\omega_5$	$5 - 4\omega_5$	$(10, 6, -2)$	-6	-14	6	6	$(-6, 2, 0)$

The integral basis for the field $\mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3 - x^2 - 9x + 5$ is

$$\{1, \alpha, 1/2(\alpha^2 + 1)\}.$$

The integral basis for the field $\mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3 - x^2 - 13x + 5$, is

$$\{1, 1/2(\alpha + 1), 1/4(\alpha^2 - 1)\}.$$

	$\sqrt{5}\cdot 89B : x^3-x^2-12x+4$				$11A\cdot 41B : x^3-x^2-9x+5$, x^3+x^2-9x+3		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_4)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2):\alpha_1$	$a_{\mathfrak{p}}(f_3):\alpha_2$
2	1	-3	2	$(0,1,0)$	3	$(0,1,0)$	$(0,1,0)$
3	2	-2	$2-\omega_{13}$	$(-4,-1,1)$	4	$(3,0,-1)$	$(3,-2,-1)$
$\sqrt{5}$	1	-1	-1	$(1,0,0)$	-2	$(1,-1,0)$	$(1,1,0)$
7	-6	6	$7-5\omega_{13}$	$(-3,-2,1)$	-10	$(0,2,0)$	$(-12,2,4)$
$11A$	0	0	$2-2\omega_{13}$	$(6,0,-2)$	-1	$(1,0,0)$	$(1,0,0)$
$11B$	0	-4	$-2+2\omega_{13}$	$(6,0,-2)$	-6	$(-1,0,1)$	$(5,-2,-1)$
$19A$	-8	0	$-1+3\omega_{13}$	$(1,0,1)$	6	$(-3,0,1)$	$(-5,0,1)$
$19B$	4	-4	$2\omega_{13}$	$(-8,-2,2)$	4	$(6,0,-2)$	$(-6,0,2)$
$29A$	-10	-10	$3-3\omega_{13}$	$(5,0,-1)$	0	$(3,-2,-1)$	$(-5,2,1)$
$29B$	-2	-2	0	$(8,2,-4)$	2	$(-5,1,0)$	$(13,-3,-4)$
$31A$	4	0	$-2+4\omega_{13}$	$(-2,-2,0)$	-10	$(5,2,-1)$	$(-3,2,1)$
$31B$	0	-8	$-1-3\omega_{13}$	$(-9,-2,5)$	4	$(-4,-2,2)$	$(8,-2,-2)$
$41A$	10	2	$-8-\omega_{13}$	$(-10,-1,3)$	-8	$(5,0,-1)$	$(3,-4,-1)$
$41B$	2	6	-6	$(10,4,-4)$	1	$(-1,0,0)$	$(-1,0,0)$
$59A$	4	-4	$-8+2\omega_{13}$	$(0,2,-2)$	0	$(-6,-4,2)$	$(-10,4,2)$
$59B$	0	12	$6-3\omega_{13}$	$(8,3,-5)$	4	$(1,-1,-2)$	$(19,-5,-6)$
$61A$	-2	-2	$3-7\omega_{13}$	$(-1,0,1)$	-14	$(14,0,-4)$	$(-10,4,4)$
$61B$	14	-6	2	$(-14,0,4)$	-4	$(-17,2,3)$	$(13,-2,-1)$
$71A$	-16	0	$-6+3\omega_{13}$	$(8,-1,-1)$	2	$(5,4,-1)$	$(-7,2,3)$
$71B$	8	16	$-12+3\omega_{13}$	$(-2,1,-1)$	-8	$(-2,0,-2)$	$(14,0,-2)$
$79A$	-4	8	$3+5\omega_{13}$	$(1,2,-1)$	14	$(3,0,-1)$	$(-17,4,5)$
$79B$	-12	4	$8-3\omega_{13}$	$(-6,-3,3)$	-8	$(-10,2,0)$	$(10,2,0)$
$89A$	-6	-6	$8+\omega_{13}$	$(-18,-5,7)$	10	$(-12,-2,4)$	$(-16,6,4)$
$89B$	1	1	-1	$(-1,0,0)$	12	$(-15,0,5)$	$(-1,-2,-1)$

The chosen integral basis for the field $\mathbb{Q}(\alpha)$, where α is a root of $x^3-x^2-12x+4$, is $\{1,\alpha,1/4(\alpha^2+\alpha+2)\}$.

The integral basis for the field $\mathbb{Q}(\alpha_1)$, where α_1 is a root of x^3-x^2-9x+5, is $\{1,\alpha_1,1/2(\alpha_1^2+1)\}$.

The integral basis for the field $\mathbb{Q}(\alpha_2)$, where α_2 is a root of x^3+x^2-9x+3, is $\{1,\alpha_2,1/2(\alpha_2^2+1)\}$.

	$461A : x^4 - 4x^3 - 12x^2 + 48x - 16$			$479A : x^4 + x^3 - 8x^2 - 11x + 1$		
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_3)$
2	$-2-\omega_5$	$-1-\omega_5$	$(-1,3,1,-1)$	1	$-1-\omega_5$	$(0,1,0,0)$
3	-1	-3	$(0,2,0,0)$	2	-3	$(2,-9,-3,4)$
$\sqrt{5}$	$3-2\omega_5$	$-3+2\omega_5$	$(3,0,-1,0)$	2	$-3+\omega_5$	$(-1,8,3,-4)$
7	$5+3\omega_5$	$8-9\omega_5$	$(-12,22,4,-6)$	10	$-7+2\omega_5$	$(4,9,1,-4)$
$11A$	$3\omega_5$	$-1-3\omega_5$	$(6,-8,-2,2)$	-2	$-3+4\omega_5$	$(0,-5,-1,2)$
$11B$	$5-\omega_5$	$-4-\omega_5$	$(-5,-1,2,0)$	0	$-1-2\omega_5$	$(4,1,-1,0)$
$19A$	$-3+3\omega_5$	$4-3\omega_5$	$(4,-6,-2,2)$	2	$-3-3\omega_5$	$(1,-6,-1,2)$
$19B$	$-3+3\omega_5$	$-4+5\omega_5$	$(4,-6,-2,2)$	-4	$1-\omega_5$	$(3,16,5,-8)$
$29A$	$-1+5\omega_5$	$4-3\omega_5$	$(-6,8,2,-2)$	-6	$-2+6\omega_5$	$(8,-6,-4,4)$
$29B$	$5-2\omega_5$	$-1+6\omega_5$	$(-10,8,3,-2)$	-6	$1-7\omega_5$	$(-3,-10,-1,4)$
$31A$	$-3+6\omega_5$	$-3+2\omega_5$	$(4,-6,-1,1)$	8	$4-3\omega_5$	$(-3,-25,-8,12)$
$31B$	$1-6\omega_5$	$-1+6\omega_5$	$(-4,-8,2,2)$	-8	$-2-\omega_5$	$(7,-7,-4,4)$
$41A$	$-7+4\omega_5$	$-1+4\omega_5$	$(8,-8,-2,2)$	6	$-3+9\omega_5$	$(-5,8,3,-4)$
$41B$	$2-\omega_5$	$-9-\omega_5$	$(10,-4,-4,2)$	4	$5-3\omega_5$	$(-1,4,1,-2)$
$59A$	-6	$-2+8\omega_5$	$(4,2,-1,-1)$	2	$-11+2\omega_5$	$(6,-11,-5,6)$
$59B$	$2-2\omega_5$	$8-6\omega_5$	$(-11,13,4,-4)$	12	$1+6\omega_5$	$(-10,-9,-1,4)$
$61A$	$6+3\omega_5$	$-3-3\omega_5$	$(-14,16,5,-4)$	-8	$4-2\omega_5$	$(-2,-10,-4,6)$
$61B$	$6-8\omega_5$	$-2-4\omega_5$	$(16,-18,-7,4)$	-12	$5-2\omega_5$	$(4,23,5,-10)$
$71A$	$-4+8\omega_5$	-12	$(-12,22,6,-6)$	-10	$2-8\omega_5$	$(-10,28,12,-14)$
$71B$	$2-10\omega_5$	$4-6\omega_5$	$(-6,6,3,-1)$	12	$1+3\omega_5$	$(-15,6,5,-4)$
$79A$	$-9+3\omega_5$	$6-11\omega_5$	$(14,-12,-6,4)$	16	$7-2\omega_5$	$(-4,25,7,-12)$
$79B$	$1+6\omega_5$	$5+2\omega_5$	$(-13,9,4,-4)$	0	$-1-5\omega_5$	$(-9,-2,3,0)$
$89A$	$-16-\omega_5$	$-3-5\omega_5$	$(10,-2,-2,0)$	-10	$-5+2\omega_5$	$(10,39,9,-16)$
$89B$	-9	$-7+8\omega_5$	$(-2,16,2,-4)$	-10	$-9+14\omega_5$	$(6,-33,-11,16)$

The chosen integral basis for the field $\mathbb{Q}(\alpha)$, where α is a roots of $x^4 - 4x^3 - 12x^2 + 48x - 16$, is

$$\{1, 1/2\alpha, 1/4\alpha^2, 1/8\alpha^3\}.$$

The chosen integral basis for the field $\mathbb{Q}(\alpha)$, where α is a roots of $x^4 + x^3 - 8x^2 - 11x + 1$, is

$$\{1, \alpha, \alpha^2, 1/4(\alpha^3 + 2\alpha^2 + 2\alpha - 1)\}.$$

	$491A : x^6 - x^5 - 13x^4$ $+16x^3 + 39x^2 - 63x + 17$		$4 \cdot 31A : x^4 + 12x^3$ $-24x^2 - 80x + 80$	
$\mathfrak{p}$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$	$a_{\mathfrak{p}}(f_1)$	$a_{\mathfrak{p}}(f_2)$
2	$-1 - \omega_5$	$(0, 1, 0, 0, 0, 0)$	0	$(0, 0, 0, 0)$
3	$-1 - 2\omega_5$	$(-7, 7, 1, -1, 0, 0)$	$-1 + \omega_3$	$(0, 1, 0, 0)$
$\sqrt{5}$	$-3 + 2\omega_5$	$(2, 3, -4, -1, 1, 0)$	$1 - \omega_3$	$(20, -11, -2, 6)$
7	$5 + 3\omega_5$	$(-3, 23, -5, -12, 1, 2)$	$-4 - 2\omega_3$	$(-6, 6, 0, -4)$
$11A$	$-2 - \omega_5$	$(4, -7, 0, 1, 0, 0)$	$1 + \omega_3$	$(2, -1, 0, 0)$
$11B$	-5	$(5, -4, 3, 1, -1, 0)$	2	$(-20, 14, 2, -8)$
$19A$	$1 + 2\omega_5$	$(-17, 33, 1, -13, 0, 2)$	$5 + \omega_3$	$(-18, 5, 2, -2)$
$19B$	$-2 + 3\omega_5$	$(18, -41, 6, 15, -2, -2)$	$-2 - 2\omega_3$	$(-32, 14, 4, -8)$
$29A$	$-6 + 7\omega_5$	$(3, -16, 4, 7, -1, -1)$	$-2 + 4\omega_3$	$(6, -8, 0, 4)$
$29B$	$3 - 2\omega_5$	$(11, -21, -1, 11, 0, -2)$	$-2 + 2\omega_3$	$(10, 0, -2, 0)$
$31A$	$-1 - 5\omega_5$	$(-1, 6, -7, -2, 2, 0)$	-1	$(1, 0, 0, 0)$
$31B$	$4 - 8\omega_5$	$(-7, 16, -3, -7, 1, 1)$	$5 - 3\omega_3$	$(10, -3, -2, 2)$
$41A$	$-1 + 7\omega_5$	$(-13, 28, 1, -12, 0, 2)$	$6\omega_3$	$(2, -2, 0, 0)$
$41B$	$-3 + 3\omega_5$	$(-17, 14, 3, -2, 0, 0)$	$-6\omega_3$	$(-30, 14, 4, -8)$
$59A$	$2 - 6\omega_5$	$(-20, 28, 2, -12, 0, 2)$	$-6 + 2\omega_3$	$(32, -14, -4, 8)$
$59B$	$1 + \omega_5$	$(-11, 14, 1, -10, 0, 2)$	-4	$(0, 0, 0, 0)$
$61A$	$-14 + \omega_5$	$(2, 19, -2, -11, 0, 2)$	$-3 + 3\omega_3$	$(40, -21, -4, 12)$
$61B$	$3 + 4\omega_5$	$(17, -33, -1, 13, 0, -2)$	$-2 + 4\omega_3$	$(-30, 16, 4, -8)$
$71A$	$7 - 7\omega_5$	$(33, -54, -3, 24, 0, -4)$	$9 - 3\omega_3$	$(30, -21, -4, 12)$
$71B$	$-6 - 4\omega_5$	$(8, -17, 10, 4, -3, 0)$	$-2 + 6\omega_3$	$(28, -6, -4, 4)$
$79A$	$-6 - 6\omega_5$	$(-18, 6, 10, 0, -2, 0)$	$-6 - 2\omega_3$	$(44, -30, -4, 16)$
$79B$	$7\omega_5$	$(11, -14, 2, 7, -1, -1)$	$4 - 4\omega_3$	$(-48, 24, 4, -12)$
$89A$	$1 - 4\omega_5$	$(-30, 69, 4, -30, -1, 5)$	$-6 - 4\omega_3$	$(70, -36, -8, 20)$
$89B$	$10 - \omega_5$	$(10, -31, 0, 13, 0, -2)$	$-8 - 6\omega_3$	$(6, -10, 0, 4)$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^6 - x^5 - 13x^4 + 16x^3 + 39x^2 - 63x + 17)$, the chosen integral basis is

$$\{1, \alpha, \alpha^2, \alpha^3, 1/2(\alpha^4 + \alpha^3 + \alpha + 1), 1/2(\alpha^5 + \alpha^3 + \alpha^2 + 1)\}.$$

For $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^4 + 12x^3 - 24x^2 - 80x + 80)$ the chosen integral basis:

$$\{1, \alpha, 1/2\alpha^2, 1/10(\alpha^3 - \alpha^2 + 4\alpha)\}.$$

8. Motives.

By a gp search, we found the list of all curves of prime conductor of norm less than 100. We would like to thank Prof. N. Elkies for his valuable help in implementing this search. We have only listed one curve for each prime as one gets the other one by Galois conjugation. For each curve $E/\mathbb{Q}(\sqrt{5})$ of conductor $\mathfrak{c}$, we have checked that all the $a_{\mathfrak{p}}(E)$ match up with the Fourier coefficients of a modular form of level $\mathfrak{c}$ given in the table. We would like to complete the list of those curves in a near future along with studing their modularity by making use of some of the results in Skinner and Wiles [**50, 51**]. In fact, we already have the modularity of the curves listed below thanks to a result in [**51**]. We begin by recalling this theorem.

Definition 1.45. *Let* $\bar{\rho}: \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *be a Galois representation, and let* $\mathfrak{p}|p$ *be a prime of* F. *We say that* $\bar{\rho}$ *is* distinguished *at* $\mathfrak{p}$ *if the semisimplification of the restriction of* $\bar{\rho}$ *to* $D_{\mathfrak{p}}$, *the decomposition group at* $\mathfrak{p}$, *is isomorphic to* $\theta_1 \oplus \theta_2$, *where* θ_1 *and* θ_2 *are two distincts characters from* $\mathrm{Gal}(\bar{F}/F)$ *to* $\overline{\mathbb{F}}_p^{\times}$.

Proposition 1.2. in Ellenberg [**22**], says that the mod p Galois representations associated to the curves listed below are $\mathfrak{p}$–distinguished for all $\mathfrak{p}$ not dividing the conductor. This is also a consequence of the corollary to Proposition 11 in Serre [**44**].

Here is the theorem we need from Skinner and Wiles [**51**] (see [**22**, Theorem 3.1]).

THEOREM 1.46. *Let F be a totally real number field, let $p > 2$ be a rational prime, let L be a finite extension of $\mathbb{Q}_p$, and let*

$$\rho: \operatorname{Gal}(\bar{F}/F) \to \operatorname{GL}_2(L)$$

be a continuous irreducible Galois representation ramified at only finitely many primes. Suppose

- $\det\rho = \psi\chi_p^{k-1}$, *where χ_p is the cyclotomic character and ψ is a finite order character;*
- *ρ is* ordinary *in the sense that, for each prime $\mathfrak{p}$ dividing p,*

$$\rho|I_{\mathfrak{p}} \cong \begin{pmatrix} \chi_p^{k-1} & * \\ 0 & 1 \end{pmatrix};$$

- *ρ is distinguished at $\mathfrak{p}$ for each prime $\mathfrak{p}$ dividing p;*
- *there exists an ordinary modular Galois representation ρ' and an isomorphism between the* mod p *representations $\bar{\rho}$ and $\bar{\rho}'$.*

Then ρ is modular.

PROOF. See [**51**]. $\square$

For the rest of this section, we let $F = \mathbb{Q}(\sqrt{5})$. Let E be an ellipitc curve defined over F. For any prime $\ell \geq 3$, let

$$\bar{\rho}_{E,\ell}: \operatorname{Gal}(\bar{F}/F) \to \operatorname{GL}_2(\mathbb{F}_\ell)$$

be the *mod ℓ* representation attached to E. The proof of Serre [**45**, Proposition 1] carries on to give the following lemma.

LEMMA 1.47. *The image of $\bar{\rho}_{E,3}$ is either $\operatorname{GL}_2(\mathbb{F}_3)$ or contained in a Borel subgroup; the latter case happens if and only if E or a 3-isogenous curve E'/F to it has a 3–torsion point defined over F.*

PROOF. Going through the proof of Proposition 1 in Serre [**45**], one sees that the only thing we need to check is that there can't be a Galois extension K of $F = \mathbb{Q}(\sqrt{5})$ such that $\mathrm{Gal}(K/F) = D_4$ or a subgroup of index 2 in D_4 and K ramifies only at 3. We recall that $D_4/\mathbb{Z}_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$. But, there can't be an abelian extension of F of degree 4 that is only ramified at 3. $\square$

Let $E/\mathbb{Q}(\sqrt{5})$ be any of the curve listed below, let $\rho_{E,3}$ be the associated 3–adic Galois representation and $\bar{\rho}_{E,3}$ its reduction mod 3. By inspecting the listed of the $a_{\mathfrak{p}}$, we see that, except for the curves of conductor of norm 49, 71 or 89, the curve E does not have a 3–torsion point; neither can it be 3–isogenous to a curve E'/F that has a 3-torsion point defined over F. Therefore by Lemma 1.47, $\bar{\rho}_{E,3}$ is irreducible, and so is $\rho_{E,3}$. As we already mentionned, E is distinguished at 3. It is also ordinary at 3, since it has good reduction at $\mathfrak{p} = 3$ and $a_3(E)$ is not divisible by 3; this follows from results in Chap. V, VIII and IX in Silverman [**49**] (on elliptic curves over finite field and reduction of elliptic curves over local or global fields.). The ordinariness in this sense is equivalent to the one in Theorem 1.46 by the theory of finite flat commutative group schemes. The determinant of $\rho_{E,3}$ being cyclotomic is also a well–known fact. So, we only need check that $\bar{\rho}_{E,3}$ is modular. But since $\mathrm{GL}_2(\mathbb{F}_3)$ is *solvable*, this follows from Langlands–Tunnell (see [**36**, Chap. I] and [**52**], see also Serre [**46**]). Therefore, E is modular.

REMARK 1.48. The curve E/F with conductor of norm 41 in our list has a 7–torsion point defined over F. So we could have used instead the Main theorem of Skinner–Wiles [**50**] to deduce its modularity. This has the advantage of avoiding to verify the modularity of the residual representation $\bar{\rho}_{E,7}$. We can also use the same result to conclude the modularity for the rest of the curves in our list which might have a reducible *mod* 3 representation.

$\mathfrak{c}$	a_1	a_2	a_3	a_4	a_6
31	1	$-1-\omega_5$	ω_5	0	0
41	0	$-\omega_5$	ω_5	0	0
7	0	ω_5	1	1	0
71	$1+\omega_5$	$-1+\omega_5$	1	0	0
79	$1+\omega_5$	$-1+\omega_5$	ω_5	0	0
89	ω_5	$-\omega_5$	1	-1	0

9. Listing of Programs.

The first program computes the Hecke eigenvalues for a given level, and second one the quaternions needed to calculate the Hecke action.

```
\\THIS COMUPTE ALL HECKE EIGENVALUES FOR A GIVEN LEVEL.


\\Initial values.
w=quadgen(5);
field=nfinit(z^2-z-1);
fieldembed(x)=Mod(real(x)+imag(x)*z, z^2-z-1);

level=fieldembed(6+w);

levelf=idealfactor(field, level);

level0=[]; res=idealfactor(field, level);\
for(j=1, matsize(res)[1], res0=[res[j,][1][1],res[j,][2]];\
    level0=concat(level0, [res0])); print(level0);
level1=length(level0);

read(icosiangroup1);
```

```
\\Working with quaternions; we represent quaternions by vectors [x,y,w,z].
hnorm(q)=q[1]^2+q[2]^2+q[3]^2+q[4]^2;
hconj(q)=[q[1],-q[2],-q[3],-q[4]];
hmult(a,b)=[a[1]*b[1]-a[2]*b[2]-a[3]*b[3]-a[4]*b[4],\
           a[1]*b[2]+a[2]*b[1]+a[3]*b[4]-a[4]*b[3],\
           a[1]*b[3]+a[3]*b[1]-a[2]*b[4]+a[4]*b[2],\
           a[1]*b[4]+a[4]*b[1]+a[2]*b[3]-a[3]*b[2] ];
hinv(q)=hconj(q)/hnorm(q);



\\This function returns the reduction of a matrix modulo level with
\\coefficients in O_K.

matredmod(u,j,res0)=res0=matrix(matsize(u)[1], matsize(u)[2]);\
    for(p1=1, matsize(u)[1], for(q1=1, matsize(u)[2],\
           res0[p1,q1]=u[p1,q1]%(level0[j][1]^level0[j][2]))); res0;

\\Be carefull in using the function vectredmod: one might have some
\\problems when the denominator is divisible by 2.

vectredmod(v, l, res1)=res1=vector(length(v));\
        for(p1=1, length(v), res2=v[p1]%(level0[l][1]^level0[l][2]);\
             res2=nfeltreduce(field, fieldembed(res2),\
             idealpow(field, levelf[l,][1], levelf[l,][2]));\
             res1[p1]=res2[1]+res2[2]*w); res1;



\\Here is the quotient ring O/level.

Quotring=vector(length(level0));\
  for(j=1, length(level0), res=[]; card=level0[j][1]^level0[j][2];\
      if(kronecker(level0[j][1],5)==-1,\
             for(i=0, card-1, for(k=0, card-1, res=concat(res, i+k*w))),\
             for(i=0, card-1, res=concat(res, i))); Quotring[j]=res);\
              print(Quotring);
```

```
Proj0=vector(level1);\
for(j=1, level1, res=[]; for(k=1, length(Quotring[j]),\
          res=concat(res, [[1, Quotring[j][k]]]));\
  if(kronecker(level0[j][1],5)==-1, for(k=1, length(Quotring[j]),\
    res0=Quotring[j][k]; if(gcd(norm(res0), level0[j][1])==1,,\
          res=concat(res, [[res0, 1]]))),\
                for(k=1, length(Quotring[j]), res0=Quotring[j][k];\
        if(gcd(res0, level0[j][1])==1,, res=concat(res, [[res0, 1]]))));\
                      Proj0[j]=res); print(Proj0);



Proj=[];\
if(level1==1, Proj=Proj0[1], res0=vector(length(Proj0[1]));\
for(j=1, length(Proj0[1]), res0[j]=[Proj0[1][j]]); res0; j=2;\
while(j<=level1, Proj=[]; for(i=1, length(res0), for(k=1, length(Proj0[j]),\
res=concat(res0[i], [Proj0[j][k]]); Proj=concat(Proj, [res])));\
res0=Proj; j=j+1)); print(Proj);


\\This routine embeds the quaternion q into the ring M_2(O/level).

\\Return 1 if x^2+1 has a root mod level and the first solution.
alpha=vector(level1);\
for(j=1, level1, res=[0];\
 k=1; while(k<=length(Quotring[j])&&\
      !((Quotring[j][k]^2+1)%(level0[j][1]^level0[j][2])==0), k=k+1);\
  if(k>length(Quotring[j]),, res=concat(1, Quotring[j][k]));\
   alpha[j]=res); alpha;


\\When x^2+1 has no root mod level, this routine returns a matrix
\\J=[a, b; b, -a] such that J^2=-[1, 0; 0, 1] mod level;

J=vector(level1);\
for(j=1,level1, res=res0=[0, 0; 0, 0];\
if(level0[j][1]==2, J[j]=matredmod([1, 2 + 9*w; 2 + 10*w, -1], j);\
J0=matredmod([27 + 23*w, 31 + 4*w; 4*w, 6 + 9*w], j),\
```

```
if(alpha[j][1]==1, , i=2; k=1;\
res=[Quotring[j][i], Quotring[j][k]; Quotring[j][k], -Quotring[j][i]];\
until(!(matredmod(res-[0, -1; 1, 0], j)==[0, 0; 0, 0]) &&\
matredmod(res^2+[1, 0; 0, 1], j)==[0, 0; 0, 0], k=k+1;\
if(k>length(Quotring[j]), i=i+1; k=1);\
res=[Quotring[j][i], Quotring[j][k]; Quotring[j][k], -Quotring[j][i]]);\
J[j]=res))); J; J0; print(J":"J0);


\\This routine embeds the quaternion q into the ring M_2(O/level).

P=[1, 0, 1/2, 0; 0, 0, 1/2, 1/2; 0, 1, 1/2, w/2; 0, 0, 1/2, conj(w)/2];

embed(q, j, res)=res=0;\
if(level0[j][1]==2, x0=matsolve(P, q~);\
                       res=matredmod(x0[1]*[1, 0; 0, 1]+x0[2]*J[j]+\
                         x0[3]*J0+x0[4]*[0, -1; 1, 0], j),\
   q0=vectredmod(q, j); if(alpha[j][1]==1,\
       res=matredmod([q0[1]+ alpha[j][2]*q0[2], -q0[3]-alpha[j][2]*q0[4];\
               q0[3]-alpha[j][2]*q0[4], q0[1]-alpha[j][2]*q0[2]], j),\
       res=matredmod(q0[1]*[1, 0; 0, 1]+q0[2]*[0, -1; 1, 0]+q0[3]*J[j]+\
               q0[4]*([0, -1; 1, 0]*J[j]), j))); res;



\\Defining the action of a quaternion on an element of P_1(O/level).

act0(q, c0, j, res3)=res3=((embed(q, j)*c0~)~)%(level0[j][1]^level0[j][2]);\
    if(res3==[0,0],, if(gcd(norm(res3[1]), level0[j][1])==1,\
            res3=(res3/res3[1])%(level0[j][1]^level0[j][2]),\
            res3=(res3/res3[2])%(level0[j][1]^level0[j][2]))); res3;

act(q, c, res5)=res5=0;\
  if(level1==1, res5=act0(q, c, 1), res5=vector(level1);\
      for(j=1, level1, res5[j]=act0(q, c[j], j))); res5;


\\This action is used to find the degenerated hecke operator at the
\\prime $p$ dividing the level.
```

```
act2(q, c, j, res4)=res4=((embed(q, j)*c~)~)%level0[j][1]; res4;

\\This routine returns the operator $U(p)$ at a prime $p$ dividing
\\the level.


print(act(icosiangroup[1], Proj[1]));


\\This returns the orbit of a point in $P^1(O_K/level)$.

orbit(x,t)=t=[];\
 for(k=1, 60, t=concat(t, [act(icosiangroup[k], x)]));\
      t=eval(Set(t)); t;

print(orbit(Proj[1]));


\\Determining the global array orbits.

orbits=[orbit(Proj[1])];\
  tt=eval(setminus(Set(Proj), Set(orbits[1])));\
    until(length(tt)==0, orbits=concat(orbits, [orbit(tt[1])]);\
        tt=eval(setminus(Set(tt), Set(orbit(tt[1]))))); print(orbits);


\\Here is a fundamental domain of the action of the icosians on
\\$P^1(O_K/level)$.

FD=[];\
for(j=1, length(orbits), FD=concat(FD, [orbits[j][1]])); print(FD);


\\Computing the action of tp.

\\We first determine the number of operators whose norm does not divide the
\\level.
```

```
Hecke=[];\
for(p=1, length(tp), heckeop=tp[p];\
       res=nfeltreduce(field, fieldembed(hnorm(tp[p][1])),\
                idealhnf(field, level));\
  if(gcd(res, idealnorm(field, level))==1,,l0=1;\
     while(gcd(res, level0[l0][1])==1, l0=l0+1)); l0; print(p":"l0);\

Tp=matrix(length(FD), length(FD));\
for(k1=1, length(FD), for(k2=1, length(FD), counter=0;\
  if(level1==1, for(k3=1, length(heckeop),\
     if(act2(heckeop[k3],FD[k1],l0)==[0,0],,\
        if(!(setsearch(Set(orbits[k2]), act(heckeop[k3],FD[k1]))==0),\
             counter=counter+1))),\
               for(k3=1, length(heckeop),\
    if(act2(heckeop[k3],FD[k1][l0],l0)==[0,0],,\
        if(!(setsearch(Set(orbits[k2]), act(heckeop[k3],FD[k1]))==0),\
             counter=counter+1)))); Tp[k1,k2]=counter)); print(Tp);\
                  Hecke=concat(Hecke, [Tp])); Hecke;



vectoalg0(u)=res; if(length(u)==1, res=u[1], res=u[1]+u[2]*w0); res;

\\This function, applied to a polynomial, returns either an irreducible factor
\\of degree 3 or the discriminant vector of all the quadratic factors.


irredfact(pol, res)=res=0;\
  polfact=factor(pol); h=1; discvector0=[];\
    while(h<=matsize(polfact)[1] && poldegree(polfact[h, 1])<=2,\
      if(poldegree(polfact[h, 1])==2, D=quaddisc(poldisc(polfact[h, 1]));\
         discvector0=concat(discvector0, D), ); h=h+1); h; discvector0;\
    if(h>matsize(polfact)[1], res=[0, discvector0], res=[1, polfact[h, 1]]);\
                                          res;


s=1; discvector=[]; factofdeg3=[];\
while(s<=length(Hecke) && irredfact(charpoly(Hecke[s], x))[1]==0,\
          discvector=concat(discvector, irredfact(charpoly(Hecke[s], x))[2]);\
```

```
                 s=s+1); s; discvector;\
 if(s>length(Hecke), discvector=eval(Set(discvector)),\
 for(k=1, length(Hecke),\
factofdeg3=concat(factofdeg3, [factor(charpoly(Hecke[k], x))])); factofdeg3);\
                 discvector; factofdeg3;

heigen=[];\
if(!(factofdeg3==[]), heigen=factofdeg3,\
    if(length(discvector)>=2, heigen=discvector,\
     if(length(discvector)==0, splt=1; spltfield=nfinit(v-1), splt=0;\
      w0=quadgen(discvector[1]); spltfield=nfinit(v^2-trace(w0)*v+norm(w0)));\

rts=nfroots(spltfield, charpoly(Hecke[1], x)); print(rts);\
res0=[vectoalg0(nfalgtobasis(spltfield, rts[1]))];\
for(i=2, length(rts),\
res0=concat(res0, vectoalg0(nfalgtobasis(spltfield, rts[i])))); res0;\
print(res0);\

P=matker(Hecke[1]-res0[1]*matid(matsize(Hecke[1])[1]));\
for(i=2, length(res0),\
P=concat(P, matker(Hecke[1]-res0[i]*matid(matsize(Hecke[1])[1])))); P;\

j=2; while(j<=length(Hecke), B=P^-1*Hecke[j]*P;\
if(matisdiagonal(B), Q=matid(matsize(Hecke[j])[1]), print(B);\
rts=nfroots(spltfield, charpoly(Hecke[j], x)); print(rts);\
res1=[vectoalg0(nfalgtobasis(spltfield, rts[1]))];\
for(i=2, length(rts),\
res1=concat(res1, vectoalg0(nfalgtobasis(spltfield, rts[i])))); res1;\
print(res1);\

Q=matker(B-res1[1]*matid(matsize(Hecke[j])[1]));\
for(i=2, length(res1),\
Q=concat(Q, matker(B-res1[i]*matid(matsize(Hecke[j])[1])))); Q);\
P=P*Q; j=j+1); P; print(P);\

heckediag=vector(length(Hecke));\
for(j=1, length(Hecke), heckediag[j]=P^-1*Hecke[j]*P); heckediag;\

test=[]; for(j=1, length(heckediag),\
```

```
test=concat(test, matisdiagonal(heckediag[j]))); test; print(test);\

lambda=vector(matsize(Hecke[1])[1], i1, []);\
for(j=1, matsize(Hecke[1])[1], for(i=1, length(Hecke),\
       lambda[j]=concat(lambda[j], heckediag[i][j, j]))); lambda;\
       heigen=concat(heigen, lambda))); print(heigen);



\\WE USE THIS ROUTINE TO COMPUTE ALL THE QUATERNIONS NEEDED TO
\\COMPUTE A GIVEN HECKE OPERATOR.

\\Initial values
w=quadgen(5);
field=nfinit(z^2-z-1);
fieldembed(x)=Mod(real(x)+imag(x)*z, z^2-z-1);


\\Working with quaternions; we represent quaternions by
\\vectors [x,y,w,z].

hnorm(q)=q[1]^2+q[2]^2+q[3]^2+q[4]^2;
hconj(q)=[q[1],-q[2],-q[3],-q[4]];
hmult(a,b)=[a[1]*b[1]-a[2]*b[2]-a[3]*b[3]-a[4]*b[4],\
           a[1]*b[2]+a[2]*b[1]+a[3]*b[4]-a[4]*b[3],\
           a[1]*b[3]+a[3]*b[1]-a[2]*b[4]+a[4]*b[2],\
           a[1]*b[4]+a[4]*b[1]+a[2]*b[3]-a[3]*b[2] ];
hinv(q)=hconj(q)/hnorm(q)


\\This function returns the primes in $O_K$ lying above any prime
\\$p$ in $\Z$.

gen(pr, res)=res=0;\
   if(kronecker(pr,5)==-1,res=pr,N0=floor(sqrt(pr-1));\
            a=N0;res1=r^2-a*r+pr-a^2;\
       while(polisirreducible(res1),a=a-1;res1=r^2-a*r+pr-a^2);\
            res=floor(real(polroots(res1)))~;\
     if(res[1]==res[2], res=a+res[1]*w, res=[a,a]+w*res)); res;
```

```
\\This function reduces any matrix modulo pr, where pr is is
\\modprinit format.

matredmodpr(m,pr, res)=res=matrix(matsize(m)[1], matsize(m)[2]);\
for(i=1, matsize(m)[1], for(j=1, matsize(m)[2],\
   res0=nfeltreducemodpr(field, fieldembed(m[i,j]), pr);\
     res0=lift(res0); res[i,j]=res0[1]+res0[2]*w)); res;

vectredmodpr(v,pr,res1)=res1=vector(length(v));\
for(l=1, length(v), res0=nfeltreducemodpr(field,\
                           fieldembed(v[l]), pr);\
         res0=lift(res0); res1[l]=res0[1]+res0[2]*w); res1;


\\Every integral quaternion in this order is then given in the form:

quat(a,b)=a+w*b;

\\We get its norm in the form $Q+R*w$, where $Q$ and $R$ are the following
\\quadratic forms.

Q(a,b)= real(hnorm(a+b*w));

R(a,b)= imag(hnorm(a+b*w));


\\This function returns the norm of the ideals lying above a prime in $O_K$.

norm0(p)=if((kronecker(p,5)==1||kronecker(p,5)==0), p, p^2);


Primes=[];\
prs=primes(1440); for(i=1, length(prs), p=prs[i];\
            if(norm0(p)<=12000), Primes=concat(Primes, p))); Primes;
```

```
heckeops=[];\
 for(n=1, length(Primes), p=Primes[n]; read(comp1);\
         heckeops=concat(heckeops, [hecke])); heckeops;

write(hecke, heckeops);


\\This is the subroutine COMP1.GP we call in our main program which
\\compute the hecke operators.


Pr=if((kronecker(p,5)==-1||kronecker(p,5)==0), gen(p), gen(p)[1]);\
print(Pr); Pr0=nfmodprinit(field,idealfactor(field, fieldembed(Pr))[1,1]);
Pr1=if(norm(Pr)==Pr^2, Pr, norm(Pr)); print(Pr1);

\\This return a root of x^2+1 has a root mod level and the first solution.

alpha=res=0;\
 if((norm(Pr)-1)%4==0, G=idealstar(field, Pr0, flag=0);\
   res=nfeltpowmodpr(field, G[3][1],G[2][1]/4, Pr0);\
     res=lift(res); alpha=res[1]+res[2]*w, alpha=0); alpha;


\\When x^2+1 has no root mod level, this routine returns a matrix
\\J=[a, b; b, -a] such that J^2=-[1, 0; 0, 1] mod level;

J=res=res0=[0, 0; 0, 0];\
if(Pr==2, J=[matredmodpr([1, 2 + 9*w; 2 + 10*w, -1], Pr0),\
 matredmodpr([27 + 23*w, 31 + 4*w; 4*w, 6 + 9*w], Pr0)],\
if(alpha==0, i=2; k=1; res=[i, k; k, -i];\
until(!(matredmodpr(res-[0, -1; 1, 0], Pr0)==[0, 0; 0, 0]) &&\
matredmodpr(res^2+[1, 0; 0, 1], Pr0)==[0, 0; 0, 0], k=k+1;\
if(k>=norm(Pr), i=i+1; k=1); res=[i, k; k, -i]); J=res)); J; print(J);



\\This routine embeds a quaternion into $M_2(O_K/pr)$.
```

```
embed(q, res2)=res2=0;\
if(Pr==2, x0=matsolve(P, q~);\
                       res2=matredmodpr(x0[1]*[1, 0; 0, 1]+x0[2]*J[1]+\
                         x0[3]*J[2]+x0[4]*[0, -1; 1, 0], Pr0),\
   q0=vectredmodpr(q, Pr0); if(alpha==0,\
       res2=matredmodpr(q0[1]*[1, 0; 0, 1]+q0[2]*[0, -1; 1, 0]+q0[3]*J+\
               q0[4]*([0, -1; 1, 0]*J), Pr0),\
       res2=matredmodpr([q0[1]+ alpha*q0[2], -q0[3]-alpha*q0[4];\
               q0[3]-alpha*q0[4], q0[1]-alpha*q0[2]], Pr0))); res2;


\\This function returns of the equivalence class of $x$ in $P^1(O_K/Pr)$
\\in the form $[0,1]$ or $[1,a]$, with $a\in F_q$.

eqcls(u,res3)=res3=0;\
 if(u[1]==0, res3=[0,1], res3=[1, u[2]/u[1]]%Pr1); res3;


N=real(Pr); N1=floor(sqrt(N));



hecke=[];\
S=[];\
for(x1=0,N1, N2=floor(sqrt(N-x1^2));\
  for(x2=0, N2, N3=floor(sqrt(N-x1^2-x2^2));\
    for(x3=-N3, N3, N4=floor(sqrt(N-x1^2-x2^2-x3^2));\
      for(x4=-N4, N4, N5=floor(sqrt(N-x1^2-x2^2-x3^2-x4^2));\
        for(y1=-N5, N5, N6=floor(sqrt(N-x1^2-x2^2-x3^2-x4^2-y1^2));\
          for(y2=-N6, N6, N7=floor(sqrt(N-x1^2-x2^2-x3^2-x4^2-y1^2-y2^2));\
             for(y3=-N7, N7, x=[x1,x2,x3,x4]; y=[y1,y2,y3,r];\
                res0=Q(x,y)-real(Pr); if(polisirreducible(res0)==0,\
                   res=floor(real(polroots(res0)));\
                for(i=1,2, y0=[y1,y2,y3,res[i]]; res4=embed(quat(x,y0))[1,];\
             if((R(x,y0)==imag(Pr))&&(!(res4==[0,0])), res4=eqcls(res4);\
          if(setsearch(S, res4)==0, S=setunion(S, Set([res4]));\
        hecke=concat(hecke, [quat(x,y0)]);\
     print(length(hecke)":"norm(Pr)-length(S)+1);\
 if(length(hecke)==norm(Pr)+1, return(hecke))))))))))));
```

```
print(hecke);
```

CHAPTER 2

Some numerical results on the Oda conjecture.

Let $\mathbf{f}$ be a newform of weight 2 and level $\mathfrak{c}$ on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with coefficients in $\mathbb{Q}$. The construction of Eichler–Shimura associates to $\mathbf{f}$ an elliptic curve $E_{\mathbf{f}}/\mathbb{Q}$ whose complex points are given by

$$E_{\mathbf{f}}(\mathbb{C}) = \mathbb{C}/\Lambda_{\mathbf{f}},$$

where $\Lambda_{\mathbf{f}}$, the *period lattice* of $\mathbf{f}$, is canonically isomorphic to $\mathrm{H}_1(E_{\mathbf{f}},\, \mathbb{Z})$ (see Knapp [**34**, Chap. XI]). The elliptic curve $E_{\mathbf{f}}$ is a quotient of the Jacobian variety of the modular curve $X_0(\mathfrak{c})$. The Taniyama–Shimura–Weil conjecture over $\mathbb{Q}$, which is now a theorem thanks to the work of Wiles [**57**] and several people, can be stated as follows.
TSW–conjecture: *For any elliptic curve E over $\mathbb{Q}$, there is a newform $\mathbf{f}$ of weight 2 and level $\mathfrak{c}_E$ on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ and an isogeny*

$$\phi:\ E_{\mathbf{f}} \to E.$$

(We refer to Diamond and Im [**20**, Theorem 13.0.5.] for several equivalent statements.)

Let F be a totally real number field of degree $g > 1$. In an attempt to generalize the Eichler–Shimura construction to Hilbert modular forms over F, Oda has proposed a conjecture that circumvents the lack of a good substitute for the Jacobian variety in higher dimension. This proceeds as follows. Let $\mathfrak{c}$ be an integral ideal in F and $X_0(\mathfrak{c})$ the modular variety of level $\mathfrak{c}$. By Deligne [**16, 17**], the middle dimensional cohomology $\mathrm{H}^g(X_0(\mathfrak{c}),\,\mathbb{Q})$ carries a mixed Hodge structure (see the text for the precise definitions). Let $(W_k)_{k\in\mathbb{Z}}$ be its weight filtration, and consider $W_g\mathrm{H}^g(X_0(\mathfrak{c}),\,\mathbb{Q})$. This has a homogeneous Hodge structure of weight g on which the Hecke operators act in a natural way. Let $\mathbf{f}$ be a newform of weight 2 and level $\mathfrak{c}$. We let $K_{\mathbf{f}}$ be the number field generated by the Fourier coefficients of $\mathbf{f}$. Then we have the natural homomorphism

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & K_{\mathbf{f}} \\ T_{\mathfrak{p}} & \longmapsto & a_{\mathfrak{p}}, \end{array}$$

where $\mathbb{T}$ is the (Hecke) $\mathbb{Q}$-algebra generated by all the operators $T_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$. Put

$$\mathrm{H}^g(M_{\mathbf{f}},\,\mathbb{Q}) = W_g\mathrm{H}^g(X_0(\mathfrak{c}),\,\mathbb{Q}) \otimes_{\mathbb{T}} K_{\mathbf{f}}.$$

This is a rational homogeneous Hodge structure of weight g with a homomorphism

$$\theta_{\mathbf{f}} : K_{\mathbf{f}} \hookrightarrow \mathrm{End}(\mathrm{H}^g(M_{\mathbf{f}},\,\mathbb{Q})).$$

Oda conjecture: *For each newform* $\mathbf{f}$ *of weight* 2 *and level* $\mathfrak{c}$*, there exists an abelian variety* $A_{\mathbf{f}}$ *of dimension* $[K_{\mathbf{f}} : \mathbb{Q}]$ *defined over* F*, with good reduction outside* $\mathfrak{c}$*, and a homomorphism*

$$\theta_{\mathbf{f}} : K_{\mathbf{f}} \hookrightarrow \mathrm{End}(A_{\mathbf{f}}) \otimes \mathbb{Q}$$

such that there is an isomorphism of $K_{\mathbf{f}}$*–Hodge structures*

$$\bigotimes_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})} \mathrm{H}^1(A^{\sigma}_{\mathbf{f}}(\mathbb{C}),\,\mathbb{Q}) \to \mathrm{H}^g(M_{\mathbf{f}},\,\mathbb{Q}).$$

When F is a quadratic field, Oda [**40**, Chap. II. sec.7] has proved that the Hodge structure $\mathrm{H}^2(M_{\mathbf{f}},\,\mathbb{Q})$ is the tensor product of two rational Hodge structures of weight 1. However the field of definition of the corresponding abelian variety is unknown. In this section, we try to provide some numerical evidence for this conjecture by making use of the computations that we have done in the first chapter. Namely, for $F = \mathbb{Q}(\sqrt{5})$, we give two examples which show that the period moduli of Oda agree to a certain precision with the period moduli of some specific elliptic curves that are known to be defined over $F = \mathbb{Q}(\sqrt{5})$. In order to do so, we need to recall the main results of Oda on his own conjecture. This requires a brief review on Hodge structures.

1. A brief review of Hodge structures.

References: Deligne [**16, 17**].

Let $\underline{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the algebraic group obtained by restriction of scalars. We denote by $w : \mathbb{G}_{m/\mathbb{R}} \to \underline{S}$ the morphism which corresponds to the natural inclusion $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$, and by $t : \underline{S} \to G_{m/\mathbb{R}}$ the inverse of the norm morphism. We have $t \circ w(x) = x^{-2}$.

Let $H_\mathbb{R}$ be a *finite* dimensional real vector space, and $H = H_\mathbb{C} = H_\mathbb{R} \otimes \mathbb{C}$ its complexification. We let $\mathcal{E}$ be a subset of $\mathbb{Z} \times \mathbb{Z}$.

Definition 2.1. *A* real Hodge structure *on $H_\mathbb{R}$ is a bigraduation*

$$H = \bigoplus_{p,q} H^{p,q}, \qquad \textit{such that} \quad \overline{H^{q,p}} = H^{p,q}.$$

We say that the Hodge structure $\{H_\mathbb{R},\, H^{p,q}\}$
a) is of type $\mathcal{E}$ *if $H^{p,q} = 0$ for $(p,\, q) \notin \mathcal{E}$;*
b) has weight n *if $H^{p,q} = 0$, for $p + q \neq n$.*
We call $h^{p,q} = \dim H^{p,q}$ the Hodge numbers *of $\{H_\mathbb{R},\, H^{p,q}\}$.*

By making use of the theory of multiplicative groups, Definition 2.1 is equivalent to the following one.

DEFINITION 2.2. *Let $H_{\mathbb{R}}$ be a* finite *dimensional real vector space. A* real Hodge structure *is a representation of real algebraic groups*

$$h : \underline{\mathrm{S}} \to \mathrm{GL}(H_{\mathbb{R}}).$$

We say that h has weight n *if $h \circ w(x) = x^n$. We call $C = h(i)$ the* Weil operator.

Now, let $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of *finite* rank, and $H = H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ its complexification. For every pair of integers (p, q), pick a complex linear subspace $H^{p,q} \subset H$.

DEFINITION 2.3. *We say that the data $\{H_{\mathbb{Z}}, H^{p,q}\}$ is a* Hodge structure *if $\{H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}, H^{p,q}\}$ is a real Hodge structure. Similarly, a* $\mathbb{Q}$-rational Hodge structure *consists of the data of a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ together with a bigraduation $H^{p,q}$, $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ of $H_{\mathbb{Q}} \otimes \mathbb{C}$ such that $\{H_{\mathbb{Q}} \otimes \mathbb{R}, H^{p,q}\}$ is a real Hodge structure.*

EXAMPLE 2.4. 1) The simplest example of Hodge structure, coming from algebraic geometry, is provided by the *Tate* Hodge structure $\mathbb{Z}(n)$ of type $(-n, -n)$ given by

$$\mathbb{Z}(n)_{\mathbb{C}} = \mathbb{C}, \quad \text{and} \quad \mathbb{Z}(n)_{\mathbb{Z}} = (2\pi i)^n \mathbb{Z} \subset \mathbb{C}.$$

2) Let X be a Kähler manifold. We let

$$H_{\mathbb{Z}} = \mathrm{H}^n(X, \mathbb{Z})/(torsion), \qquad H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C} \cong \mathrm{H}^n(X, \mathbb{C}),$$

and $H^{p,q}$ be the cohomology of type (p, q) in the Hodge decomposition of $\mathrm{H}^n(X, \mathbb{C})$. Then $\{H_{\mathbb{Z}}, H^{p,q}\}$ is a Hodge structure of weight n. (See [**27**, Chap.0, sec.7]).

3) Let A be a complex abelian variety, and $A(\mathbb{C})$ its complex points. Since A is projective, $A(\mathbb{C})$ is a Kähler manifold; so the cohomology group $H_{\mathbb{Z}} = \mathrm{H}^1(A, \mathbb{Z})$ carries a Hodge structure of weight 1. This is a polarized Hodge structure, and the assignment $A \mapsto \mathrm{H}^1(A, \mathbb{Z})$ is an equivalence of categories (see Section 4 for more details).

From Definition 2.2, one gets a natural way to define the notion of morphisms and tensor products of Hodge structures. Indeed, a *morphism* of Hodge structures will correspond to the morphism of the underlying representations; likewise for tensor product. The category of $\mathbb{Q}$–Hodge structures is abelian.

DEFINITION 2.5. *A* polarization *of a Hodge structure of weight n is a morphism of Hodge structures $\psi : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}(-n)$ such that, on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$, the real bilinear form $(x, y) \mapsto (2\pi i)^n \psi(x, Cy)$ is symmetric positive definite. Note that ψ is then symmetric or antisymmetric according to the parity of the weight n.*

In terms of Definition 2.2, a polarization of a real Hodge structure $H_{\mathbb{R}}$ of weight n is a bilinear form ψ on $H_{\mathbb{R}}$, invariant by the compact subgroup $\ker(t)$ of $\underline{\mathrm{S}}$, such that the form $(x, y) \mapsto \psi(x, Cy)$ is symmetric positive definite. So, a polarization of a Hodge structure $H_{\mathbb{Z}}$ of weight n is a polarization ψ of the real Hodge structure $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$ which is *integral valued* on $H_{\mathbb{Z}}$.

EXAMPLE 2.6. Let A be a complex abelian variety. The intersection form gives a polarization of the homogeneous Hodge structure $\mathrm{H}^1(A, \mathbb{Z})$. Let ψ be polarization of $\mathrm{H}^1(A, \mathbb{Z})$. It is a perfect pairing; therefore it induces a $\mathbb{Z}$–linear injection of $\mathrm{H}^1(A, \mathbb{Z})$ into its $\mathbb{Z}$–dual. As a result, we get a polarization of A.

REMARK 2.7. In general, for any Noetherian subring A of $\mathbb{R}$, one defines an *A-Hodge structure of weight n* to be a finitely generated A-module H_A such that $H_{\mathbb{R}} = H_A \otimes_A \mathbb{R}$ has a real Hodge structure of weight n. A *polarization* of an A–Hodge structure H_A is a polarization of the underlying real Hodge structure which is A–valued on H_A.

Now, let G be a reductive group over $\mathbb{Q}$, and t and w two $\mathbb{Q}$-morphisms

$$\mathbb{G}_m \xrightarrow{w} G \xrightarrow{t} \mathbb{G}_m$$

with $t \circ w(x) = x^{-2}$ and w central. Furthermore, assume we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{G}_{m/\mathbb{R}} & \xrightarrow{w} & \underline{\mathbb{S}} & \xrightarrow{t} & \mathbb{G}_{m/\mathbb{R}} \\ \| & & \downarrow{\scriptstyle h} & & \| \\ \mathbb{G}_{m/\mathbb{R}} & \xrightarrow{w} & \mathbb{G}_{\mathbb{R}} & \xrightarrow{t} & \mathbb{G}_{m/\mathbb{R}}. \end{array}$$

Let $\rho: G \to \mathrm{GL}(V_{\mathbb{Q}})$ be a representation defined over $\mathbb{Q}$. We say that ρ is *homogeneous of weight* n if $\rho \circ w(x) = x^n \cdot 1_V$. Every representation is a sum of homogeneous representations.

We can let G act on $\mathbb{Q}(n)_{\mathbb{Q}}$ by the rule $g * x = t(g)^n \cdot x$. This definition is made compatible with the commutative square above as $\underline{\mathbb{S}}$ acts on $\mathbb{Q}(n)_{\mathbb{R}}$ in a similar way.

Definition 2.8. *A* polarization *of $\rho: G \to \mathrm{GL}(V_{\mathbb{Q}})$ is a G-invariant form*

$$(2\pi i)^n \psi : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}(n)$$

such that $(x, \rho \circ h(y))$ is symmetic positive definite on $V_{\mathbb{R}}$.

Proposition 2.11 in Deligne [**17**] gives necessary and sufficient conditions for a representation to be polarizable. We see that, if $V_{\mathbb{Q}}$ has a rational Hodge structure, any polarization of a representation $V_{\mathbb{Q}}$ induces a polarization of this rational Hodge structure on $V_{\mathbb{Q}}$.

Definition 2.9. *A* mixed Hodge structure *is a triple $(H_{\mathbb{Z}}, W, F)$ which consists of:*
(i) A finitely generated $\mathbb{Z}$-module $H_{\mathbb{Z}}$.
(ii) A finite increasing filtration W_n of $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ called the weight

filtration.
(iii) A finite decreasing filtration F^p *of* $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ *called the* Hodge filtration. *We require that the filtration induced on* $\mathrm{Gr}_n^W H_{\mathbb{C}}$ *by* F *provides* $\mathrm{Gr}_n^W H_{\mathbb{Q}}$ *with a homogeneous Hodge structure of weight* n.

In [**17**], Deligne proved that the category of mixed Hodge structures is abelian and that the cohomology of any finite scheme over $\mathbb{C}$ has a mixed Hodge structure and that this construction is functorial. The main example of a mixed Hodge structure we will be dealing with is the subject of the next section.

2. Cohomology of Hilbert modular surfaces.

Reference Oda [**40**, Chap. I.] and [**41**].

We assume in this section that the modular surface $X_0(\mathfrak{c})$ is connected but this restriction could be removed. Recall that we have intersection pairing

$$\langle\, , \rangle :\ H_2(X_0(\mathfrak{c})) \times H_2(X_0(\mathfrak{c})) \to \mathbb{Q}$$

which is non-degenerate. By making use of Deligne [**18**, sec. 8], Oda computed the weight filtration of the Hodge structure of $H^2(X_0(\mathfrak{c}), \mathbb{Q})$:

$$W_k H^2(X_0'(\mathfrak{c}), \mathbb{Q}) = 0, \text{ for } k \leq 1,$$
$$W_2 H^2(X_0'(\mathfrak{c}), \mathbb{Q}) = W_3 H^2(X_0'(\mathfrak{c}), \mathbb{Q}) = j^* H^2(X_0(\mathfrak{c}), \mathbb{Q}),$$

where j is the natural inclusion. We put

$$\mathbb{H}^2(X_0(\mathfrak{c}), \mathbb{Q}) = j^* H^2(X_0(\mathfrak{c}), \mathbb{Q}).$$

One gets the Hodge filtration on $\mathbb{H}^2(X_0(\mathfrak{c}),\, \mathbb{Q})$ by using its description in terms of Hilbert modular forms. To this end, let ε_1 (resp. ε_2) be the involution on $X_0(\mathfrak{c})$ induced by the maps $((z_1,\, z_2) \mapsto (\bar{\varepsilon}\bar{z}_1,\, \varepsilon z_2))$ and $((z_1,\, z_2) \mapsto (\varepsilon z_1,\, \bar{\varepsilon}\bar{z}_2))$, where ε is a unit in $\mathcal{O}_F$ such that $\varepsilon > 0$ and $\bar{\varepsilon} < 0$. Let $\mathbf{f}$ be in $S_2(\mathfrak{c})$. We can define a holomorphic 2-form on $X_0(\mathfrak{c})$, by putting

$$\omega_{\mathbf{f}} = (2\pi i)^2 \mathbf{f}(z_1,\, z_2) dz_1 \wedge dz_2 \quad \text{on } \Gamma\backslash\mathcal{H}^2.$$

This gives a linear map $S_2(\mathfrak{c}) \to \mathbb{H}^2(X_0(\mathfrak{c}),\, \mathbb{C})$. We define

$$\eta_{\mathbf{f}} = \varepsilon_1^* \omega_{\mathbf{f}},$$
$$\eta'_{\mathbf{f}} = \varepsilon_2^* \omega_{\mathbf{f}}.$$

Oda obtained the Hodge filtration of $\mathbb{H}^2(X_0(\mathfrak{c}),\, \mathbb{Q})$.

THEOREM 2.10. [**40**, Chap.I, sec.1] $\mathbb{H}^2(X_0(\mathfrak{c}),\, \mathbb{C})$ *is the direct sum of*
(i) its $(2,\, 0)$*-component* $\{\omega_{\mathbf{f}} :\, \mathbf{f} \in S_2(\mathfrak{c})\}$
(ii) its $(0,\, 2)$*-component* $\{\bar{\omega}_{\mathbf{f}} :\, \mathbf{f} \in S_2(\mathfrak{c})\}$
(iii) its $(1,\, 1)$*-component* $\{\eta_{\mathbf{f}} + \eta'_{\mathbf{g}} :\, \mathbf{f},\, \mathbf{g} \in S_2(\mathfrak{c})\} \oplus W$,
where W is the 2-dimensional space span by the forms

$$w_1 = -\frac{1}{2\pi}\frac{dx_1 \wedge dy_1}{y_1^2} \quad \text{and} \quad w_2 = -\frac{1}{2\pi}\frac{dx_2 \wedge dy_2}{y_2^2}$$

.

We are only interested in the orthogonal complement of W which arises from cusp forms. We call it the *cuspidal cohomology* and denote it by $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})$. We denote by $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$ the part of $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})$ that is described by newforms and call it the *new* part of $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})$. $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})$ and $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$ are both equipped with some homogeneous Hodge structures polarized by the restriction of the bilinear form coming from Poincaré duality.

REMARK 2.11. Our definition of the new part of the cuspidal cohomology is equivalent to the one in Oda [**41**], by Proposition 3.3 in [**41**].

3. The Hodge structure attached a Hilbert newform.

In the previous section, we obtained a homogeneous Hodge structure together with a bilinear pairing

$$\Psi : \mathbb{H}^2_{sp}(X_\Gamma, \mathbb{Q})^{new} \times \mathbb{H}^2_{sp}(X_\Gamma, \mathbb{Q})^{new} \to \mathbb{Q}$$

coming from Poincaré duality. We will now define an action of the Hecke algebra on $\mathbb{H}^2_{sp}(X_\Gamma, \mathbb{Q})^{new}$. This action is compatible with the polarization Ψ. As a consequence, we will get that $\mathbb{H}^2_{sp}(X_\Gamma, \mathbb{Q})^{new}$ decomposes as a direct sum of polarized Hodge structures.

The action of the Hecke algebra is given as usual via algebraic correspondences. Let $\mathfrak{p}$ be a prime ideal in F such that $(\mathfrak{p}, \mathfrak{c}) = 1$. Then, as in the one dimensional case, the diagram

$$\begin{tikzcd} & X'_0(\mathfrak{p}\mathfrak{c}) \arrow[dl] \arrow[dr] & \\ X'_0(\mathfrak{c}) & & X'_0(\mathfrak{c}) \end{tikzcd}$$

with arrows corresponding to natural projections, gives rise to a correspondence on $X'_0(\mathfrak{c}) \times X'_0(\mathfrak{c})$ which extends to a correspondence C on $X_0(\mathfrak{c}) \times X_0(\mathfrak{c})$. Let $p_1, p_2 : C \to X_0(\mathfrak{c})$ be the canonical projections. They are finite flat morphisms, étale outside a finite set of points. Therefore, we have an action of $T_\mathfrak{p}$ on $H_2(X'_0(\mathfrak{c}), \mathbb{Q})$ given by

$$T_\mathfrak{p} \cdot c = p_2(p_1^{-1} c), \qquad c \in H_2(X'_0(\mathfrak{c}), \mathbb{Q}),$$

and an action on $H^2_{DR}(X'_0(\mathfrak{c}),\,\mathbb{Q})$ given by

$$T_{\mathfrak{p}}\cdot\omega=(p_2)_*(p_1^*\omega),\qquad \omega\in H^2_{DR}(X'_0(\mathfrak{c}),\,\mathbb{Q}),$$

where p_1^* and $(p_2)_*$ are the usual pull back and trace maps respectively. To avoid problems with quotient singularities, we take a torsion free normal subgroup Γ' of Γ and put

$$H^2_{DR}(X'_0(\mathfrak{c}),\,\mathbb{Q})=H^2_{DR}(X_{\Gamma'},\,\mathbb{Q})^{\Gamma/\Gamma'}.$$

This definition does not depend on Γ' as one can check. To ensure self-adjointness, we restrict to primes $\mathfrak{p}$ not dividing the level $\mathfrak{c}$. Let $\mathbf{T}$ be the image of the Hecke $\mathbb{Q}$–algebra in $\mathrm{End}(\mathbb{H}^2(X_0(\mathfrak{c}),\,\mathbb{Q}))$. By symmetry of Hecke correspondences, we have

$$\Psi(T\cdot\omega_1,\,\omega_2)=\Psi(\omega_1,\,T\cdot\omega_2),\qquad \omega_1,\,\omega_2\in H^2_{DR}(X_0(\mathfrak{c}),\,\mathbb{Q}),\ T\in\mathbf{T}.$$

THEOREM 2.12. [**40**, Theorem 1.12] $\mathbf{T}$ *acts on the homogenous polarized Hodge structure* $\mathbb{H}^2(X_0(\mathfrak{c}),\,\mathbb{Q})$ *of weight* 2 *as endomorphisms of Hodge structures. This action preserves* $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\,\mathbb{Q})$ *and* $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\,\mathbb{Q})^{new}$.

As a consequence of the Multiplicity one Theorem (see Miyake [**39**]) and the fact that $S_2(\mathfrak{c})^{new}$ has a basis of eigenforms we have the following lemma.

LEMMA 2.13. [**41**, Proposition 3.3.] $\mathbf{T}$ *is a semi-simple* $\mathbb{Q}$*-algebra.*

Since $\mathbf{T}$ is a semi-simple $\mathbb{Q}$-algebra, we can write it as a direct sum of number fields

$$\mathbf{T}=\bigoplus_{i=1}^{n}K_i.$$

Note that since each $T \in \mathbf{T}$ is self–adjoint with respect to the Peterson inner product, it has real eigenvalues, therefore each K_i is totally real. Let

$$\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{C})^{new} = H^{2,\,0} \oplus H^{1,\,1} \oplus H^{0,\,2}$$

be the Hodge decomposition of $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$. By Theorems 2.10 and 2.12, each summand has the same isotypic components as $S_2(\mathfrak{c})^{new}$ as a $\mathbb{T}$-module. And, since $S_2(\mathfrak{c})^{new}$ admits a basis of eigenforms, putting $[K_i : \mathbb{Q}] = d_i$, we get $\sum d_i = \dim S_2(\mathfrak{c})^{new}$. We choose a set of primitive idempotents $\{e_1,\, \ldots,\, e_n\}$ such that $K_i = e_i\mathbf{T} = \mathbf{T}e_i = e_i\mathbf{T}e_i$. Then,

$$\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new} = \bigoplus_{i=1}^{n} e_i\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}.$$

Since the action of $\mathbf{T}$ on $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$ preserves Hodge type, each $e_i\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$ is a sub-Hodge structure. Moreover, by symmetry of Hecke correspondences, we see that the above decomposition is in fact a direct sum of polarized Hodge structures, the polarization of $e_i\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$ being obtained by restriction of the form Ψ. We denote by

$$\theta_i :\; K_i \to \mathrm{End}(e_i\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new})$$

the action of K_i on $e_i\mathbb{H}^2_{sp}(X_0(\mathfrak{c}),\, \mathbb{Q})^{new}$. Take an idempotent $e_i \in \{e_1,\, \ldots,\, e_n\}$ and let $\mathbf{f} \in e_iS_2(\mathfrak{c})^{new}$ be a normalized eigenform. Then $\mathbf{f}$ defines an embedding $\sigma :\; K_i \to \mathbb{C}$ by $t(\mathbf{f}) = \sigma(t)\mathbf{f}$. Since $K_i \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_\sigma \mathbb{C}$ with the summands corresponding to the embeddings $\sigma :\; K_i \to \mathbb{C}$, we have

$$e_iS_2(\mathfrak{c})^{new} = \bigoplus_{\sigma} \mathbb{C}\mathbf{f}^\sigma,$$

where the $\mathbf{f}^\sigma$ are the normalized eigenforms with $t(\mathbf{f}^\sigma) = \sigma(t)\mathbf{f}^\sigma$. We call the $\mathbf{f}^\sigma$ the *companions* of $\mathbf{f}$. Now, we choose a set $E = \{\mathbf{f}_1,\, \ldots,\, \mathbf{f}_n\}$ of normalized eigenforms such that $\mathbf{f}_i \in e_iS_2(\mathfrak{c})^{new}$. Each $\mathbf{f}_i$ then corresponds to a unique field K_i. So, for a normalized eigenform $\mathbf{f} \in e_i S_2(\mathfrak{c})^{new}$, we can put $e_i = e_{\mathbf{f}}$, $K_i = K_{\mathbf{f}}$, $\Psi_i = \Psi_{\mathbf{f}}$, $\theta_i = \theta_{\mathbf{f}}$ and define

$$H^2(M_{\mathbf{f}}, \mathbb{Q}) := e_{\mathbf{f}} \mathbb{H}^2_{sp}(X_0(\mathfrak{c}), \mathbb{Q})^{new}.$$

This symbol $M_{\mathbf{f}}$ though very suggestive as it seems to refer to a certain conjectural motive associated to $\mathbf{f}$, of which we only study the Hodge realization, will be used only in combination with $H^2(M_{\mathbf{f}}, \mathbb{Q})$.

THEOREM 2.14. [**41**, Lemma 3.5.] *Corresponding to* $\mathbf{T} = \bigoplus_{\mathbf{f} \in E} K_{\mathbf{f}}$, *there is a direct sum decomposition*

$$\mathbb{H}^2_{sp}(X_0(\mathfrak{c}), \mathbb{Q})^{new} = \bigoplus_{\mathbf{f}} H^2(M_{\mathbf{f}}, \mathbb{Q})$$

of polarized Hodge structures. Moreover, if $K_{\mathbf{f}}$ *acts on* $H^2(M_{\mathbf{f}}, \mathbb{Q})$ *via* $\theta_{\mathbf{f}}$, *then*

$$\Psi_{\mathbf{f}}(\theta_{\mathbf{f}}(t)x, y) = \Psi_{\mathbf{f}}(x, \theta_{\mathbf{f}}(t)y)$$

with $\Psi_{\mathbf{f}}$ *the polarization of* $H^2(M_{\mathbf{f}}, \mathbb{Q})$.

The involutions ε_1 and ε_2 defined on $X_0(\mathfrak{c})$ and $Y_0(\mathfrak{c})$ induce an action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{H}^2_{sp}(X_0(\mathfrak{c}), \mathbb{Q})$ which commutes with the action of $\mathbb{T}$. Therefore, we can split $H^2(M_{\mathbf{f}}, \mathbb{Q})$ as

$$H^2(M_{\mathbf{f}}, \mathbb{Q}) = \bigoplus_{s, s' \in \{+,-\}} H^2(M_{\mathbf{f}}, \mathbb{Q})_{ss'},$$

where ε_1^* (resp. ε_2^*) acts on $H^2(M_{\mathbf{f}}, \mathbb{Q})$ as $s \cdot id$ (resp. $s' \cdot id$).

PROPOSITION 2.15. [**24**, Proposition 6.3] *For every newform* $\mathbf{f} \in S_2(\mathfrak{c})$ *we have* $\operatorname{rank}_{K_{\mathbf{f}}} H^2(M_{\mathbf{f}}, \mathbb{Q})_{ss'} = 1$. *Furthermore,* $\Psi_{\mathbf{f}} = \operatorname{Tr}_{K_{\mathbf{f}}/\mathbb{Q}} \psi_{\mathbf{f}}$ *with*

$$\psi_{\mathbf{f}} : H^2(M_{\mathbf{f}}, \mathbb{Q}) \times H^2(M_{\mathbf{f}}, \mathbb{Q}) \to K_{\mathbf{f}}$$

which, with respect to a suitable basis $e_{ss'} \in H^2(M_{\mathbf{f}}, \mathbb{Q})_{ss'}$, *is of the form*

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We call such a basis $(e_{ss'})$ a *canonical* basis.

Define

$$\mathbb{H}_2^{sp}(X_0(\mathfrak{c}), \mathbb{Q}) = \Big\{\gamma \in \mathrm{Im}(H_2(X_0'(\mathfrak{c}), \mathbb{Q}) \to H_2(Z_0(\mathfrak{c}), \mathbb{Q})) : \int_\gamma w = 0, \forall w \in W\Big\}.$$

By duality, we have a direct sum

$$\mathbb{H}_2^{sp}(X_0(\mathfrak{c}), \mathbb{Q})^{new} = \bigoplus_{\mathbf{f}} H_2(M_{\mathbf{f}}, \mathbb{Q}),$$

and each $H_2(M_{\mathbf{f}}, \mathbb{Q})$ splits into four eigenspaces $H_2(M_{\mathbf{f}}, \mathbb{Q})_{ss'}$.

DEFINITION 2.16. *Let* $(e_{ss'})$ *be a canonical basis of* $H^2(M_{\mathbf{f}}, \mathbb{Q})$, $(\gamma_{ss'})$ *a dual basis in* $H_2(M_{\mathbf{f}}, \mathbb{Q})$ *and* $\mathbf{g}$ *a companion of* $\mathbf{f}$. *The* periods *of* $\mathbf{g}$ *are given by*

$$\Omega_{ss'}(\mathbf{g}) = \int_{\gamma_{ss'}} \omega_{\mathbf{g}}.$$

The following period relation which is due to Oda will be needed later.

THEOREM 2.17 (Hodge–Riemann bilinear relation). [**40**, Theorem 4. 4.]*:*

$$\Omega_{++}(\mathbf{g})\Omega_{--}(\mathbf{g}) = -\Omega_{+-}(\mathbf{g})\Omega_{-+}(\mathbf{g}) = \frac{1}{4}\int_{X_0'(\mathfrak{c})} \omega_{\mathbf{g}} \wedge \bar{\omega}_{\mathbf{g}}.$$

4. Hilbert-Blumenthal abelian varieties.

We start with a totally real number field K of degree d, and let $\mathcal{O}_K$ be its ring of integers. With this notation, we have in mind the field $K_{\mathbf{f}}$ generated by the eigenvalues a newform $\mathbf{f}$.

DEFINITION 2.18. *A* complex Hilbert-Blumenthal *abelian variety of dimension d is a pair (A, θ), where:*
(i) A is a complex abelian variety,
(ii) $\theta : \mathcal{O}_K \hookrightarrow \mathrm{End}(A)$ defines an action of $\mathcal{O}_K$ on A.

The Lie algebra of any complex Hilbert-Blumenthal abelian variety satisfies the following lemma.

LEMMA 2.19. *Let A be a complex Hilbert-Blumenthal abelian variety of dimension d. Then,*
$\mathrm{Lie}(A/\mathbb{C})$ *is locally free $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}$-module of rank 1.*

Let A be a complex abelian variety. From the general theory of complex abelian varieties, we know that $\mathrm{Lie}(A/\mathbb{C})$ is the universal covering space of $A(\mathbb{C})$ so that

$$\mathrm{Lie}(A/\mathbb{C})/\Lambda \cong A(\mathbb{C}),$$

where Λ is a lattice in $\mathrm{Lie}(A/\mathbb{C})$ canonically isomorphic to $H_1(A, \mathbb{Z})$. Moreover $H_1(A, \mathbb{Z})$ admits a Riemann form which, by Poincaré duality, gives a polarization of the Hodge structure of type $\{(1, 0), (0, 1)\}$ on $H^1(A, \mathbb{Z})$. We have the following well known theorem.

THEOREM 2.20 (Riemann). *The map $A \mapsto H^1(A, \mathbb{Z})$ is an equivalence between the category of abelian varieties and the one of polarized Hodge structures of type $\{(0, 1), (1, 0)\}$.*

Now, let (A, θ) be a Hilbert-Blumenthal abelian variety. We let ψ be the polarization of A given by the Riemann form which defines the algebraic structure on A. Then, the Rosati involution defined by ψ acts trivially on $\mathcal{O}_K$. So letting

$$\psi^* : H^1(A, \mathbb{Z}) \times H^1(A, \mathbb{Z}) \to \mathbb{Z}$$

be the bilinear pairing induced from ψ, we get that

$$\psi^*(\theta^*(a)\omega_1, \omega_2) = \psi^*(\omega_1, \theta^*(a)\omega_2).$$

To avoid certain technical problems, we assume that the pairing ψ^* is obtained as the composition of the trace map with some $\mathcal{O}_K$–valued pairing.

THEOREM 2.21. *The map $(A, \theta) \mapsto (H^1(A, \mathbb{Z}), \theta^*)$ is an equivalence between the category of Hilbert-Blumenthal abelian varities with RM by $\mathcal{O}_K$ and the one of polarized Hodge structures of type $\{(0, 1), (1, 0)\}$ with $\mathcal{O}_K$-action.*

We would like to determine the periods of the complex Hilbert-Blumenthal abelian variety (A, θ). To this end, let ω be a nowhere vanishing invariant differential form on A. Then, under the action of $\mathcal{O}_K$, we have the decomposition

$$\underline{\omega}_{A/\mathbb{C}} = H^0(A, \Omega^1_{A/\mathbb{C}}) = \bigoplus_{\sigma} \omega^{\sigma},$$

where σ runs over all the complex embeddings of K. As $\mathbb{C}$-vector spaces, we know that $\mathrm{Lie}(A/\mathbb{C})$ and $\underline{\omega}_{A/\mathbb{C}}$ have dimension d, and they are duals of each other. Therefore, the differential form ω defines an $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}$ isomorphism:

$$\omega : \mathrm{Lie}(A/\mathbb{C}) \cong K \otimes \mathbb{C}.$$

The period lattice $\mathcal{L}$ of A is then given by the image of $H_1(A, \mathbb{Z})$ under that isomorphism. More explicitly, let ω_1, ω_2 be an $\mathcal{O}_K$-basis of $H^1(A, \mathbb{Z})$ such that $\psi^*(\omega_1, \omega_2) = 1$, and γ_1, γ_2 its dual basis in $H_1(A, \mathbb{Z})$. For each embedding $\sigma : K \hookrightarrow \mathbb{C}$, and $i = 1, 2$, put

$$\Omega_i^\sigma = \int_{\gamma_i} \omega^\sigma.$$

Then, by Riemann period relations, the point $\tau = (\tau_\sigma)_\sigma \in (\mathbb{C} - \mathbb{R})^d$, given by

$$\tau_\sigma = \Omega_2^\sigma / \Omega_1^\sigma,$$

belongs to $\mathcal{H}^d$. It determines the lattice $\mathcal{L}$. The point $\tau \in \mathcal{H}^d$ we just constructed does not depend on the choice of ω, however, it depends on the one of ω_1 and ω_2. Indeed, let ω_1', ω_2' be another $\mathcal{O}_K$-basis of $H^1(A, \mathbb{Z})$ such that $\psi^*(\omega_1', \omega_2') = 1$. Then there exists $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

The resulting point which corresponds to (ω_1', ω_2') is then $\gamma \cdot \tau$. Therefore, $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathcal{H}^d$ parametrizes the period moduli of all Hilbert-Blumenthal abelian varieties with $\mathcal{O}_K$-action. In fact, the compactification of $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathcal{H}^d$ is the analytic space of the moduli stack M_K which parametrizes all semiabelian Hilbert-Blumenthal varieties with real multiplication by $\mathcal{O}_K$. If we start instead with the cohomology $H_1(A, \mathbb{Q})$, the abelian variety we get is then defined only up to $\mathbb{Q}$-isogeny. So the above discussion then needs some modifications (see Oda [**40**, Chap. II.]).

REMARK 2.22. Note that the variety M_K is pretty much distinct from the Hilbert modular variety $X_0(\mathfrak{c})$. Indeed, each point on M_K parametrizes an isomorphism class of Hilbert–Blumenthal abelian varieties with multiplication by $\mathcal{O}_K$. So when $K = K_{\mathbf{f}}$, the field generated by the Fourier coefficients of a newform $\mathbf{f}$, the piece $\mathrm{H}^g(M_{\mathbf{f}}, \mathbb{Q})$ will correspond to some point on $\mathrm{M}_{K_{\mathbf{f}}}$, but each point on $\mathrm{M}_{K_{\mathbf{f}}}$ will not necessarily come from such a piece on $\mathrm{H}^g(X_0(\mathfrak{c}), \mathbb{Q})$.

5. The abelian variety associated to a Hilbert newform.

References Deligne [**17**] and Oda [**40**, Chap. II. sec. 1.].

Let A be a ring, E a free A-module and Q a quadratic form on E. We consider the subfunctor

$$F : A - \mathbf{Alg} \to \mathbf{Set}$$

of $B \mapsto \mathrm{Hom}_{A-\mathbf{mod}}(E, B)$, where $F(B)$ consists of all the A-linear maps $u : E \to B$ such that $u(x)^2 = Q(x)$, for all $x \in E$. We recall that the functor F is represented by the *Clifford algebra* we denote by $C(E, Q)$ or $C(E)$. We will denote by $C^+(E, Q)$ or $C^+(E)$ the even part of $C(E, Q)$. For any bilinear symmetric form B on V, we will denote by $C(E, B)$ or $C(E)$ the Clifford algebra of the corresponding quadratic form.

Now, assume that $A = k$ is a field of characteristic 0 and that Q is non-degenerate. Then E injects into $C(E)$. Let $\mathrm{CSpin}(E)$ be the *Clifford group*. It is the algebraic group of all invertible elements g in $C^+(E)$ such that $gEg^{-1} = E$. This gives a morphism φ of $\mathrm{CSpin}(E)$ onto $\mathrm{SO}(E)$ by letting $\varphi(g)$ acts on E by $\varphi(g) \cdot x = gxg^{-1}$. The kernel of φ is $\mathbb{G}_m$, so one has an exact squence

$$0 \longrightarrow \mathbb{G}_m \xrightarrow{w} \mathrm{CSpin}(E) \xrightarrow{\varphi} \mathrm{SO}(E) \longrightarrow 0.$$

Let $C^+(E)^\times$ be the algebraic group of all invertible elements in $C^+(E)$. By definition, we have an inclusion

$$\alpha:\ \mathrm{CSpin}(E) \hookrightarrow C^+(E)^\times.$$

This induces a representation of the algebraic group $\mathrm{CSpin}(E)$ on $C^+(E)$ by:

$$g * x = \alpha(g)\cdot x.$$

We denote this representation by $C^+(E)_s$. We let $(C^+(E))_{ad}$ be the representation of $\mathrm{CSpin}(E)$ on $C^+(E)$ given by

$$\mathrm{Ad}(g) * x = \alpha(g)\cdot x \cdot \alpha(g)^{-1}.$$

The action of $\mathrm{CSpin}(E)$ on $C^+(E)_s$ is compatible with the *right* $C^+(E)$-module structure of $C^+(E)$; and since $\mathrm{End}_{C^+(E)}(C^+(E))$ consists of multiplication to the *left* by elements of $C^+(E)$, one easily sees that the representation obtained on $\mathrm{End}_{C^+(E)}(C^+(E))$ from $C^+(E)_s$ is nothing else but $C^+(E)_{ad}$; that is, we have an isomorphism of representations:

$$\mathrm{End}_{C^+(E)}(C^+(E)_s) = (C^+(E))_{ad}.$$

Also, from the structure theorem of Clifford algebras, we have an isomorphism of representations

$$(C^+(E))_{ad} = \oplus \wedge^{2i} E.$$

Now, let $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank $n+2$ endowed with a bilinear form of non-zero discriminant

$$B:\ H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}.$$

We assume that B has signature $(n,\,2)$. Let $H_{\mathbb{Q}}$ be a rational Hodge structure of weight 0 polarized by B and having Hodge numbers:

$$h^{-1,\,1} = h^{1,\,-1} = 1, \quad \text{and} \quad h^{0,\,0} = n.$$

Then, $H_{\mathbb{Q}}$ is given by a morphism of $\mathbb{R}$-groups $h : \underline{\mathrm{S}} \to \mathrm{SO}(H_{\mathbb{R}})$, which lifts uniquely to a morphism $\tilde{h} : \underline{\mathrm{S}} \to \mathrm{CSpin}(H_{\mathbb{R}})$ fitting into the commutative diagram

$$\begin{array}{ccccc}
\mathbb{G}_{m/\mathbb{R}} & \xrightarrow{w} & \underline{\mathrm{S}} & \xrightarrow{t} & \mathbb{G}_{m/\mathbb{R}} \\
\| & & \downarrow{\scriptstyle \tilde{h}} & & \| \\
\mathbb{G}_{m/\mathbb{R}} & \xrightarrow{w} & \mathrm{CSpin}(H_{\mathbb{R}}) & \xrightarrow{t} & \mathbb{G}_{m/\mathbb{R}}.
\end{array}$$

By Proposition 2.11 in Deligne [**17**], every representation of $\mathrm{CSpin}(H_{\mathbb{Q}})$ is polarizable with respect to $\tilde{h}$. As a consequence, one gets the following.

PROPOSITION 2.23. *(*[**17**, Deligne]*) The Hodge structure of $C^+(H_{\mathbb{Q}})_{ad}$ defined by $\tilde{h}$ has type*
$\{(-1,\,1),\,(0,\,0),\,(1,\,-1)\}$.

PROPOSITION 2.24. [**17**, Proposition 4. 5] *The Hodge structure of $C^+(H_{\mathbb{Q}})_s$ defined by $\tilde{h}$ has type $\{(0,\,1),\,(1,\,0)\}$. The Hodge structure $C^+(H_{\mathbb{Q}})_s$ together with the lattice $C^+(H_{\mathbb{Z}})_s$ is a polarized Hodge structure; therefore it defines an abelian variety.*

REMARK 2.25. If we start with a rational Hodge structure $H_{\mathbb{Q}}$, the resulting abelian variety is given up to $\mathbb{Q}$-isogeny as this involves choosing a lattice $H_{\mathbb{Z}}$ in $H_{\mathbb{Q}}$ on which the restriction of B is integral valued.

Now, let us assume instead that we are given a Hodge structure H_K of weight 0 and rank $n+2$ over the totally real number field K with a non–degenerate bilinear form

$$\psi :\, H_K \otimes H_K \to K.$$

Moreover, we assume that K acts on H_K as endomorphisms of Hodge structures. Then, we have

$$H_K \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{i=1}^{g} H_K \otimes_{K,\,\tau_i} \mathbb{R},$$

and

$$H_K \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{i=1}^{g} H_K \otimes_{K,\,\tau_i} \mathbb{C}.$$

We assume that, for each $i = 1, \ldots, g$, the extension of scalars $\psi \otimes_{\tau_i} \mathbb{R}$ of ψ has signature $(n,\, 2)$ and defines a polarization of $H_K \otimes_{K,\,\tau_i} \mathbb{R}$, and that the Hodge number of $H_K \otimes_{K,\,\tau_i} \mathbb{R}$ are

$$h^{1,\,-1} = h^{-1,\,1} = 1 \quad \text{and} \quad h^{0,\,0} = n.$$

By Definition 2.2, this datum defines a morphism of real algebraic groups

$$h : \underline{\mathbb{S}} \to \mathrm{SO}(H_K \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{SO}(H_K))_{\mathbb{R}}$$

of weight 0. By making use of the functoriality of $\mathrm{Res}_{K/\mathbb{Q}}$, one easily sees that h lifts to a unique morphism

$$\tilde{h} : \underline{\mathbb{S}} \to \mathrm{CSpin}(H_K \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{CSpin}(H_K))_{\mathbb{R}} = G_{\mathbb{R}}$$

which fits in the commutative diagram

$$\begin{array}{ccccc}
\mathbb{G}_{m/\mathbb{R}} & \xrightarrow{w} & \underline{\mathbb{S}} & \xrightarrow{t} & \mathbb{G}_{m/\mathbb{R}} \\
\| & & \big\downarrow \tilde{h} & & \| \\
\mathbb{G}_{m/\mathbb{R}} & \xrightarrow{w} & G_{\mathbb{R}} & \xrightarrow{t} & \mathbb{G}_{m/\mathbb{R}},
\end{array}$$

where G is the $\mathbb{Q}$-algebraic group $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{CSpin}(H_K))$ and $t \circ w(x) = x^{-2}$. The representations $C^+(H_K)_s$ and $C^+(H_K)_{ad}$ of $\mathrm{CSpin}(H_K)$ restrict functorially to $\mathbb{Q}$-rational representations of the $\mathbb{Q}$-group G on the $\mathbb{Q}$-spaces $\mathrm{Res}_{K/\mathbb{Q}}(C^+(H_K)_s)$ and $\mathrm{Res}_{K/\mathbb{Q}}(C^+(H_K)_{ad})$ respectively. Now, Proposition 4. 5 in [**17**] can be restated as follows.

PROPOSITION 2.26. [**17**, Proposition 4. 5] *The Hodge structure of* $\mathrm{Res}_{K/\mathbb{Q}}(C^+(H_K)_s)$ *defined by* $\tilde{h}$ *is a polarized Hodge structure of type* $\{(0,1),(1,0)\}$.

By choosing an $\mathcal{O}_K$-lattice $H_{\mathcal{O}_K}$ of H_K to which the restiction of ψ is $\mathcal{O}_K$-valued, we get a lattice $\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}}(C^+(H_{\mathcal{O}_K})_s)$ in $\mathrm{Res}_{K/\mathbb{Q}}(C^+(H_K)_s)$. This then defines a polarized abelian variety. Any other choice of lattice gives an F-isogenous variety.

Applying this construction to the Hodge structure $H^2(M_{\mathbf{f}},\mathbb{Q})(1)$, we get a Hilbert-Blumenthal abelian variety $A_{\mathbf{f}}$. We call it the Hilbert-Blumenthal abelian variety attached to $\mathbf{f}$. By making use of the fact that $\Psi_{\mathbf{f}}$ is a kernel form, one can be a bit more specific about $A_{\mathbf{f}}$.

THEOREM 2.27. [**40**, sec. 5.5] *The* $K_{\mathbf{f}}$*-rational Hodge structure* $H^2(M_{\mathbf{f}},\mathbb{Q})$ *is the tensor product of two* $K_{\mathbf{f}}$*-rational Hodge structures of weight 1:*

$$H^2(M_{\mathbf{f}},\mathbb{Q}) = W^{(1)} \otimes_{K_{\mathbf{f}}} W^{(2)}.$$

Therefore the variety $A_{\mathbf{f}}$ *is of the form* $A_{\mathbf{f}} = A^1_{\mathbf{f}} \times A^1_{\mathbf{f}} \times A^2_{\mathbf{f}} \times A^2_{\mathbf{f}}$.

The period moduli of those varieties are given by the following result of Oda.

THEOREM 2.28. [**40**, Theorem 6.6] *Let* $\mathbf{f} \in S_2(\mathfrak{c})$ *be a normalized eigenform, and* $\mathbf{f}^\sigma$ *the companions of* $\mathbf{f}$*, where* σ *runs over all the real embeddings of* $K_{\mathbf{f}}$*. Choose a canonical basis of* $H^2(M_{\mathbf{f}},\mathbb{Q})$ *and form the periods*

$$\Omega_{ss'}(\mathbf{f}^\sigma).$$

Then, the period moduli of the two Hilbert-Blumenthal abelian varieties $A^1_{\mathbf{f}}$ *and* $A^2_{\mathbf{f}}$ *are given by the points* $\tau_1, \tau_2 \in (\mathbb{C}-\mathbb{R})^d$ *given by*

$$\tau_1 = (\Omega_{-+}(\mathbf{f}^\sigma)/\Omega_{++}(\mathbf{f}^\sigma))_\sigma, \quad \tau_2 = (\Omega_{+-}(\mathbf{f}^\sigma)/\Omega_{++}(\mathbf{f}^\sigma))_\sigma.$$

6. Computations.

In this section, we explain, in a specific example, how one can compute the period modulus of the Hilbert-Blumenthal abelian variety associated to a newform $\mathbf{f} \in S_2(\mathfrak{c})$. One way of doing this is to exploit the relation given by Oda [**40**] between such periods and special values of twists of L–functions. Those methods can then be applied easily when $p_g(X_0(\mathfrak{c})) = 1$.

In the rest of this section, we let $F = \mathbb{Q}(\sqrt{5})$ and $\omega = (1+\sqrt{5})/2$. Let $\mathfrak{n} = (\nu)$ be an integral ideal and $V \subset \mathcal{O}_F^{\times +}$ a subgroup of finite index such that $V \subset 1 + \mathfrak{n}$. For any $\alpha \in \mathcal{O}_F$, we define

$$D_{\alpha/\nu} = \{z \in \mathcal{H}^2 : \mathrm{Re}(z) = \alpha/\nu\}.$$

We have a map $i : V\backslash D_{\alpha/\nu} \to \Gamma\backslash\mathcal{H}^2$, where V acts on $D_{\alpha/\nu}$ by $z \mapsto \varepsilon z + (1-\varepsilon)\alpha/\nu$. The quotient $V\backslash D_{\alpha/\nu}$ is topologically a cylinder $S^1 \times \mathbb{R}$ and its image in X_Γ can be compactified by adding the orbits of α/ν and ∞ in $\mathbf{P}^1(F)$. This gives a 2-cycle $\gamma_{\alpha/\nu, V}$ in X_Γ. (Replacing V by a smaller subgroup V' multiplies this cycle by $[V : V']$). The following lemma is from Oda [**40**].

Lemma 2.29. [**40**, Lemma 15.2] *If $\chi : (\mathcal{O}_F/\mathfrak{n})^\times \to \mathbb{C}^\times$ is a non-trivial character, then*

$$\gamma_{\chi, V} = \sum_{\alpha \in \mathcal{O}_F/\mathfrak{n}} \chi(\alpha)\gamma_{\alpha/\nu, V} \in \mathrm{Im}(H_2(X'_\Gamma, \mathbb{Q}) \to H_2(X_\Gamma, \mathbb{Q})).$$

By projection, $\gamma_{\chi, V}$ defines a class in $H_2^{sp}(X_\Gamma, \mathbb{Q})$).

LEMMA 2.30. [**40**, Lemma 15.4] *Let* $\chi : (\mathcal{O}_F/\mathfrak{n})^\times \to \mathbb{C}^\times$ *be a primitive character. Then, we have*

$$\gamma_{\chi,V} \in \begin{cases} H_2^{sp}(X_\Gamma,\, \mathbb{Q}))_{++} \oplus H_2^{sp}(X_\Gamma,\, \mathbb{Q}))_{--} & \text{if } \chi(-1) = 1, \\ H_2^{sp}(X_\Gamma,\, \mathbb{Q}))_{+-} \oplus H_2^{sp}(X_\Gamma,\, \mathbb{Q}))_{-+} & \text{if } \chi(-1) = -1. \end{cases}$$

Moreover, $\gamma_{\chi,V} \in H_2^{sp}(X_\Gamma,\, \mathbb{Q}))_{ss'}$ *if* $\chi(\omega) = s$ *and* $\chi(\bar\omega) = s'$.

Oda had shown that integrating along the cycle $\gamma_{\chi,V}$, one gets special values of twisted L-functions.

THEOREM 2.31. [**40**, Theorem 16.3] *Let* $\mathbf{f} \in S_2(\mathfrak{c})$ *be a normalized eigenform. Then*

$$\mathrm{L}(\mathbf{f},\, 1,\, \chi) = \frac{1}{[\mathcal{O}_F^{\times +} : V]} \frac{1}{(2\pi)^2} \frac{1}{G(\bar\chi)} \int_{\gamma_{\bar\chi},V} \omega_{\mathbf{f}},$$

where V *is of sufficiently large index and the orientation of* $\gamma_{\bar\chi,V}$ *is appropriately chosen, and* $G(\chi)$ *the Gauss sum*

$$G(\chi) = \sum_{\alpha \in \mathcal{O}_F/(\nu)} \chi(\alpha) \exp(2\pi \mathrm{Tr}(\omega\alpha/\nu\sqrt{D})).$$

This theorem gives us a way to compute the period moduli of the abelian varieties $A^1_{\mathbf{f}}$ and $A^2_{\mathbf{f}}$. Let us fix a normalized eigenform $\mathbf{f} \in S_2(\mathfrak{c})$ and a primitive character χ of conductor $\mathfrak{n} = (\nu)$. We let W_A be the Atkin-Lehner involution given by

$$W_A : z = (z_1,\, z_2) \mapsto (-\frac{1}{Az_1},\, -\frac{1}{\bar A z_2}),$$

where A is a totally positive generator of $\mathfrak{c}$. Let

$$\begin{aligned} \mathbf{f}(z_1,\, z_2) &= \sum_{\mu \in \mathcal{O}_F^+} c((\mu)) \exp[2\pi i(\mathrm{Tr}(\frac{\mu\omega z}{\sqrt{D}})] \\ &= \sum_{\mu \in \mathcal{O}_F^+/\mathcal{O}_F^{\times +}} c((\mu)) \sum_{\varepsilon \in \mathcal{O}_F^{\times +}} \exp[2\pi i \mathrm{Tr}(\frac{\varepsilon\mu\omega z}{\sqrt{D}})] \end{aligned}$$

be the q-expansion of $\mathbf{f}$. Then $\mathbf{f}_\chi = \mathbf{f} \otimes \chi \in S_2(\mathfrak{c}\mathfrak{n}^2)$ and its q-expansion is given by

$$\begin{aligned}
\mathbf{f} \otimes \chi(z_1,\, z_2) &= \sum_{\mu \in \mathcal{O}_F^+} c((\mu))\chi(\mu) \exp[2\pi i \mathrm{Tr}(\frac{\mu\omega z}{\sqrt{D}})] \\
&= \sum_{\mu \in \mathcal{O}_F^+/\mathcal{O}_F^{\times +}} c((\mu))\chi(\mu) \sum_{\varepsilon \in \ \mathcal{O}_F^{\times +}} \exp[2\pi i \mathrm{Tr}(\frac{\varepsilon\mu\omega z}{\sqrt{D}})].
\end{aligned}$$

LEMMA 2.32. *Let* $\mathbf{f} \in S_2(\mathfrak{c})$ *be an eigenform.*
If $W_A\mathbf{f} = -\mathbf{f}$*, then* $\mathrm{L}(\mathbf{f}) = 0$*; otherwise*

$$\begin{aligned}
\mathrm{L}(\mathbf{f},\, 1) &= -\frac{D}{2\pi^2} \sum_{\mu \in \mathcal{O}_F^+} \frac{c((\mu))}{\mathrm{N}(\mu)} \left[1 - \exp\left(-\frac{2\pi\mu\omega}{\sqrt{DA}}\right)\right] \times \\
&\qquad \exp\left[\frac{2\pi}{\sqrt{D}}\left(\frac{\bar{\mu}\bar{\omega}}{\sqrt{\bar{A}}} - \frac{\mu}{\sqrt{A}}\right)\right].
\end{aligned}$$

PROOF. By definition,

$$\mathrm{L}(\mathbf{f},\, 1) = \int\int_{\mathcal{O}_F^{\times +}\backslash\mathbb{R}_+^2} \mathbf{f}(iy_1,\, iy_2)dy_1dy_2 = \int_{\tau_0}^{\omega^2\tau_0} \int_0^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2,$$

with τ_0 arbitrarily chosen on the imaginary axis. But choosing $\tau_0 = i/\omega\sqrt{A}$, and making the change of variable $z \mapsto W_A(z)$, we get

$$\begin{aligned}
\mathrm{L}(\mathbf{f},\, 1) &= \int_{\tau_0}^{\omega^2\tau_0} \int_0^{i\infty} \mathbf{f}\|W_A(z_1,\, z_2)dz_1dz_2 \\
&= \varepsilon_A \mathrm{L}(\mathbf{f},\, 1).
\end{aligned}$$

This gives the first part of the lemma. To get the second one, keeping the same choice of τ_0 as before, we choose C on the imaginary axis and split the integral as

$$\mathrm{L}(\mathbf{f},\, 1) = \int_{\tau_0}^{\omega^2\tau_0} \int_0^{C} \mathbf{f}(z_1,\, z_2)dz_1dz_2 + \int_{\tau_0}^{\omega^2\tau_0} \int_C^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2.$$

Again choosing $C = i/\sqrt{A}$ and making the change of variable $z \mapsto W_A(z)$ in the first integral of the sum, we get that

$$\mathrm{L}(\mathbf{f},\,1) \;=\; (1+\varepsilon_A)\int_{\tau_0}^{\omega^2\tau_0}\int_C^{i\infty}\mathbf{f}(z_1,\,z_2)dz_1dz_2.$$

We then conclude the proof of the lemma by an integration term by term. □

REMARK 2.33. Let $\mathbf{f} \in S_2(\mathfrak{c})$ be an eigenform with $W_A\mathbf{f} = \varepsilon_A\mathbf{f}$, and χ a *quadratic* character of conductor $\mathfrak{n} = (\nu)$ with $(\mathfrak{c},\,\mathfrak{n}) = 1$. When χ is a quadratic character, $\mathbf{f}_\chi \in S_2(\mathfrak{c}\mathfrak{n}^2)$ and $W_{A\nu^2}\mathbf{f}_\chi = \varepsilon_A\chi(-A)\mathbf{f}_\chi$. Therefore, by the above Lemma, when $\varepsilon_A\chi(-A) = 1$,

$$\begin{aligned}\mathrm{L}(\mathbf{f},\,1,\,\chi) \;=\; & -\frac{D}{2\pi^2}\sum_{\mu\in\mathcal{O}_F^+}\frac{c((\mu))}{\mathrm{N}(\mu)}\chi(\mu)\left[1-\exp\left(-\frac{2\pi\mu\omega}{\nu\sqrt{DA}}\right)\right]\times \\ & \exp\left[\frac{2\pi}{\sqrt{D}}\left(\frac{\bar{\mu}\bar{\omega}}{\bar{\nu}\sqrt{\bar{A}}}-\frac{\mu}{\nu\sqrt{A}}\right)\right].\end{aligned}$$

By using the fact that every totally positive element is of the form ω^{2k}, $k \in \mathbb{Z}$, this series can be rearranged as

$$\begin{aligned}&\mathrm{L}(\mathbf{f},\,1,\,\chi) = \\ &\quad -\frac{D}{2\pi^2}\sum_{\mu\in\mathcal{O}_F^+/\mathcal{O}_F^{+\times}}\frac{c((\mu))}{\mathrm{N}(\mu)}\chi(\mu)\times\sum_{k\in\mathbb{Z}}\left[1-\exp\left(-\frac{2\pi\mu\omega^{2k+1}}{\nu\sqrt{DA}}\right)\right]\times \\ &\quad\quad \exp\left[\frac{2\pi}{\sqrt{D}}\left(\frac{\bar{\mu}\bar{\omega}^{2k+1}}{\bar{\nu}\sqrt{\bar{A}}}-\frac{\mu\omega^{2k}}{\nu\sqrt{A}}\right)\right].\end{aligned}$$

When χ is not a quadratic character, we need to resort to a different approach to express $\mathrm{L}(\mathbf{f},\,1,\,\chi)$ as a series. This involves the calculation of the partial L-functions that appear in Oda's Theorem. As we will see shortly, this has the inconvenient of bringing some extra factor in the denominator of the exponential sum with the bad effect of slowing down the convergence of the series. To start, we recall that

$$\mathbf{f}_\chi = G(\bar{\chi})^{-1} \sum_{\alpha \in \mathcal{O}_F/\mathfrak{n}} \overline{\chi(\alpha)} \mathbf{f} \| \gamma_{\alpha,\nu} \qquad \text{with } \gamma_{\alpha,\nu} = \begin{pmatrix} \nu & \alpha \\ 0 & \nu \end{pmatrix},$$

so that

$$\mathrm{L}(\mathbf{f},\, 1,\, \chi) = G(\bar{\chi})^{-1} \sum_{\alpha \in \mathcal{O}_F/\mathfrak{n}} \overline{\chi(\alpha)} \mathrm{L}(\mathbf{f} \| \gamma_{\alpha,\nu},\, 1).$$

The partial L-functions are then computed using the following procedure.

Lemma 2.34. *Let* $\mathbf{f}$ *be a normalized eigenform, with* $\mathbf{f}\|W_A = \varepsilon_A \mathbf{f}$. *We let* $\mathcal{P} = \{\mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1),\, \alpha \in (\mathcal{O}_F/\mathfrak{n})^\times\}$ *be the set of all partial L-functions that contribute to* $\mathrm{L}(\mathbf{f},\, 1,\, \chi)$, *and* $w_A : (\mathcal{O}_F/\mathfrak{n})^\times \to (\mathcal{O}_F/\mathfrak{n})^\times$ *the isomorphism* $\alpha \mapsto -1/A\alpha$.
Then the map $\mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) \mapsto \mathrm{L}(\mathbf{f}\|\gamma_{w_A(\alpha),\nu},\, 1)$ *is a permutation of* $\mathcal{P}$. *Moreover,* $\mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) = \varepsilon_A \mathrm{L}(\mathbf{f}\|\gamma_{w_A(\alpha),\nu},\, 1)$ *for all* α, *and*

$$\begin{aligned} \mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) = &-\frac{D}{(2\pi)^2} \sum_{\mu \in \mathcal{O}_F^+} \frac{c((\mu))}{\mathrm{N}(\mu)} \left[1 - \exp\left(-\frac{2\pi\mu\omega}{\varepsilon_0 \nu \sqrt{DA}}(\varepsilon_0^2 - 1)\right)\right] \times \\ &\left[\exp\left(-2\pi i \mathrm{Tr}\left(\frac{\mu\omega\alpha}{\nu\sqrt{D}}\right)\right) + \varepsilon_A \exp\left(-2\pi i \mathrm{Tr}\left(\frac{\mu\omega w_A(\alpha)}{\nu\sqrt{D}}\right)\right)\right] \times \\ &\exp\left[\frac{2\pi}{\sqrt{D}}\left(\frac{\bar{\mu}\bar{\omega}}{\bar{\nu}\sqrt{\bar{A}}} - \frac{\mu\omega}{\varepsilon_0 \nu \sqrt{A}}\right)\right], \end{aligned}$$

where $\varepsilon_0 \in \mathcal{O}_F^\times$ *is chosen such that* $\varepsilon_0 > 0$ *and* $\varepsilon_0^2 - 1 \in (\nu)$.

Proof. Let $\beta \in \mathcal{O}_F$ be such that $A\alpha\beta = -1 \mod (\nu)$. Then, there exists $\gamma_1 = \begin{pmatrix} \nu & \alpha \\ A\beta & d \end{pmatrix} \in \Gamma_0(\mathfrak{c})$ such that $\gamma_1(-\alpha/\nu) = 0$. Take

$\gamma_0 = \gamma_{\alpha,\nu}^{-1} \begin{pmatrix} \varepsilon_0^2 & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\alpha,\nu}$ in $\Gamma_0(\mathfrak{c})$ and put $\gamma = \gamma_1\gamma_0$. Then, by definition,

$$\begin{aligned} \mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) &= \int\int_{V\setminus D_{\alpha,\,\nu}} \mathbf{f}(z_1,\, z_2)dz_1dz_2 \\ &= \int_{\tau_0}^{\gamma_0(\tau_0)} \int_{-\bar{\alpha}/\bar{\nu}}^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2, \end{aligned}$$

where we choose $\tau_0 \in \mathcal{H}$ with $\mathrm{Re}(\tau_0) = -\alpha/\nu$. By making the substitution $z \mapsto \gamma_1^{-1}W_A(z)$, we get that

$$\mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) = \int_{-1/A\gamma(\tau_0)}^{-1/A\gamma_1(\tau_0)} \int_{-\bar{\beta}/\bar{\nu}}^{i\infty} \mathbf{f}\|W_A(z_1,\, z_2)dz_1dz_2.$$

Taking $\tau_0 = -\alpha/\nu + i/\varepsilon_0\nu\sqrt{A}$, this gives

$$\mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) = \int_{\tau_1}^{\gamma_0'(\tau_1)} \int_{-\bar{\beta}/\bar{\nu}}^{i\infty} \mathbf{f}\|W_A(z_1,\, z_2)dz_1dz_2,$$

where $\tau_1 = -\beta/\nu + i/\varepsilon_0\nu\sqrt{A}$ and $\gamma_0' = \gamma_{\beta,\nu}^{-1} \begin{pmatrix} \varepsilon_0^2 & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\beta,\nu}$. This just says that

$$\begin{aligned} \mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) &= \varepsilon_A \mathrm{L}(\mathbf{f}\|\gamma_{\beta,\nu},\, 1) \\ &= \varepsilon_A \mathrm{L}(\mathbf{f}\|\gamma_{w_A(\alpha),\nu},\, 1), \end{aligned}$$

and the first part of the lemma then follows. To the second one, we take $C \in \mathcal{H}$ with $\mathrm{Re}(C) = -\bar{\alpha}/\bar{\nu}$, and split the integral as

$$\begin{aligned} \mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) &= \int_{\tau_0}^{\gamma_0(\tau_0)} \int_{-\bar{\alpha}/\bar{\nu}}^{C} \mathbf{f}(z_1,\, z_2)dz_1dz_2 \\ &\quad + \int_{\tau_0}^{\gamma_0(\tau_0)} \int_{C}^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2 \\ &= \int_{\gamma_1(\tau_0)}^{\gamma_1\gamma_0(\tau_0)} \int_{0}^{\bar{\gamma}_1(C)} \mathbf{f}(z_1,\, z_2)dz_1dz_2 \\ &\quad + \int_{\tau_0}^{\gamma_0(\tau_0)} \int_{C}^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2, \end{aligned}$$

where the second equality is obtained by making the change of variable $z \mapsto \gamma_1^{-1}(z)$ in the first integral and using the $\Gamma_0(\mathfrak{c})$-invariancy of $\mathbf{f}$. The substitution $z \mapsto W_A(z)$ then gives:

$$\begin{aligned} \mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) &= \int_{-1/A\gamma(\tau_0)}^{-1/A\gamma_1(\tau_0)} \int_{-1/\bar{A}\bar{\gamma}_1(C)}^{i\infty} \mathbf{f}\|W_A(z_1,\, z_2)dz_1dz_2 \\ &\quad + \int_{\tau_0}^{\gamma_0(\tau_0)} \int_{C}^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2. \end{aligned}$$

Again, choosing $\tau_0 = -\alpha/\nu + i/\varepsilon_0\nu\sqrt{A}$ and $C = -\bar{\alpha}/\bar{\nu} + i/\bar{\nu}\sqrt{\bar{A}}$, we obtain that

$$\begin{aligned} \mathrm{L}(\mathbf{f}\|\gamma_{\alpha,\nu},\, 1) &= \varepsilon_A \int_{\tau_1}^{\gamma_0'(\tau_1)} \int_{-1/\bar{A}\bar{\gamma}_1(C)}^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2 \\ &\quad + \int_{\tau_0}^{\gamma_0(\tau_0)} \int_{C}^{i\infty} \mathbf{f}(z_1,\, z_2)dz_1dz_2, \end{aligned}$$

with $\mathrm{Im}(\tau_0) = \mathrm{Im}(\tau_1) = 1/\varepsilon_0\nu\sqrt{A}$, $\mathrm{Im}(\gamma_0(\tau_0)) = \mathrm{Im}(\gamma_0'(\tau_1)) = \varepsilon_0/\nu\sqrt{A}$ and $\mathrm{Im}(C) = \mathrm{Im}(-1/\bar{A}\bar{\gamma}_1(C)) = 1/\bar{\nu}\sqrt{\bar{A}}$. We then complete the lemma by integrating term by term. $\square$

By making use of the identity

$$\overline{\chi}(\mu)G(\chi) = \sum_{\alpha \in \mathcal{O}_F/\mathfrak{n}} \chi(\alpha) \exp\left(2\pi i \mathrm{Tr}\left(\frac{\mu\omega\alpha}{\sqrt{D}}\right)\right)$$

and the fact that $\overline{\chi}(-A\alpha\beta) = 1$, we see that

$$\begin{aligned} \sum_{\alpha \in \mathcal{O}_F/(\nu)} \overline{\chi}(\alpha) &\left[\exp\left(-2\pi i \mathrm{Tr}\left(\frac{\mu\omega\alpha}{\nu\sqrt{D}}\right)\right)\right. \\ &\left. + \varepsilon_A \exp\left(-2\pi i \mathrm{Tr}\left(\frac{\mu\omega w_A(\alpha)}{\nu\sqrt{D}}\right)\right)\right] \\ &= \chi(\mu)G(\overline{\chi}) + \varepsilon_A\chi(-A)\overline{\chi}(\mu)G(\chi). \end{aligned}$$

Thus reversing the order of the summation in $\mathrm{L}(\mathbf{f},\, 1,\, \chi)$, we get the following proposition.

PROPOSITION 2.35. *Let* $\mathbf{f} \in S_2(\mathfrak{c})$ *be an eigenform with* $W_A\mathbf{f} = \varepsilon_A\mathbf{f}$, *and* χ *a character of conductor* $\mathfrak{n} = (\nu)$ *with* $(\mathfrak{c}, \mathfrak{n}) = 1$. *Then,*

$$\mathrm{L}(\mathbf{f}, 1, \chi) = \mathrm{S}(\mathbf{f}, 1, \chi) + \varepsilon_A\chi(-A)\frac{G(\chi)}{G(\overline{\chi})}\mathrm{S}(\mathbf{f}, 1, \overline{\chi}),$$

where

$$\mathrm{S}(\mathbf{f}, 1, \chi) = -\frac{D}{(2\pi)^2}\sum_{\mu\in\mathcal{O}_F^+}\frac{c((\mu))}{\mathrm{N}(\mu)}\chi(\mu)\times$$
$$\left[1 - \exp\left(-\frac{2\pi\mu\omega}{\varepsilon_0\nu\sqrt{DA}}(\varepsilon_0^2 - 1)\right)\right]\exp\left[\frac{2\pi}{\sqrt{D}}\left(\frac{\bar{\mu}\bar{\omega}}{\bar{\nu}\sqrt{\bar{A}}} - \frac{\mu\omega}{\varepsilon_0\nu\sqrt{A}}\right)\right].$$

REMARK 2.36. Again making use of the fact that every totally positive element is of the form ω^{2k}, $k \in \mathbb{Z}$, we get

$$\mathrm{S}(\mathbf{f}, 1, \chi) = -\frac{D}{2\pi^2}\sum_{\mu\in\mathcal{O}_F^+/\mathcal{O}_F^{\times +}}\frac{c((\mu))}{\mathrm{N}(\mu)}\chi(\mu)\times$$
$$\sum_{k\in\mathbb{Z}}\left[1 - \exp\left(-\frac{2\pi\mu\omega^{2k+1}}{\varepsilon_0\nu\sqrt{DA}}(\varepsilon_0^2 - 1)\right)\right]\times$$
$$\exp\left[\frac{2\pi}{\sqrt{D}}\left(\frac{\bar{\mu}\bar{\omega}^{2k+1}}{\bar{\nu}\sqrt{\bar{A}}} - \frac{\mu\omega^{2k+1}}{\varepsilon_0\nu\sqrt{A}}\right)\right].$$

REMARK 2.37. The rate of convergence of those two series is controlled by the exponential sum. So to improve the convergence of our algorithm we must choose the character χ with conductor as small as possible. This turns out to be very delicate as a satisfactory error term requires to compute all Hecke operators for primes $\mathfrak{p}$ such that the ratio $\mathrm{N}(\mathfrak{p})/D\mathrm{N}(\mathfrak{c}\mathfrak{n}^2)$ is large enough.

EXAMPLE 2.38. For $\mathfrak{c} = 31A = (5+2\omega)$, we take $\mathfrak{n}_1 = (3)$ and $\mathfrak{n}_2 = (4)$. Then $(\mathcal{O}_F/\mathfrak{n}_1)^\times = \mathbb{F}_9^\times = \langle\omega\rangle$ and $(\mathcal{O}_F/\mathfrak{n}_2)^\times = \langle -1, \omega\rangle$. We let χ_1 be the unique quadratic character of conductor $\mathfrak{n}_1$ and χ_2 the quadratic character given by

$$\chi_2(\omega) = -1 = \chi_2(-1).$$

Then $\gamma_{\chi_1, V_1} \in H_2(X_0(\mathfrak{c}), \mathbb{Q})_{--}$ with $V_1 = \langle\omega^8\rangle$ and $\gamma_{\chi_2, V_2} \in H_2(X_0(\mathfrak{c}), \mathbb{Q})_{-+}$ with $V_2 = \langle\omega^6\rangle$. We complete γ_{χ_1, V_1} and

γ_{χ_2, V_2} to get a canonical basis. Then, Ω_{++} and Ω_{+-} are completely determined by the Riemann–Hodge relation. We get

$$\Omega_{--} = -67.464569223618835646128485047426897$$

and

$$\Omega_{-+} = -128.75794400309976871557728718976371i.$$

This gives

$$\tau' = -\Omega_{--}/\Omega_{-+} = 0.52396432504385645846658571918312i$$

The curve $E/\mathbb{Q}(\sqrt{5}) : y^2 + xy + \omega y = x^3 - (1+\omega)x^2$ has conductor $31A = (5 + 2\omega)$. The PARI procedure on elliptic curves yields the following periods for this curve:

$$\omega_1 = 3.0521731533572582695976003439575002$$

and

$$\omega_2 = 2.3988447693237222273201739886086 2i.$$

We get

$$\tau = \frac{\omega_2}{\omega_1} = 0.78594648756578468769989532640147425i.$$

The ratio

$$\frac{\tau}{\tau'} = 1.5000000000000000000000003196$$

is close to $3/2$ with 21 decimals accuracy. (Note that this is consistent with the fact that E has some 2–torsion).

EXAMPLE 2.39. For $\mathfrak{c} = 41A = (6 + \omega)$, we take $\mathfrak{n} = (4)$. Then, $(\mathcal{O}_F/\mathfrak{n})^\times = \langle -1, \omega\rangle$. We let χ_1 the quadratic character given by

$$\chi_1(\omega) = -1 = \chi_1(-1),$$

and χ_2 the quadratic character given by

$$\chi_2(\omega) = -1 \quad \text{and} \quad \chi_2(-1) = 1.$$

Then $\gamma_{\chi_1,V} \in H_2(X_0(\mathfrak{c}),\mathbb{Q})_{-+}$ and $\gamma_{\chi_2,V} \in H_2(X_0(\mathfrak{c}),\mathbb{Q})_{--}$ with $V = \langle \omega^6 \rangle$. We complete $\gamma_{\chi_1,V}$ and $\gamma_{\chi_2,V}$ to get a canonical basis. We get

$$\Omega_{--} = -54.400185806959564311393839632 07$$

and

$$\Omega_{-+} = -160.12204086937231354613169017024i.$$

This gives

$$\tau' = -\Omega_{--}/\Omega_{-+} = 0.33974202122079669553192582647 30i.$$

The curve $E/\mathbb{Q}(\sqrt{5}) : y^2 + \omega y = x^3 - \omega x^2$ has conductor $41A = (6+\omega)$. Again, the PARI procedure yields the real and imaginary periods

$$\omega_1 = 7.2271574783846882931990959990 68$$

and

$$\omega_2 = 2.4553690893874103055043782282 62i.$$

This gives

$$\tau = \frac{\omega_2}{\omega_1} = 0.33974202122079669553215993921i.$$

The ratio

$$\frac{\tau}{\tau'} = 1.0000000000000000000000689$$

is close to 1 with 22 decimals accuracy.

7. Program Listing

We use this program to compute the Oda periods.

```
\\THIS ROUTINE COMPUTE THE ODA PERIODS.

\\Initializing parameters.
w=quadgen(5);
field=nfinit(z^2-z-1);
fieldembed(x)=Mod(real(x)+imag(x)*z, z^2-z-1);
level=6+w;

read(uniformizers1);

read(41Acoeffs2_10979);

\\This function returns the primes in $O_K$ lying above\\
\\any prime $p$ in $\Z$.

gen(pr, res)=res=0;\
   if(kronecker(pr,5)==-1,res=pr,N0=floor(sqrt(pr-1));\
                                   a=N0;res1=r^2-a*r+pr-a^2;\
       while(polisirreducible(res1),a=a-1;res1=r^2-a*r+pr-a^2);\
            res=floor(real(polroots(res1)))~;\
     if(res[1]==res[2], res=a+res[1]*w, res=[a,a]+w*res)); res;




\\Creating the group of unit in $O_K/I$ for some ideal $I$. We are only
\\interested in the reduction of the torsion free part. We call $nu$ a
\\generator of $I$.

\\nu=w^10-1;
nu=3;

nu1=[]; res=idealfactor(field, fieldembed(nu));\
for(j=1, matsize(res)[1], res0=nfmodprinit(field, res[j,][1]);\
```

```
            nu1=concat(nu1, [res0])); nu1;

nu0=vector(length(nu1), i1, 0); for(j=1, length(nu1),\
                                res=idealnorm(field, nu1[j]);\
       if(issquare(res)==1, nu0[j]=sqrtint(res), nu0[j]=res)); nu0;

nu2=3;
nu3=4;



G=vector(3);\
G[1]=1; for(j=1, length(nu0), if(kronecker(nu0[j],5)==-1,\
             G[1]=G[1]*(nu0[j]^2-1), G[1]=G[1]*(nu0[j]-1))); G[1];\
G[2]=[];\
for(j=1, length(nu0), if(kronecker(nu0[j],5)==-1, res=nu0[j]^2-1,\
                     res=nu0[j]-1); G[2]=concat(G[2], res)); G[2];\
G[3]=[];\
for(j=1, length(G[2]), s=divisors(G[2][j]);\
if(kronecker(nu0[j],5)==-1, i=0; k=1; n=1;\
                        res=Mod(i, nu0[j])+Mod(k, nu0[j])*w;\
  until(n==length(s), n=1; k=k+1; if(k>=nu0[j], k=1; i=i+1);\
                          res=Mod(i,nu0[j])+Mod(k, nu0[j])*w;\
     while((!(res^s[n]==Mod(1, nu0[j])))&&n<length(s), n=n+1));\
       res=lift(res); G[3]=concat(G[3], res),\
     l=1; until(n==length(s), n=1; l=l+1; res=Mod(l, nu0[j]);\
     while((!(res^s[n]==Mod(1, nu0[j])))&&n<length(s), n=n+1));\
         res=lift(res); G[3]=concat(G[3], res))); print(G);

G0=vector(3);\
G0[1]=G[1]; G0[2]=G[2]; G0[3]=[]; for(j=1, length(G[2]),\
  if(!(imag(G[3][j])==0), res=Mod(real(G[3][j]), nu0[j])+Mod(imag(G[3][j]),\
               nu0[j])*w, res=Mod(G[3][j], nu0[j]));\
                G0[3]=concat(G0[3], res)); print(G0);


\\In the cyclic group of order $G[2][j]$, this function returns the
\\order of any element $g$.

ord(g, j, s)=s=1; divs=divisors(G[2][j]); k=1;\
```

```
g0=nfeltreducemodpr(field, fieldembed(g), nu1[j]);\
     g0=g0[1]+g0[2]*w;\
     while((!(g0^divs[k]==Mod(1, nu0[j])))&&k<length(divs), k=k+1);\
          s=divs[k]; s;

\\Writing any element of the local factor at the j^th primes as \\
\\a power of the cyclic generator $G[3][j]$.

decomp(g, j, res0)=res0=1; res1=G[2][j]/ord(g,j); i=0;\
g0=nfeltreducemodpr(field, fieldembed(g), nu1[j]); g0=g0[1]+g0[2]*w;\
  while(!(G0[3][j]^i==g0), i=i+1); res0=i; res0;


\\ This function returns the quadratic residue character modulo $\nu$.
\\We first define the local residue character at each prime. The global
\\character is just the product of those one.

quad0(g, j, res)=res=1; res1=(G[2][j]/ord(g,j))%2; res=(-1)^res1; res;

quad(g, res0)=res0=0;\
  if(!(idealadd(field, fieldembed(g), fieldembed(nu))==[1, 0; 0, 1]),\
        res0=0, res0=1; for(j=1, length(G[2]), res0=res0*quad0(g, j))); res0;

print([quad(-1), quad(w), quad(1-w), quad(-level)]);


\\Here we create the quadratic character over $O_K/(4)$. We use the fact that
\\$O_K/(4)^*$ is generated by $w$ and $-1$.

quad2(g, res3)=res3=0;\
  if(!(idealadd(field, fieldembed(g), [4, 0; 0, 4])==[1, 0; 0, 1]),\
    res3=0, res3=1; if(g==-1||g==w, res3=-1, res0=g%4;\
    j=1; until((w^j%4==res0||-w^j%4==res0), j=j+1);\
      if(w^j%4==res0, res3=(-1)^j, res3=(-1)^(j+1)))); res3;

print([quad2(-1), quad2(w), quad2(1-w), quad2(-level)]);


quad3(g, res3)=res3=0;\
```

```
  if(!(idealadd(field, fieldembed(g), [4, 0; 0, 4])==[1, 0; 0, 1]),\
    res3=0, res3=1; if(g==-1||g==w, if(g==w, res3=-1, res3=1), res0=g%4;\
    j=1; until((w^j%4==res0||-w^j%4==res0), j=j+1); res3=(-1)^j)); res3;

print([quad3(-1), quad3(w), quad3(1-w), quad3(-level)]);




\\This function gives the factorization of all ideals of norm less than
\\$N$ in terms of uniformizer given by the gen function above.

N=10979;

ideals=[];\
ids=ideallist(field, N);\
ids0=[]; for(j=2, length(ids), ids0=concat(ids0, ids[j]));\
for(j=1, length(ids0), res0=idealfactor(field, ids0[j]);\
              res3=matrix(matsize(res0)[1], 2);\
for(k=1, matsize(res0)[1], res2=res0[k,1][1];\
 if(kronecker(res2,5)==-1, res3[k,]=[res2, res0[k,2]],\
  if(kronecker(res2,5)==0, res3[k,]=[gen(5), res0[k,2]],\
    if(kronecker(res2,5)==1 &&\
             idealfactor(field, fieldembed(gen(res2)[1]))[1,1]==res0[k,1],\
         res3[k,]=[gen(res2)[1], res0[k,2]], res3[k,]=\
                  [conj(gen(res2)[1]), res0[k,2]]))));\
            ideals=concat(ideals, [res3])); ideals;



\\For an ideal given in the above form, this function returns a generator.

idgen(id, res4)=res4=1;\
 for(k1=1, matsize(id)[1], res4=res4*id[k1,][1]^id[k1,][2]); res4;



\\This function returns the Hecke eigenvalue associated to any prime\\
power given in the above list.

hcoeff0(id0,res0)=res0=1;\
 j1=1; while(!(id0[1]==unifs[j1]), j1=j1+1); j1;\
```

```
   if(denominator(level/unifs[j1])==1, res0=ap[j1]^id0[2],\
     if(id0[2]==1, res0=ap[j1], res0=res2=ap[j1]; res1=1;\
       for(k2=2, id0[2], res0=ap[j1]*res2-norm(unifs[j1])*res1;\
                                res1=res2; res2=res0))); res0;


\\This function compute the Hecke eigenvalue associated to any ideal\\
\\given in the the list above.

hcoeff(id, res5)=res5=1;\
  for(l=1, matsize(id)[1], res5=res5*hcoeff0(id[l,])); res5;




\\This compute the exponential sum corresponding to the residue
\\$\alpha$ modulo $\nu$.

subsum(id1, nus, sum0)=sum0=0;\
M=[0,0]; M0=-1;\
  while(abs(2*Pi*conj(idgen(id1))*(1-w)^(2*M0+1)/(conj(nus)*(2*w-1)*\
      sqrt(conj(level))))<=10^5, M0=M0-1); M[1]=M0;\
M0=1; while(2*Pi*idgen(id1)*w^(2*M0+1)/(nus*(2*w-1)*\
                     sqrt(level))<=10^5, M0=M0+1); M[2]=M0;\
sum0=sum(k3=M[1], M[2], (1-exp(-2*Pi*idgen(id1)*\
                w^(2*k3+1)/(nus*(2*w-1)*sqrt(level))))*\
   exp((2*Pi/(2*w-1))*(conj(idgen(id1)*w^(2*k3+1))/(conj(nus)*\
        sqrt(conj(level)))- idgen(id1)*w^(2*k3)/(nus*sqrt(level))))); sum0;


\\This returns an approximation of the L-function...???



L0=res6=subsum(Mat([1,1]), nu3);\
    res6=res6+sum(m=1, length(ideals), hcoeff(ideals[m])*\
        quad3(idgen(ideals[m]))/norm(idgen(ideals[m]))*\
             subsum(ideals[m], nu3)); L0=-res6;

print(L0);
```

```
Wmm=sqrt(abs(norm(nu)))*L0; print(Wmm);


L=res7=subsum(Mat([1,1]), nu3);\
    res7=res7+sum(m=1, length(ideals),\
          (hcoeff(ideals[m])/norm(idgen(ideals[m])))*\
              quad2(idgen(ideals[m]))*subsum(ideals[m], nu3)); L=-res7;

print(L);

Wmp=sqrt(abs(norm(nu2)))*L; print(Wmp);
```

8. Conclusion.

The theory of algebraic modular forms in Gross [**29**] gives a very nice way to compute Hilbert modular eigenforms of weight 2 over $F = \mathbb{Q}(\sqrt{5})$ by transferring such computations to the Hamilton quaternion algebra by means of the Jacquet–Langlands correspondence. The same approach can be used over any totally real field F (of narrow class number one) and for more general (including non-parrallel) weights (see the recent paper of Consani and Scholten [**12**]). The major requirement of the algorithms we have used is finding an efficient way to compute the Hecke operators. This becomes an imperative issue as the norm of the level increases, in order to get a satisfactory precision on the period moduli of the computed newforms. So it would be interesting to see wether or not the icosian ring admits a Euclidean algorithm. If affirmative, such a result will have the dramatic effect of increasing the efficiency of our algorithm. This is an issue that we intend to explore.

In contrast to the one dimensional case, the computation of periods in higher dimension seems to be a delicate matter. Indeed, from the examples we discussed, the computation of the Oda periods requires a large number of Hecke eigenvalues. Indeed, for the two smallest primes in our list, namely 31 and 41, we needed the Hecke eigenvalues for all primes of norm less than or equal to 10000. But this could only give us an accuracy of no more than 20 decimals; and this does not seem to be enough to determine either an integral or a rational model in the corresponding isogeny classes. The main difficulty lies in the fact that the computation of the Oda periods implicitly involves the choice of a lattice in the rational cohomology group $H^2(M_{\mathbf{f}}, \mathbb{Q})$ associated to the newform $\mathbf{f}$, and this is a big difference with the one dimensional case. Indeed, when $F = \mathbb{Q}$, not only is it known that the elliptic curve $E_{\mathbf{f}}$ is rational, but also a result of Edixhoven [**21**] on the Manin constant, says that the modular symbols computations, as done in Cremona [**14**], directly leads to an integral model. This greatly cuts down the amount of computations needed in this case. The modular parametrization of elliptic curves play a rôle in the definition of the Manin constant. In our setting, one is then tempted to consider its natural analogue which is the parametrization of elliptic curves by Shimura curves. Due to the lack of q–expansion in this situation, one then asks for the right substitute to the Manin constant and how this will ensure us with a good model for the isogeny class of the elliptic curve corresponding to our newform. There is no doubt that this question is related to the one of accessing the fundamental periods of the newform $\mathbf{f}$. As this is a very active field in the theory of Hilbert modular forms, the few results in our second chapter should be seen more as a work in progress. We would like to explore the issue of overcoming the torsion in $H^2(M_{\mathbf{f}}, \mathbb{Q})$ in order to obtain an integral model for the computed isogeny class. We would also like to extend the PARI search that we started with the help of Prof. N. Elkies and see if we can make use of results in the spirit of Faltings–Serre or the more recent results of Skinner–Wiles to prove the modularity of the curves we have obtained (and will obtain).

Bibliography

[1] A. Ash and L. Rudolph, The modular symbol and continued fractions in higher dimensions. *Invent. Math.* **55** (1979), no. 3, 241–250.

[2] A. Ash and A. Borel, Generalized modular symbols. Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), 57–75, *Lecture Notes in Math.*, **1447**, Springer, Berlin, 1990.

[3] A. Ash, D. Ginzburg and S. Rallis, Vanishing periods of cusp forms over modular symbols. *Math. Ann.* **296** (1993), no. 4, 709–723.

[4] M. Bertolini and H. Darmon, Heegner points on Mumford-Tate curves. Inventiones Math. **126** (1996), pp. 413-456.

[5] *Modular functions of one variable. IV.* Proceedings of the International Summer School on Modular Functions of One Variable and Arithmetical Applications, RUCA, University of Antwerp, Antwerp, July 17–August 3, 1972. Edited by B. J. Birch and W. Kuyk. Lecture Notes in Mathematics, Vol. 476. Springer-Verlag, Berlin-New York, 1975.

[6] A. Borel and H. Jacquet, Automorphic forms and automorphic representations. Proc. Sympos. Pure Math., vol **33**, part I. pp. 189-202.

[7] C. Breuil; B. Conrad; F. Diamond and R. Taylor, On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.* **14** (2001), no. 4, 843–939.

[8] D. Bump, Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics 55. Cambridge University Press, 1997.

[9] P. Cartier, Representations of p–adic groups: a survey. Proc. Sympos. Pure Math., vol **33**, part I. pp. 111-155.

[10] W. Casselman, One some results of Atkin and Lehner, Math. Ann. **201** (1973), pp. 301-313.

[11] Coates, J.; Wiles, A. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.* **39** (1977), no. 3, 223–251.

[12] C. Consani and J. Scholten, Arithmetic on a Quintic treefold, preprint.

[13] J. H. Conway and Sloane, Sphere packings, lattice and groups. Second edition. Springer Verlag, New York, 1993.

[14] Cremona, J. E. *Algorithms for modular elliptic curves.* Second edition. Cambridge University Press, Cambridge, 1997.

[15] H. Darmon, Rigid local systems, Hilbert modular forms, and Fermat's last theorem. *Duke Math. J.* **102** (2000), no. 3, 413–449.

[16] P. Deligne, Formes modulaire et représentations de GL_2. Modular functions on variable II, Lecture Notes in Math., vol. 349, pp. 55-105, Springer-Verlag, 1973.

[17] P. Deligne, La oonjecture de Weil pour les surfaces $K3$. Inventiones math. vol **15** (1972), pp.206-226.

[18] P. Deligne, Théorie de Hodge II et III. Publ. Math. IHES, **40** (1971), pp. 5-58; **44** (1974), pp. 5-77.

[19] P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 247–289, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[20] F. Diamond and J. Im, Modular forms and modular curves. Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994), 39–133, CMS Conf. Proc., **17**, Amer. Math. Soc., Providence, RI, 1995.

[21] B. Edixhoven, On the Manin constants of modular elliptic curves. Arithmetic algebraic geometry (Texel, 1989), 25–39, *Progr. Math.*, **89**, Birkhuser Boston, Boston, MA, 1991.

[22] J. Ellenberg, Serre's conjectures over $\mathbb{F}_9$; preprint.

[23] D. Flath, Decomposition of representations into tensor products. Proc. Sympos. Pure Math., vol **33**, part I. pp. 179-183.

[24] G. van der Geer. *Hilbert modular surfaces.* Springer-Verlag 1988.

[25] S. S. Gelbart, Automorphic forms on adele groups. *Annals of Maths. Studies* **83**, Princeton Univ. Press, 1975.

[26] R. Godement, Notes on Jacquet-Langlands' theory. IAS, 1970.

[27] P. Griffiths and J. Harris, Principles of Algebraic Geometry. *Wiley Classics Library*, John Wiley and Sons, New York, 1994.

[28] B. H. Gross, Heights and the Special values of L-series. CMS conference Proceedings, vol.7 (1987), pp. 115-187.

[29] Gross, Benedict H. Algebraic modular forms. *Israel J. Math.* **113** (1999), 61–93.

[30] Harish-Chandra, Automorphic forms on semisimple Lie groups. Lecture Notes in Math., vol. 62, Springer-Verlag, New York, 1968.

[31] F. Hirzebruch, Hilbert modular surfaces. L'Ens. Math. **19** (1973), pp. 183-281.

[32] H. Jacquet and R. P. Langlands, Automorphic forms on GL(2). Lectures Notes in Math., vol. 114, Springer-Verlag, Berlin and New York, 1970.

[33] Kamal Khuri–Makdisi, On the curves associated to certain rings of automorphic forms. Canad. J. Math. **53** (2001), No. 1, pp. 98–121.

[34] Knapp, Anthony W. *Elliptic curves.* Mathematical Notes, **40**. Princeton University Press, Princeton, NJ, 1992.

[35] Kobayashi, Toshiyuki; Oda, Takayuki A vanishing theorem for modular symbols on locally symmetric spaces. *Comment. Math. Helv.* **73** (1998), no. 1, 45–70.

[36] Langlands, Robert P. *Base change for* GL(2). Annals of Mathematics Studies, **96**. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.

[37] A. Pizer, An algorithm for computing modular forms on $\Gamma_0(N)$. Journal of Algebra **64** (1980), pp. 340–390.

[38] T. Miyake, *Modular forms.* Springer-Verlag, Berlin Heidelberg, 1989.

[39] T. Miyake, On automorphic forms on GL_2 and Hecke operators. *Ann. Math.* **94**, pp. 174–189 (1971).

[40] T. Oda, *Periods of Hilbert Modular Forms.* Progress in Mathematics **19**, Birkhäuser, 1982.

[41] T. Oda, The Riemann-Hodge period relation for Hilbert modular forms of weight 2. Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), 261–286, *Lecture Notes in Math.*, **1447**, Springer, Berlin, 1990.

[42] I. Reiner, Maximal Orders. Academic Press 1975.

[43] Ribet, K. A. On modular representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms. *Invent. Math.* **100** (1990), no. 2, 431–476.

[44] Serre, Jean-Pierre; Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.* **15** (1972), no. 4, 259–331.

[45] Serre, Jean-Pierre; Travaux de Wiles (et Taylor, ...). I. Séminaire Bourbaki, Vol. 1994/95. *Astérisque* No. **237** (1996), Exp. No. 803, 5, 319–332.

[46] Serre, Jean-Pierre; Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. *Duke Math. J.* **54** No. 1, pp. 179–230 (1987).

[47] G. Shimura, The special values of the zeta functions associated to Hilbert modular forms. *Duke Math. J.* **45**, No. 3 (1978), pp. 637-679.

[48] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Princeton Univ. Press 1971.

[49] Silverman, Joseph H. *The arithmetic of elliptic curves.* Graduate Texts in Mathematics, 106. Springer-Verlag, New York, 1986.

[50] Skinner, C. M.; Wiles, A. J. Residually reducible representations and modular forms. *Inst. Hautes tudes Sci. Publ. Math.* No. **89** (1999), 5–126 (2000).

[51] Skinner, C. M.; Wiles, A. J. Nearly ordinary deformations of irreducible residual representations. To appear, Ann. Fac. Sci. Toulouse Math. (6).

[52] J. Tunnell, Artin's conjecture for representations of octahedral type. *Bull. Amer. Math. Soc.* (N.S.) **5** (1981), no. 2, 173–175.

[53] R. Taylor and A. Wiles, Ring theoretic properties of certain Hecke algebras. *Annals of Maths.* **141** (1995), pp. 553–572.

[54] M.–F. Vignéras, Arithmétique des algèbres de quaternions. Lecture Notes in Math., vol. **800**, Springer–Verlag, New York, 1981.

[55] Guého, Marie France Le théorème d'Eichler sur le nombre de classes d'idéaux d'un corps de quaternions totalement défini et la mesure de Tamagawa.

Journées Arithmétiques (Grenoble, 1973), pp. 107–114. *Bull. Soc. Math. France Mem.*, No. **37**, Soc. Math. France, Paris, 1974.

[56] N. R. Wallach, Real reductive group I. *Pure and applied mathematics*, vol. 132, Academic press, 1988.

[57] A. Wiles, Modular elliptic curves and Fermat Last Theorem. *Annals of Maths.* **141** (1995), pp. 443–551.