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Automorphic forms and Galois representations

A p-adic Gross-Zagier formula for Garrett triple product L-functions

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Joint work with Victor Rotger
(+ earlier work with Massimo Bertolini and Kartik Prasanna.)

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The original Gross-Zagier formula

f=eigenform of weight 2 on $\Gamma_0(N)$;

Example: f has rational fourier coefficients, hence corresponds to an elliptic curve $E/\mathbb{Q}$.

K= quadratic imaginary field.

Heegner hypothesis: There is an ideal $\mathfrak{N} \subset \mathcal{O}_K$, with $\mathcal{O}_K/\mathfrak{N} = \mathbb{Z}/N\mathbb{Z}$.

Consequence: the sign in the functional equation of $L(f/K, s)$ is -1, and therefore $L(f/K, 1) = 0$.

BSD conjecture predicts that rank$(E(K)) \geq 1$.

Heegner points

Let $A_1, \ldots, A_h$= elliptic curves with CM by $\mathcal{O}_K$.

The pairs $(A_1, A_1[\mathfrak{N}]), \ldots, (A_h, A_h[\mathfrak{N}])$ correspond to points

$$P_1, \ldots, P_h \in X_0(N)(H).$$

(H=Hilbert class field of K.)

Let $P_K :=$ Image of the divisor

$$P_1 + \cdots + P_h - h(\infty)$$

in $E(K)$.

The Gross-Zagier formula

Theorem (Gross-Zagier)

In the setting above,

$$L'(E/K,1) = C_{E,K} \times \langle P_K, P_K \rangle,$$

where

- $C_{E,K}$ *is an explicit, non-zero "fudge factor";*
- $\langle \ , \ \rangle$ *is the Néron-Tate canonical height.*

In particular, the point P_K *is of infinite order if and only if* $L(E/K,s)$ *has a* simple zero *at* $s=1$.

p-adic analogues

Question: formulate p-adic analogues of the Gross-Zagier theorem, replacing the classical L-function $L(E/K, s)$ by a p-adic avatar.

General framework: Given an L-function like

$$L(E/K, s) = L(V_{E,K}, s), \qquad \text{where } V_{E,K} := H^1_{\text{et}}(E_{\bar{K}}, \mathbb{Q}_p)(1),$$

realise $V_{E,K}$ as a specialisation of a p-adic family of p-adic representations of G_K, and interpolate the (critical) L-values that arise.

p-adic L-functions

One of the charms of the p-adic world is that it affords more room for p-adic variation of a p-adic Galois representation V:

- The family $V(n)$ of cyclotomic twists: the “cyclotomic variable” n corresponds to the variable s in the complex theory;
- The “weight variables” arising in Hida theory. These have no immediate counterpart in the complex setting.

Hida families

$\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \simeq \mathbb{Z}_p[[T]]^{p-1}$: "extended" Iwasawa algebra.

Weight space: $W = \hom(\Lambda, \mathbb{C}_p) \subset \hom(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$.

The integers form a dense subset of W via $k \leftrightarrow (x \mapsto x^k)$.

Classical weights: $W_{\mathrm{cl}} := \mathbb{Z}^{\geq 2} \subset W$.

If $\tilde{\Lambda}$ is a finite extension of Λ, let $\tilde{\mathcal{X}} = \hom(\tilde{\Lambda}, \mathbb{C}_p)$ and let

$$\kappa : \tilde{\mathcal{X}} \longrightarrow W$$

be the natural projection to weight space.

Classical points: $\tilde{\mathcal{X}}_{\mathrm{cl}} := \{x \in \tilde{\mathcal{X}} \text{ such that } \kappa(x) \in W_{\mathrm{cl}}\}$.

Hida families, cont'd

Definition

A *Hida family* of tame level N is a triple $(\Lambda_f, \Omega_f, \underline{f})$, where

1. Λ_f is a finite extension of Λ;
2. $\Omega_f \subset \mathcal{X}_f := \hom(\Lambda_f, \mathbb{C}_p)$ is a non-empty open subset (for the p-adic topology);
3. $\underline{f} = \sum_n \mathbf{a}_n q^n \in \Lambda_f[[q]]$ is a formal q-series, such that $\underline{f}(x) := \sum_n x(\mathbf{a}_n) q^n$ is the q series of the *ordinary p-stabilisation* $f_x^{(p)}$ of a normalised eigenform, denoted f_x, of weight $\kappa(x)$ on $\Gamma_1(N)$, for all $x \in \Omega_{f,\mathrm{cl}} := \Omega_f \cap \mathcal{X}_{f,\mathrm{cl}}$.

Hida's theorem

f = normalised eigenform of weight $k \geq 2$ on $\Gamma_1(N)$.

$p \nmid N$ an ordinary prime for f (i.e., $a_p(f)$ is a p-adic unit).

Theorem (Hida)

There exists a Hida family $(\Lambda_f, \Omega_f, \underline{f})$ and a classical point $x_0 \in \Omega_{f,\mathrm{cl}}$ satisfying

$$\kappa(x_0) = k, \qquad f_{x_0} = f.$$

As x varies over $\Omega_{f,\mathrm{cl}}$, the specialisations f_x give rise to a "p-adically coherent" collection of classical newforms on $\Gamma_1(N)$, and one can hope to construct p-adic L-functions by interpolating classical special values attached to these eigenforms.

Back to Gross-Zagier: Rankin L-functions

Key insight in Gross-Zagier's evaluation of $L(f/K, s)$: it is a *Rankin convolution* L-series:

$$L(f/K, s) = L(f \otimes \theta_K, s),$$

where θ_K is a weight one theta series attached to K.

We obtain p-adic analogues of $L(f \otimes \theta_K, s)$ by considering p-adic L-functions arising from the Hida families $\underline{f}$ and $\underline{\theta}_K$ satisfying

$$f_{x_0} = f, \quad \theta_{K,y_0} = \theta_K, \qquad \text{for some } x_0 \in \Omega_{f,\mathrm{cl}}, \quad y_0 \in \Omega_{\theta,\mathrm{cl}}.$$

p-adic variants of $L(f \otimes \theta_\chi, s)$

Two different p-adic L-functions arise naturally.

1. The first, denoted

$$L_p^f(\underline{f} \otimes \underline{\theta}_K, x, y, s) : \Omega_f \times \Omega_\theta \times W \longrightarrow \mathbb{C}_p,$$

   interpolates the critical values

$$\frac{L(f_x \otimes \theta_y, s)}{*\langle f_x, f_x \rangle} \in \bar{\mathbb{Q}}, \qquad \kappa(y) \leq s \leq \kappa(x) - 1;$$

2. The second, denoted $L_p^\theta(\underline{f} \otimes \underline{\theta}, x, y, s)$, interpolates the critical values

$$\frac{L(f_x \otimes \theta_y, s)}{*\langle \theta_y, \theta_y \rangle}, \qquad \kappa(x) \leq s \leq \kappa(y) - 1.$$

Perrin-Riou's p-adic Gross-Zagier formula

The p-adic L-function $L_p^f(\underline{f} \otimes \underline{\theta}_K, x, y, s)$, evaluated at $(x_0, y_0, 1)$, is equal to a simple multiple of $L(f \otimes \theta_K, 1)$ since $(x_0, y_0, 1)$ lies in the range of classical interpolation defining it.

In the setting of the Gross-Zagier formula, this special value is therefore 0.

Theorem (Perrin-Riou)

$$\frac{d}{ds} L_p^f(\underline{f} \otimes \underline{\theta}_\chi, x_0, y_0, s)_{s=1} = * \times \langle P_K, P_K \rangle_p,$$

where $\langle \ , \ \rangle_p$ is the cyclotomic p-adic height on $E(K)$.

Nekovar: analogue for forms of higher weight.

A second p-adic Gross-Zagier formula

The p-adic L-function $L_p^\theta(\underline{f} \otimes \underline{\theta}_K, x, y, s)$, evaluated at $(x, y, s) = (x_0, y_0, 1)$, is not directly related to the associated classical value, since $(x_0, y_0, 1)$ now lies *outside* the range of classical interpolation.

Theorem (Bertolini-Prasanna-D)

$$L_p^\theta(\underline{f} \otimes \underline{\theta}_K, x_0, y_0, 1) = * \times \log_p^2(P_K),$$

where $\log_p : E(\bar{\mathbb{Q}}_p) \longrightarrow \bar{\mathbb{Q}}_p$ *is the p-adic formal group logarithm.*

Massimo Bertolini, Kartik Prasanna, HD. *Generalised Heegner cycles and p-adic Rankin L-series*, submitted. (`http://www.math.mcgill.ca/darmon/pub/pub.html`)

Diagonal cycles

The Gross-Zagier formula admits a higher dimensional analogue, relating

1. Null homologous codimension 2 *diagonal cycles* in the product of three modular curves;
2. Garrett-Rankin L-functions attached to the convolution of three modular forms.

Goal of the work with Rotger: Prove the counterpart of the p-adic formula of Bertolini-Prasanna-D in this setting.

The Garrett-Rankin triple convolution of eigenforms

Definition

A triple of eigenforms

$$f \in S_k(\Gamma_0(N_f), \varepsilon_f), \quad g \in S_\ell(\Gamma_0(N_g), \varepsilon_g), \quad h \in S_m(\Gamma_0(N_h), \varepsilon_h)$$

is said to be *self-dual* if

$$\varepsilon_f \varepsilon_g \varepsilon_h = 1;$$

in particular, $k + \ell + m$ is even.

A ‘Heegner-type” hypothesis

Triple product L-function $L(f \otimes g \otimes h, s)$ has a functional equation

$$\Lambda(f \otimes g \otimes h, s) = \epsilon(f, g, h)\Lambda(f \otimes g \otimes h, k + \ell + m - 2 - s).$$

$$\epsilon(f, g, h) = \pm 1, \qquad \epsilon(f, g, h) = \prod_{q|N\infty} \epsilon_q(f, g, h).$$

Key assumption: $\epsilon_q(f, g, h) = 1$, for all $q|N$.

This assumption is satisfied when, for example:

- $\gcd(N_f, N_g, N_h) = 1$, or,
- $N_f = N_g = N_h = N$ and $a_p(f)a_p(g)a_p(h) = -1$ for all $p|N$.

Diagonal cycles on triple products of Kuga-Sato varieties.

Hence, for (f, g, h) balanced, $L(f \otimes g \otimes h, c) = 0$. $(c = \frac{k+\ell+m-2}{2})$

$$k = r_1 + 2, \quad \ell = r_2 + 2, \quad m = r_3 + 2, \quad r = \frac{r_1 + r_2 + r_3}{2}.$$

$\mathcal{E}^r(N) = r$-fold Kuga-Sato variety over $X_1(N)$; $\dim = r + 1$.

$$V = \mathcal{E}^{r_1}(N_f) \times \mathcal{E}^{r_2}(N_g) \times \mathcal{E}^{r_3}(N_h), \quad \dim V = 2r + 3.$$

Generalised Gross-Kudla-Schoen cycle: there is an *essentially unique* interesting way of embedding $\mathcal{E}^r(N)$ as a null-homologous cycle in V.

Cf. Rotger, D. Notes for the AWS, Chapter 7.

Definition of $\Delta_{k,\ell,m}$

Let A, B, C be subsets of $\{1, \ldots, r\}$ of sizes r_1, r_2 and r_3, such that each $1 \le i \le r$ belongs to *precisely two* of A, B and C.

$$\mathcal{E}^r \longrightarrow \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3},$$

$$(x, P_1, \ldots, P_r) \mapsto ((x, (P_j)_{j \in A}), (x, (P_j)_{j \in B}), (x, (P_j)_{j \in C})).$$

Fact: If $k, \ell, m > 2$, the image of $\mathcal{E}^r$ is a null-homologous cycle.

$$\Delta_{k,\ell,m} = \mathcal{E}^r \subset V, \quad \Delta \in CH^{r+2}(V).$$

Gross-Kudla-Schoen cycle: $(k, \ell, m) = (2, 2, 2)$:

$$\Delta = X_{123} - X_{12} - X_{13} - X_{23} + X_1 + X_2 + X_3.$$

Diagonal cycles and L-series

Gross-Kudla. The height of the (f, g, h)-isotypic component $\Delta^{f,g,h}$ of the diagonal cycle Δ should be related to the central critical derivative

$$L'(f \otimes g \otimes h, r+2).$$

Work of **Yuan-Zhang-Zhang** represents substantial progress in this direction, when $r_1 = r_2 = r_3 = 0$.

For more general (k, ℓ, m), there are (at present) no such archimedean results in the literature.

p-adic Abel-Jacobi maps

Complex Abel-Jacobi map (**Griffiths, Weil**):

$$\mathrm{AJ} : \mathrm{CH}^{r+2}(V)_0 \longrightarrow \frac{H^{2r+3}_{\mathrm{dR}}(V/\mathbb{C})}{\mathrm{Fil}^{r+2}\, H^{2r+3}_{\mathrm{dR}}(V/\mathbb{C}) + H^{2r+3}_{B}(V(\mathbb{C}),\mathbb{Z})} = \frac{\mathrm{Fil}^{r+2}\, H^{2r+3}_{\mathrm{dR}}(V/\mathbb{C})^{\vee}}{H_{2r+3}(V(\mathbb{C}),\mathbb{Z})}.$$

$$\mathrm{AJ}(\Delta)(\omega) = \int_{\partial^{-1}\Delta} \omega.$$

p-adic Abel-Jacobi map:

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(V/\mathbb{Q}_p)_0 \longrightarrow \mathrm{Fil}^{r+2}\, H^{2r+3}_{\mathrm{dR}}(V/\mathbb{Q}_p)^{\vee}.$$

Goal: relate $\mathrm{AJ}_p(\Delta)$ to Rankin triple product p-adic L-functions,

Triple product p-adic Rankin L-functions

They interpolate the *central critical values*

$$\frac{L(\underline{f}_x \otimes \underline{g}_y \otimes \underline{h}_z, c)}{\Omega(f_x, g_y, h_z)} \in \bar{\mathbb{Q}}.$$

Four *distinct* regions of interpolation in $\Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$:

1. $\Sigma_f = \{(x,y,z) : \kappa(x) \geq \kappa(y) + \kappa(z)\}$. $\Omega = *\langle f_x, f_x \rangle^2$.
2. $\Sigma_g = \{(x,y,z) : \kappa(y) \geq \kappa(x) + \kappa(z)\}$. $\Omega = *\langle g_y, g_y \rangle^2$.
3. $\Sigma_h = \{(x,y,z) : \kappa(z) \geq \kappa(x) + \kappa(y)\}$. $\Omega = *\langle h_z, h_z \rangle^2$.
4. $\Sigma_{\mathrm{bal}} = (\mathbb{Z}^{\geq 2})^3 - \Sigma_f - \Sigma_g - \Sigma_h$.
   $\Omega(f_x, h_y, g_z) = *\langle f_x, f_x \rangle^2 \langle g_y, g_y \rangle^2 \langle h_z, h_z \rangle^2$.

Resulting p-adic L-functions: $L_p^f(\underline{f} \otimes \underline{g} \otimes \underline{h})$, $L_p^g(\underline{f} \otimes \underline{g} \otimes \underline{h})$, and $L_p^h(\underline{f} \otimes \underline{g} \otimes \underline{h})$ respectively.

Garrett's formula

Let (f, g, h) be an unbalanced triple of eigenforms

$$k = \ell + m + 2n, \qquad n \geq 0.$$

Theorem (Garrett, Harris-Kudla)

The central critical value $L(f, g, h, c)$ is a simple multiple of

$$\langle f, g\delta_m^n h\rangle^2, \qquad \textit{where}$$

$$\delta_k = \frac{1}{2\pi i}\left(\frac{d}{d\tau} + \frac{k}{\tau - \bar{\tau}}\right) : S_k(\Gamma_1(N))^! \longrightarrow S_{k+2}(\Gamma_1(N))^!$$

is the Shimura-Maass operator on "nearly holomorphic" modular forms, and

$$\delta_m^n := \delta_{m+2n-2} \cdots \delta_{m+2}\delta_m.$$

The p-adic L-function

Theorem (Hida, Harris-Tilouine)

There exists a (unique) element $\mathscr{L}_p{}^f(\underline{f}, \underline{g}, \underline{h}) \in Frac(\Lambda_f) \otimes \Lambda_g \otimes \Lambda_h$ *such that, for all* $(x, y, z) \in \Sigma_f$*, with* $(k, \ell, m) := (\kappa(x), \kappa(y), \kappa(z))$ *and* $k = \ell + m + 2n$*,*

$$\mathscr{L}_p{}^f(\underline{f}, \underline{g}, \underline{h})(x, y, z) = \frac{\mathscr{E}(f_x, g_y, h_z)}{\mathscr{E}(f_x)} \frac{\langle f_x, g_y \delta_m^n h_z \rangle}{\langle f_x, f_x \rangle},$$

where, after setting $c = \frac{k+\ell+m-2}{2}$*,*

$$\begin{aligned} \mathscr{E}(f_x, g_y, h_z) &:= \left(1 - \beta_{f_x}\alpha_{g_y}\alpha_{h_z}p^{-c}\right) \times \left(1 - \beta_{f_x}\alpha_{g_y}\beta_{h_z}p^{-c}\right) \\ &\quad \times \left(1 - \beta_{f_x}\beta_{g_y}\alpha_{h_z}p^{-c}\right) \times \left(1 - \beta_{f_x}\beta_{g_y}\beta_{h_z}p^{-c}\right), \\ \mathscr{E}(f_x) &:= \left(1 - \beta_{f_x}^2 p^{-k}\right) \times \left(1 - \beta_{f_x}^2 p^{1-k}\right). \end{aligned}$$

More notations

$$\omega_f = (2\pi i)^{r_1+1} f(\tau) dw_1 \cdots dw_{r_1} d\tau \in \mathrm{Fil}^{r_1+1} H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1}).$$

$\eta_f \in H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1}/\bar{\mathbb{Q}}_p)$ = representative of the f-isotypic part on which Frobenius acts as a p-adic unit, normalised so that

$$\langle \omega_f, \eta_f \rangle = 1.$$

Lemma

If (k, ℓ, m) is balanced, then the (f_k, g_ℓ, h_m)-isotypic part of the $\bar{\mathbb{Q}}_p$ vector space $\mathrm{Fil}^{r+2} H^{2r+2}_{\mathrm{dR}}(V/\bar{\mathbb{Q}}_p)$ is generated by the classes of

$$\omega_{f_k} \otimes \omega_{g_\ell} \otimes \omega_{h_m}, \quad \eta_{f_k} \otimes \omega_{g_\ell} \otimes \omega_{h_m}, \quad \omega_{f_k} \otimes \eta_{g_\ell} \otimes \omega_{h_m}, \quad \omega_{f_k} \otimes \omega_{g_\ell} \otimes \eta_{h_m}.$$

A p-adic Gross-Kudla formula

Given $(x_0, y_0, z_0) \in \Sigma_{\mathrm{bal}}$, write $(f, g, h) = (f_{x_0}, g_{y_0}, h_{z_0})$, and $(k, \ell, m) = (\kappa(x_0), \kappa(y_0), \kappa(z_0))$.

Recall that $\mathrm{sign}(L(f \otimes g \otimes h, s)) = -1$, hence $L(f \otimes g \otimes h, c) = 0$.

Theorem (Rotger-D)

$$\mathscr{L}_p{}^f(\underline{f} \otimes \underline{g} \otimes \underline{h}, x_0, y_0, z_0) = \frac{\mathscr{E}(f, g, h)}{\mathscr{E}(f)} \times \mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta_f \otimes \omega_g \otimes \omega_h),$$

and likewise for $\mathscr{L}_p{}^g$ *and* $\mathscr{L}_p{}^h$.

Conclusion: The Abel-Jacobi image of $\Delta_{k,\ell,m}$ encodes the special values of the *three distinct* p-adic L-functions attached to $(\underline{f}, \underline{g}, \underline{h})$ at the points in Σ_{bal}.

A few words on the proof

Assume $(k, \ell, m) = (2, 2, 2)$, $\quad N_f = N_g = N_h = N$.

Step 1. A formula for $\mathrm{AJ}_p(\Delta)(\eta_f \otimes \omega_g \otimes \omega_h)$.

$$\mathcal{A} \subset X_0(N)(\mathbb{C}_p) = \text{ ordinary locus;}$$

$$\mathcal{W}_\epsilon = \text{ "wide open neighbourhood" of } \mathcal{A}, \quad \epsilon > 0.$$

$$\mathcal{A} \subset \mathcal{W}_\epsilon \subset X_0(N)(\mathbb{C}_p).$$

The cohomology of X over $\mathbb{C}_p$

$$H^1_{\mathrm{dR}}(X/K) = \frac{\Omega^1_{\mathrm{mer}}(X/K)^{II}}{dK(X)};$$

Fact: Restriction induces an isomorphism

$$H^1_{\mathrm{dR}}(X/\mathbb{C}_p) \longrightarrow \frac{\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/\mathbb{C}_p)^{II}}{d\mathcal{O}_{\mathcal{W}_\epsilon}}.$$

Action of Frobenius on H^1_{dR}: "canonical" lift of Frobenius

$$\Phi : \Omega^1(\mathcal{W}_\epsilon) \longrightarrow \Omega^1(\mathcal{W}_{\epsilon/p}).$$

The recipe for $\mathrm{AJ}_p(\Delta)$

This builds on ideas arising in **Coleman**'s *p*-adic integration theory.

$$\Phi = \Phi_1\Phi_2 = \text{Frobenius on } \mathcal{W}_\epsilon \times \mathcal{W}_\epsilon \subset X \times X.$$

There exists a polynomial P such that

$$P(\Phi)([\omega_f \otimes \omega_h]) = 0,$$

hence there exists $\xi_{g,h,P} \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$, satisfying

$$d\xi_{g,h,P} = P(\Phi)(\omega_g \otimes \omega_h),$$

which is well-defined up to closed forms in $\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$.

$$\delta : X \hookrightarrow X \times X, \quad \delta_1 : X \hookrightarrow X \times \{\infty\} \subset X \times X, \quad \delta_2 : X \hookrightarrow \{\infty\} \times X.$$

$$\rho_{g,h,P} := \delta^*\xi_{g,hP} - \delta_1^*\xi_{g,h,P} - \delta_2^*\xi_{g,h,P}$$

is an element of $\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$, which is well-defined modulo *exact* one-forms.

The recipe for $\mathrm{AJ}_p(\Delta)$, cont'd

Suppose that $\Phi(\eta_f) = \alpha\eta_f$, and let $\beta = p/\alpha$.

Main formula:

$$\mathrm{AJ}_p(\Delta)(\eta_f \otimes \omega_g \otimes \omega_h) = \frac{1}{P(\beta)} \langle \eta_f, \rho_{g,h,P} \rangle.$$

Remarks

1. Can assume all roots of P are Weil numbers of weight 2, hence $P(\beta) \neq 0$.

2. The final result does not depend on P.

Explicit calculation

$$P(x) = (x - \alpha_g\alpha_h)(x - \alpha_g\beta_h)(x - \beta_g\alpha_h)(x - \beta_g\beta_h)$$

To solve

$$d\xi = P(\Phi_1\Phi_2)\omega_g\omega_h,$$

we use

$$P(\Phi_1\Phi_2) = A(\Phi_1, \Phi_2)(\Phi_1 - \alpha_g)(\Phi_1 - \beta_g) + B(\Phi_1, \Phi_2)(\Phi_2 - \alpha_h)(\Phi_2 - \beta_h),$$

which implies that

$$P(\Phi_1\Phi_2)(\omega_g\omega_h) = A(\Phi_1, \Phi_2)\omega_g^{[p]}\omega_h + B(\Phi_1, \Phi_2)\omega_g\omega_h^{[p]},$$

where

$$\omega_g^{[p]} = \sum_{p\nmid n} a_n q^n \frac{dq}{q},$$

Explicit calculation

$$\xi_{g,h,P} = A(\Phi_1, \Phi_2)G\omega_h + B(\Phi_1, \Phi_2)\omega_g H,$$

where

$$G = \sum_{p \nmid n} \frac{a_n}{n} q^n, \quad H = \sum_{p \nmid n} \frac{b_n}{n} q^n.$$

$G, H = p$-adic (overconvergent) modular forms of weight 0.

$$\mathrm{AJ}_p(\Delta)(\eta_f \omega_g \omega_h) = \langle \eta_f, A(\Phi_1, \Phi_2)G\omega_h \rangle + \langle \eta_f, B(\Phi_1, \Phi_2)\omega_g H)$$
$$= \Xi(f, g, h)\langle \eta_f, G\omega_h \rangle$$

Where $\Xi(f, g, h)$ is a (a priori complicated!) polynomial in $a_p(f), a_p(g), a_p(h)$. This follows from a *tedious*, but elementary, calculation.

End of the proof

$$\begin{aligned}
\langle \eta_f, G\omega_h \rangle &= \langle \eta_f, d^{-1}\omega_g^{[p]}\omega_h \rangle \\
&= \lim_{x \to x_0} \langle \eta_{f_x^{(p)}}, d^{\frac{\kappa(x)-4}{2}} \omega_g^{[p]}\omega_h \rangle \\
&= \lim_{x \to x_0} \langle \eta_{f_x^{(p)}}, e(d^{\frac{\kappa(x)-4}{2}} \omega_g^{[p]}\omega_h) \rangle \\
&= \lim_{x \to x_0} \langle \overline{f_x^{(p)}}, (\delta^{\frac{\kappa(x)-4}{2}} \omega_g^{[p]}\omega_h) \rangle \langle \bar{f}_x^{(p)}, f_x^{(p)} \rangle^{-1} \\
&= \lim_{x \to x_0} \mathcal{E}(f_x, g, h) \langle \bar{f}_x, \delta^{\frac{\kappa(x)-4}{2}} \omega_g \omega_h) \rangle ||f_x||^{-1} \\
&= \lim_{x \to x_0} \mathscr{L}_p^f(\underline{f}, \underline{g}, \underline{h})(x, y_0, z_0) \\
&= \mathscr{L}_p^f(\underline{f}, \underline{g}, \underline{h})(x_0, y_0, z_0).
\end{aligned}$$

p-adic heights and derivatives of L-series

One could also envisage a *fourth* type of p-adic L-series

$$L_p^{\mathrm{bal}}(\underline{f} \otimes \underline{g} \otimes \underline{h}) : \Omega_f \times \Omega_g \times \Omega_h \times W \longrightarrow \mathbb{C}_p,$$

interpolating

$$L(f_x, g_y, h_z, s), \qquad (x, y, z) \in \Sigma_{\mathrm{bal}}, \qquad 1 \leq s \leq \kappa(x)+\kappa(y)+\kappa(z)-3.$$

(But not their square roots...)

This p-adic L-function is not (to our knowledge) available in the literature.

p-adic heights and derivatives of L-series, cont'd

Under the hypothesis

$$\epsilon_q(f, g, h) = 1, \qquad \text{for all } q|N$$

that was imposed on the local signs, we see

$$L_p^{\text{bal}}(\underline{f}, \underline{g}, \underline{h})(x, y, z, c) = 0, \qquad \text{for all } (x, y, z) \in \Sigma_{\text{bal}},$$

because $L(f_x, g_y, h_z, c) = 0$.

Expectation:

$$\frac{d}{ds} L_p^{\text{bal}}(\underline{f}, \underline{g}, \underline{h})(x, y, z, s)_{s=c} \stackrel{?}{=} * \times \text{ht}_p(\Delta_{f_x, g_y, h_z}).$$

p-adic heights and derivatives of L-series, cont'd

The just-alluded to formula for $\frac{d}{ds}L_p^{\mathrm{bal}}(\underline{f},\underline{g},\underline{h})(x,y,z,c)$ would be a "more direct"

1. p-adic counterpart of Gross-Kudla/Yuan-Zhang-Zhang,
2. "diagonal cycles" counterpart of Perrin-Riou/Nekovar's p-adic Gross-Zagier formulae.

Final comments

The p-adic Gross-Zagier formula for $\mathscr{L}_p^f(\underline{f}, \underline{g}, \underline{h})$, $\mathscr{L}_p^g(\underline{f}, \underline{g}, \underline{h})$ and $\mathscr{L}_p^h(\underline{f}, \underline{g}, \underline{h})$

1. admits proofs that are relatively simple;
2. seems well-adapted to studying the Euler system properties of p-adic families of diagonal cycles.

Thank you for your attention.