ARTICLE IN PRESS YJNTH:6686

Journal of Number Theory ••• (••••) •••–•••

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Journal of Number Theory

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JNT Prime

# Arithmetic intersections of modular geodesics

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ARTICLE INFO

*Article history:*
Received 4 June 2020
Received in revised form 4 December 2020
Accepted 14 December 2020
Available online xxxx
Communicated by L. Smajlovic

*MSC:*
11G18
14G35

*Keywords:*
Explicit class field theory
Real quadratic fields
Quaternion algebras
Shimura curves

ABSTRACT

The *arithmetic*, *p-arithmetic*, and *incoherent* intersections between pairs of closed geodesics on a modular or Shimura curve are defined, and some of their expected algebraicity and factorisation properties are examined. These properties follow in a special case from the conjectures of [DV] on the RM values of rigid meromorphic cocycles, and are inspired from the recent work of James Rickards [Ri] and Xavier Guitart, Marc Masdeu and Xavier Xarles [GMX] in the quaternionic setting.

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https://doi.org/10.1016/j.jnt.2020.12.012
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Please cite this article in press as: H. Darmon, J. Vonk, Arithmetic intersections of modular geodesics, J. Number Theory (2021), https://doi.org/10.1016/j.jnt.2020.12.012

## 0. Introduction

Let $\mathscr{H}$ denote the Poincaré upper half plane endowed with its usual action of $\mathrm{SL}_2(\mathbb{Z})$ by Möbius transformations. The arithmetic quotient

$$X := \mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H} = \mathrm{Spec}(\mathbb{C}[j])$$

is equipped with a rich collection of cycles indexed by primitive integral binary quadratic forms

$$F(x,y) = ax^2 + bxy + cy^2 = a(x-\tau y)(x-\tau' y), \qquad \tau \in \mathscr{H} \cup \mathbb{R} \cup \{\infty\}.$$

When $F$ has negative discriminant $D := b^2 - 4ac$, its root $\tau \in \mathscr{H}$ is a CM point whose associated $j$-value generates an abelian extension of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. When $F$ has positive discriminant, it gives rise to the oriented geodesic on $\mathscr{H}$ going from $\tau$ to $\tau'$, where

$$\tau = \frac{-b+\sqrt{D}}{2a}, \qquad \tau' = \frac{-b-\sqrt{D}}{2a}, \qquad \sqrt{D} > 0.$$

This open geodesic maps to a closed *modular geodesic* on $X$, which is of real dimension one. Since it is not an algebraic cycle on $X$, its relevance to the generation of class fields of $\mathbb{Q}(\sqrt{D})$ is less immediately apparent.

The goal of this note is to propose an *arithmetic intersection theory* for modular geodesics, attaching to a pair of such geodesics certain numerical invariants that are rich enough to (ostensibly) generate class fields of real quadratic fields. The predicted algebraicity of these quantities is a by-product of the approach of [DV] to explicit class field theory based on the RM values of rigid meromorphic cocycles, but avoids the latter notion and offers a somewhat complementary perspective. The authors hope that this perspective might appeal to certain readers and, by casting a different light on the conjectures of [DV], make them more amenable to other types of generalisations.

**Topological intersections**. Let $\gamma_1 = (\tau_1, \tau_1')$ and $\gamma_2 = (\tau_2, \tau_2')$ be two distinct geodesics on $\mathscr{H}$ attached to a pair of indefinite binary quadratic forms $F_1$ and $F_2$. The fixing of an orientation on $\mathscr{H}$ determines the signed *topological intersection* of $\gamma_1$ and $\gamma_2$, defined as

$$(\gamma_1 \cdot \gamma_2) := \begin{cases} 0 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ do not intersect;} \\ 1 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ intersect positively;} \\ -1 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ intersect negatively,} \end{cases}$$

where the orientation conventions are illustrated in Fig. 1.

If $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is any congruence subgroup, then the stabiliser subgroups $\Gamma_1$ and $\Gamma_2$ of $\tau_1$ and $\tau_2$ are infinite cyclic modulo torsion. Let $\Sigma$ and $\Sigma_{12}$ be the double coset spaces

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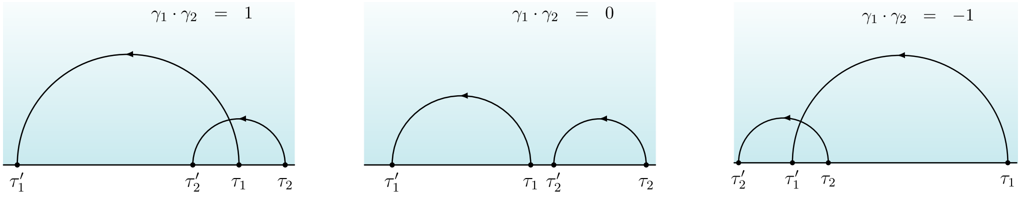


**Fig. 1.** The topological intersection number $\gamma_1 \cdot \gamma_2$.

$$\Sigma := \Gamma \backslash (\Gamma \times \Gamma)/(\Gamma_1 \times \Gamma_2), \qquad \Sigma_{12} := \Gamma_1 \backslash \Gamma / \Gamma_2,$$

which are in bijection via the map $(g_1, g_2) \mapsto g_1^{-1} g_2$. The sum

$$(\gamma_1 \cdot \gamma_2)_\Gamma \quad := \sum_{(g_1,g_2) \in \Sigma} (g_1\gamma_1 \cdot g_2\gamma_2) = \sum_{g \in \Sigma_{12}} (\gamma_1 \cdot g\gamma_2) \tag{1}$$

represents the topological intersection of the oriented closed geodesics arising from the image of $\gamma_1$ and $\gamma_2$ in the quotient Riemann surface $X_\Gamma := \Gamma\backslash\mathscr{H}$. (In particular, this quantity vanishes when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, since $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}$ has genus zero.) Although the index set $\Sigma_{12}$ is infinite, the sums on the right of (1) involve only finitely many non-zero terms. Basic facts about modular geodesics and their topological intersections, when $\Gamma$ is any discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathscr{H}$ with compact or finite volume quotient, are recalled in Section 1.

**Arithmetic intersections**. The cross-ratio

$$(\gamma_1 \; ; \; \gamma_2) := (\tau_1, \tau_1' \; ; \; \tau_2, \tau_2') := \frac{(\tau_1 - \tau_2)(\tau_1' - \tau_2')}{(\tau_1 - \tau_2')(\tau_1' - \tau_2)} \in \mathbb{R}^\times$$

attached to $\gamma_1 = (\tau_1, \tau_1')$ and $\gamma_2 = (\tau_2, \tau_2')$ is negative if and only if $\gamma_1 \cdot \gamma_2 \neq 0$. Its logarithm can be envisaged as an arithmetic intersection between the degree zero divisors $(\tau_1) - (\tau_1')$ and $(\tau_2) - (\tau_2')$. Combining this quantity with the topological intersection of modular geodesics leads to the "multiplicative arithmetic intersection"

$$(\gamma_1 \star \gamma_2) := (\tau_1, \tau_1' \; ; \; \tau_2, \tau_2')^{(\gamma_1 \cdot \gamma_2)} \tag{2}$$

of $\gamma_1$ and $\gamma_2$. Replacing topological by arithmetic intersections in (1) yields a quantity

$$(\gamma_1 \star \gamma_2)_\Gamma \quad := \prod_{(g_1,g_2) \in \Sigma} (g_1\gamma_1 \star g_2\gamma_2) = \prod_{g \in \Sigma_{12}} (\gamma_1 \star g\gamma_2), \tag{3}$$

which can be viewed as an arithmetic (multiplicative) variant of the topological intersection of modular geodesics on the quotient $\Gamma\backslash\mathscr{H}$. When $\gamma_1$ and $\gamma_2$ are not $\Gamma$-equivalent, it belongs to $\mathbb{Q}(\tau_1, \tau_2)^\times$, since the product in (3) is finite.

Section 2 associates an arithmetic intersection to any pair of distinct closed geodesics in $\Gamma\backslash\mathscr{H}$, when $\Gamma$ is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$ arising from the multiplicative group of an order in an indefinite quaternion algebra over $\mathbb{Q}$.

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**$p$-Arithmetic intersections**. A richer class of numerical invariants is obtained by choosing a prime $p$ and replacing the arithmetic group $\Gamma$ by a *$p$-arithmetic* counterpart $\Gamma_p$. The simplest instance, which was treated at length in [DV], is to replace $\mathrm{SL}_2(\mathbb{Z})$ by Ihara's $p$-modular group

$$\Gamma_p := \mathrm{SL}_2(\mathbb{Z}[1/p]).$$

Such a $\Gamma_p$ operates on the set of modular geodesics on $\mathscr{H}$. Letting

$$\Gamma_1^p := \mathrm{Stab}_{\Gamma_p}(\tau_1), \quad \Gamma_2^p := \mathrm{Stab}_{\Gamma_p}(\tau_2), \qquad \Sigma_{12}^p := \Gamma_1^p \setminus \Gamma_p \,/\, \Gamma_2^p,$$

the formal product

$$(\gamma_1 \star \gamma_2)_{\Gamma_p} := \prod_{g \in \Sigma_{12}^p} (\gamma_1 \star g\gamma_2) \tag{4}$$

now involves *infinitely many* non-trivial factors.

Assume that the geodesics $\gamma_1$ and $\gamma_2$ are inequivalent under $\Gamma_p$. More strongly, it will be assumed that their associated discriminants $D_1$ and $D_2$ are fundamental and satisfy

$$\gcd(D_1, D_2) = 1, \qquad \left(\frac{D_1}{p}\right) = \left(\frac{D_2}{p}\right) = -1.$$

The terms that appear (4) all belong to $\mathbb{Q}(\sqrt{D_1 D_2})^\times$, and can be viewed as elements of $\mathbb{Q}_p^\times$ after choosing a $p$-adic square root of $D_1 D_2$ in this field. The first main result of this note is:

**Theorem 1.** *The infinite product in* (4) *converges absolutely $p$-adically.*

This theorem is proved in Section 3 as a special case of a more general result involving $p$-arithmetic subgroups of indefinite quaternion algebras over $\mathbb{Q}$. The $p$-adic number $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ is called the *$p$-arithmetic intersection number* of the geodesics $\gamma_1$ and $\gamma_2$.

The main interest of the $p$-arithmetic intersection number lies in its relevance to explicit class field theory for real quadratic fields. Let $H_1$ and $H_2$ be the narrow ring class fields attached to the discriminants $D_1$ and $D_2$ respectively, and let $H_{12}$ denote their compositum, viewed as a subfield of $\bar{\mathbb{Q}}_p$ after fixing a $p$-adic embedding of this field.

Section 3 formulates a general conjecture on the algebraicity of $p$-arithmetic intersections, and Section 4 explains how it follows from the conjectures of [DV] in the following special case:

**Conjecture 2.** *If $p = 2, 3, 5, 7,$ or 13, i.e., if the modular curve $X_0(p)$ has genus zero, then $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ belongs to $H_{12}$.*

For example, let $\tau_1 = (1+\sqrt{5})/2$ be the golden ratio, and let $\tau_2 = (15+\sqrt{321})/8$ be a real quadratic irrationality of discriminant $321 = 3\cdot 107$. The discriminants 5 and 321 have narrow class numbers 1 and 6 respectively, and the primitive integral binary quadratic form with root $\tau_2$ generates the narrow class group of discriminant 321. The smallest prime $p$ for which $(\frac{5}{p}) = (\frac{321}{p}) = -1$ is $p = 7$, and Conjecture 2 asserts the algebraicity of the 7-arithmetic intersection of $\gamma_1 = (\tau_1, \tau_1')$ and $\gamma_2 = (\tau_2, \tau_2')$. The quantity $(\gamma_1 \star \gamma_2)_{\Gamma_7}$ was computed to 300 digits of 7-adic precision, and a rational recognition algorithm suggests that it is the square of a quantity satisfying the palindromic polynomial

$$g(t) = 7881253325449 \cdot t^{12} + a_{11}t^{11} + \ldots + a_1 t \ + 7881253325449, \tag{5}$$

whose other coefficients are listed below:

| $n$ | $a_n = a_{12-n}$ | $n$ | $a_n = a_{12-n}$ | $n$ | $a_n = a_{12-n}$ |
|---|---|---|---|---|---|
| 1 | 16711393316898 | 2 | 15580968918207 | 3 | 1349942214176 |
| 4 | −8232873610095 | 5 | −7408349176266 | 6 | −4016169195897 |

The fact that the splitting field of $g(t)$ is contained in the compositum of $\mathbb{Q}(\sqrt{5})$ and the narrow Hilbert class field of $\mathbb{Q}(\sqrt{321})$ provides convincing evidence for Conjecture 2.

Experiments like this indicate that $p$-arithmetic intersection numbers are typically non-trivial. Theoretical insights into how often this happens can be obtained by seeking to understand the prime factorisations of $p$-arithmetic intersection numbers, a theme which is touched on in Section 6.

For instance, the constant coefficient in (5) factors as

$$a_0 = 7881253325449 = 7^4 \cdot 23^2 \cdot 47^2 \cdot 53^2.$$

It is no coincidence that the primes that occur in this factorisation are inert in both $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{321})$, and arise among the factors of the quantity $(5\cdot 321 - t^2)/(4\cdot 7)$ when it is an integer, which are listed in the table below:

| $t$ | $(5\cdot 321 - t^2)/(4\cdot 7)$ | $t$ | $(5\cdot 321 - t^2)/(4\cdot 7)$ |
|---|---|---|---|
| 3 | $3\cdot 19$ | 11 | 53 |
| 17 | 47 | 25 | $5\cdot 7$ |
| 31 | 23 | 39 | 3 |

This suggests that $p$-arithmetic intersection numbers ought to admit explicit factorisations just like those obtained by Gross and Zagier [GZ] for differences of singular moduli.

Let $\mathfrak{m}_p$ be a prime of $\mathbb{Q}(\sqrt{D_1 D_2})$ above $p$. Since $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ is defined $p$-adically, its $\mathfrak{m}_p$-adic valuation can be understood directly by elementary manipulations, of the kind used to prove [DV, Thm. 3.26]. These arguments prove the identity

$$\mathrm{ord}_{\mathfrak{m}_p}((\gamma_1 \star \gamma_2)_{\Gamma_p}) = \mathrm{ord}_{\mathfrak{m}_p}((\gamma_1 \star \gamma_2)_{\Gamma}).$$

The $q$-adic valuation of the $p$-arithmetic intersection when $q \neq p$ lies deeper, and is ostensibly related to an arithmetic intersection on a discrete subgroup of a suitable indefinite quaternion algebra. More precisely, let $B(pq)$ be the indefinite quaternion algebra ramified at $p$ and $q$, and let $R(pq)$ be a maximal order in $B(pq)$, which is unique up to conjugation. After fixing an identification of $B(pq) \otimes \mathbb{R}$ with $M_2(\mathbb{R})$, the group $\Gamma(pq) := R(pq)_1^\times$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. If the primes $p$ and $q$ are both non-split in $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$, Section 6 explains how the geodesics $\gamma_1$ and $\gamma_2$, or their hyperbolic conjugacy classes, can be transferred to similar objects $\gamma_1^\flat, \gamma_2^\flat$ for $\Gamma(pq)$, depending on the choice of primes $\mathfrak{m}_p$ and $\mathfrak{m}_q$ of $\mathbb{Q}(\sqrt{D_1 D_2})$ above $p$ and $q$ respectively. One can then consider the arithmetic intersection number

$$(\gamma_1^\flat \star \gamma_2^\flat)_{\Gamma(pq)} \in \mathbb{Q}(\sqrt{D_1 D_2})^\times.$$

**Conjecture 3.** *Assume as before that $\Gamma_p = \mathrm{SL}_2(\mathbb{Z}[1/p])$ with $p = 2, 3, 5, 7,$ or $13$, so that $H^2(\Gamma, \mathbb{Q}) = 0$. If at least one of $D_1$ or $D_2$ is a non-zero square modulo $p$ or $q$, then*

$$\mathrm{ord}_{\mathfrak{Q}}\left((\gamma_1 \star \gamma_2)_{\Gamma_p}\right) = 0,$$

*for all primes $\mathfrak{Q}$ of $H_{12}$ above $\mathfrak{m}_q$. Otherwise, $\mathfrak{Q}$ can be chosen so that*

$$\mathrm{ord}_{\mathfrak{Q}}\left((\gamma_1 \star \gamma_2)_{\Gamma_p}\right) = \mathrm{ord}_{\mathfrak{m}_q}\left((\gamma_1^\flat \star \gamma_2^\flat)_{\Gamma(pq)}\right).$$

An extension of Conjecture 3 to the setting of general primes $p$ and general $p$-arithmetic groups arising from quaternion algebras over $\mathbb{Q}$ is described in Section 6, where it is formulated as a relation between arithmetic intersection numbers and certain conjectural "incoherent intersection numbers" between compatible systems of geodesics on an "incoherent collection" of Shimura curves.

## 1. Topological intersections

The topological intersections described in the introduction extend to more general discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, such as those arising from indefinite quaternion algebras over $\mathbb{Q}$, and the article will place itself in this more general setting throughout. This is not merely done for the sake of extra generality. Arithmetic intersections on quaternion algebras arise naturally in the factorisation of $p$-arithmetic intersections on $\mathrm{SL}_2(\mathbb{Z})$ predicted by Conjecture 3 of the Introduction, and formulating the theory for all quaternion algebras at once leads to a richer, more coherent picture.

Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$, viewed as s subring of $M_2(\mathbb{R})$ by fixing an identification $B \otimes \mathbb{R} = M_2(\mathbb{R})$. The set $S$ of rational primes $p$ for which $B \otimes \mathbb{Q}_p$ is a division ring is of finite, even cardinality. The $\mathbb{Q}$-vector space $B_0$ of elements of $B$ of trace zero, equipped with the trace form

$$\langle b_1, b_2 \rangle = \frac{1}{2}\mathrm{Trace}(b_1 b_2), \quad \text{for } b_1, b_2 \in B_0,$$

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is a non-degenerate quadratic space over $\mathbb{Q}$ of real signature $(2,1)$, on which the group $B^\times$ acts isometrically by conjugation. Let $R$ be a maximal order in $B$, which is unique up to conjugation by $B^\times$. The group $\Gamma := R_1^\times$ of elements of reduced norm one preserves the lattice $R_0 := R \cap B_0$. It also acts discretely on $\mathscr{H}$ by Möbius transformations via the chosen inclusion of $B$ in $M_2(\mathbb{R})$. The quotient $\Gamma\backslash\mathscr{H}$ has finite volume, and is even a compact Riemann surface when $B \neq M_2(\mathbb{Q})$.

Let $\tilde{\gamma} = (\tau, \tau')$ be the open geodesic on $\mathscr{H}$ joining the endpoints $\tau, \tau' \in \mathbb{R}$, and suppose that it maps onto a closed geodesic $\gamma$ on $\Gamma\backslash\mathscr{H}$. Let $L_\tau$ and $L_{\tau'}$ be the lines in $\mathbb{R}^2$ spanned by the column vectors $\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \tau' \\ 1 \end{pmatrix}$. The ring

$$\mathcal{O}_\gamma := \{a \in R \subset M_2(\mathbb{R}) \text{ such that } a \text{ preserves } L_\tau \text{ and } L_{\tau'}\ \} \tag{6}$$

is isomorphic to a real quadratic order and is equipped with a canonical map $\mathcal{O}_\gamma \hookrightarrow \mathbb{R}\oplus\mathbb{R}$ sending $a \in \mathcal{O}_\gamma$ to its eigenvalues on the two eigen-lines $L_\tau$ and $L'_\tau$. The discriminant of $\mathcal{O}_\gamma$ is called the *discriminant* of the geodesic $\gamma$. The primes $q \in S$ are non-split in $\mathcal{O}_\gamma$, since $\mathbb{Q}_q \times \mathbb{Q}_q$ is not a subring of $B \otimes \mathbb{Q}_q$ when $q \in S$. It will be frequently assumed, in order to lighten the exposition, that the $q \in S$ are *inert*, i.e., the ring $\mathcal{O}_\gamma/q$ is isomorphic to the field $\mathbb{F}_{q^2}$ with $q^2$ elements.

For each $q \in S$, the set of elements whose norm is divisible by $q$ is a maximal ideal in $R$, whose associated quotient is $\mathbb{F}_{q^2}$. After choosing a reduction map $\nu_q : R \longrightarrow \mathbb{F}_{q^2}$ for each $q \in S$, the collection

$$\eta_q(\gamma) := \nu_q(\sqrt{D}) \in \mathbb{F}_{q^2}^\times$$

where $\sqrt{D} \in \mathcal{O}_\gamma$ is the "positive" square root of $D$ relative to the chosen real embedding, is an invariant of $\gamma$, called the *orientation* of $\gamma$ at $q \in S$. After fixing, for each $q \in S$, an element $\delta_q \in \mathbb{F}_{q^2}^\times$ satisfying $\delta_q^2 = D$, we can then consider the set $\Pi_D$ of geodesics of discriminant $D$ in $\Gamma\backslash\mathscr{H}$ having orientation $\delta_q$ at $q$, for each $q \in S$. The group $\Gamma$ acts naturally on $\Pi_D$.

**Lemma 4.** *The quotient* $\Gamma\backslash\Pi_D$ *has cardinality* $h(D) := \#\mathrm{Pic}^+(\mathcal{O}_D)$.

**Sketch of proof.** The set $\Pi_D$ is in bijection with the set of oriented optimal embeddings of $\mathcal{O}_D$ into $R$. When $D$ is a negative discriminant and $R$ is a maximal order in a definite quaternion algebra, the set of such embeddings is endowed with a simply transitive action of the class group of $\mathcal{O}_D$, as described in [Gr1, §3]. These definitions adapt to our situation, with the following differences:

(1) Since $B$ is an indefinite quaternion algebra, it satisfies the so-called *Eichler condition* and there is a single $B^\times$ conjugacy class of maximal orders in $B$. Hence any embedding of $\mathcal{O}_D$ into some maximal order can be conjugated into an embedding in $R$.

(2) The orientation on a geodesic is reversed when it is translated by a principal ideal of $\mathcal{O}_D$ having a generator of negative norm, and hence it is the class group in the narrow sense that acts simply transitively on $\Gamma\backslash\Pi_D$. □

Let $\gamma_1$ and $\gamma_2$ be two geodesics on $\Gamma\backslash\mathscr{H}$ attached to $(\tau_1, \tau_1')$ and $(\tau_2, \tau_2')$. The topological intersection of $\gamma_1$ and $\gamma_2$ on $\Gamma\backslash\mathscr{H}$ is given by formula (1) of the introduction:

$$(\gamma_1 \cdot \gamma_2)_\Gamma = \sum_{g \in \Sigma_{12}} \gamma_1 \cdot g\gamma_2. \tag{7}$$

The topological interpretation of this sum reveals that it involves only finitely many non-zero terms.

For each $q \in S$, the pair $(\gamma_1, \gamma_2)$ of geodesics gives rise to a canonical invariant

$$\eta_q(\gamma_1, \gamma_2) := \eta_q(\gamma_1)\eta_q(\gamma_2) \in \mathbb{F}_q^\times.$$

Unlike the invariants $\eta_q(\gamma_1)$, it does not depend on the choice of the reduction map $\nu_q : R \longrightarrow \mathbb{F}_{q^2}$ that was made in order to define it. It determines a distinguished ideal of $\mathbb{Q}(\sqrt{D_1D_2})$ above each prime $q \in S$. Note that these primes are split since they are assumed to be inert in both $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$.

## 2. Arithmetic intersections

The definition of the arithmetic intersection rests on the following explicit expression for the cross-ratio attached to the pair $(\gamma_1, \gamma_2)$ of modular geodesics of relatively prime discriminants $D_1$ and $D_2$. Let $b_1, b_2 \in R_0$ denote the images of the positive square roots $\sqrt{D_1} \in \mathcal{O}_{\gamma_1}$ and $\sqrt{D_2} \in \mathcal{O}_{\gamma_2}$ respectively.

**Lemma 5.** *The cross-ratio attached to $(\gamma_1, \gamma_2)$ is given by*

$$(\gamma_1 \; ; \; \gamma_2) := (\tau_1, \tau_1' \; ; \; \tau_2, \tau_2') = \frac{\langle b_1, b_2\rangle + \sqrt{D_1D_2}}{\langle b_1, b_2\rangle - \sqrt{D_1D_2}}.$$

*The two geodesics intersect if and only if $\langle b_1, b_2\rangle^2 \leq D_1D_2$, i.e., the elements $b_1$ and $b_2$ generate a positive definite subspace of $B_0$ relative to the trace form.*

**Proof.** This lemma follows from a direct calculation, as in [Ri] for example. For instance, the second assertion can be seen by noting that two geodesics on $\mathscr{H}$ intersect if and only if their associated cross-ratio is negative, by exploiting the $\mathrm{SL}_2(\mathbb{R})$-invariance of the cross-ratio to reduce to the case where $\gamma_1 = (0, \infty)$. □

Thanks to Lemma 5, the arithmetic intersection $(\gamma_1 \star \gamma_2)_\Gamma$ generalises directly to the case where $\Gamma$ is the group of norm one elements of the multiplicative group of a maximal

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order in an indefinite quaternion algebra over $\mathbb{Q}$, as in the previous section. Namely, with notations as in (7), we extend (2) and (3) by setting

$$(\gamma_1 \star \gamma_2) := (\tau_1, \tau_1' \,;\, \tau_2, \tau_2')^{(\gamma_1 \cdot \gamma_2)}, \qquad (\gamma_1 \star \gamma_2)_\Gamma = \prod_{g \,\in\, \Sigma_{12}} (\gamma_1 \star g\gamma_2). \tag{8}$$

Lemma 5 shows that $(\gamma_1 \star \gamma_2)_\Gamma$ belongs to the real quadratic field $\mathbb{Q}(\sqrt{D_1 D_2})$, and is of norm one in this field.

Let $\tilde{\Gamma} := R^\times$, which contains $\Gamma$ with index two.

**Lemma 6.** *The arithmetic intersection $(\gamma_1 \star \gamma_2)_\Gamma$ satisfies the identities*

$$\begin{array}{lll} (g\gamma_1 \star g\gamma_2)_\Gamma & = (\gamma_1 \star \gamma_2)_\Gamma^{\mathrm{sgn}(\det(g))} & \text{for all } g \in \tilde{\Gamma}, \\ (\gamma_2 \star \gamma_1)_\Gamma & = (\gamma_1 \star \gamma_2)_\Gamma^{-1} & \\ (\gamma_1' \star \gamma_2)_\Gamma & = (\gamma_1 \star \gamma_2)_\Gamma. & \end{array} \tag{9}$$

**Proof.** The first identity follows from the observation that the cross-ratio of any quadruple is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$, while $\gamma_1 \cdot \gamma_2$ is only invariant under the action of the orientation-preserving group $\mathrm{SL}_2(\mathbb{R})$, and is negated by orientation-reversing elements. The second identity follows from the symmetry of the cross-ratio and the antisymmetry of the topological intersection, and the last follows from the fact that replacing $\gamma_1$ by $\gamma_1'$ sends the resulting cross-ratio to its inverse, and negates the topological intersection of the modular geodesics. □

## 3. $p$-Arithmetic intersections

As explained in the introduction, the $p$-arithmetic intersection is obtained by replacing the arithmetic group $\Gamma$ by its $p$-arithmetic counterpart $\Gamma_p$. With notations as in the previous section, if $p$ is a prime which does not divide the discriminant of the indefinite quaternion algebra $B$, and $\Gamma = R_1^\times$ where $R$ is an order of $B$ of discriminant prime to $p$, then $\Gamma_p := (R[1/p])_1^\times$. The groups $\Gamma$ and $\Gamma_p$ are both viewed as subgroups of $\mathrm{SL}_2(\mathbb{R})$ by fixing an identification of $B \otimes \mathbb{R}$ with $M_2(\mathbb{R})$, as before.

Let

$$\Gamma_1^p := \mathrm{Stab}_{\Gamma_p}(\gamma_1), \quad \Gamma_2^p := \mathrm{Stab}_{\Gamma_p}(\gamma_2), \qquad \Sigma_{12}^p := \Gamma_1^p \backslash \Gamma_p / \Gamma_2^p$$

be the $p$-arithmetic analogues of $\Gamma_1$, $\Gamma_2$, and $\Sigma_{12}$ attached to the geodesics $\gamma_1$ and $\gamma_2$. The $p$-arithmetic intersection of $\gamma_1$ and $\gamma_2$ is obtained by considering the *infinite* product

$$(\gamma_1 \star \gamma_2)_{\Gamma_p} := \prod_{g \in \Sigma_{12}^p} (\gamma_1 \star g\gamma_2). \tag{10}$$

In order to make sense of this infinite product, it is essential to view it as a $p$-adic quantity: to this end, fix an embedding of the real quadratic field $\mathbb{Q}(\sqrt{D_1 D_2})$ into $\mathbb{Q}_p$.

**Theorem 7.** *The infinite product (10) converges absolutely p-adically.*

The proof of Theorem 7 rests on the study of the quantity

$$n_g := \langle b_1, gb_2 \rangle \in \mathbb{Z}[1/p]$$

attached to any $g \in \Sigma_{12}^p$. Let $v_p(n) := p^{-\mathrm{ord}_p(n)}$ denote the normalised $p$-adic norm of $n \in \mathbb{Q}$. We claim that the set

$$\Sigma_{12}^{\leq N} := \{g \in \Sigma_{12}^p \text{ such that } \quad (\gamma_1 \ ; \ g\gamma_2) < 0 \quad \text{and} \quad v_p(n_g) \leq p^N\}$$

is finite. Indeed, the inequality $(\gamma_1 \ ; \ g\gamma_2) < 0$ holds if and only if $n_g^2 < D_1 D_2$. But there are finitely many such $n_g \in \mathbb{Z}[1/p]$ with $v_p(n_g) \leq p^N$, and the finiteness of $\Sigma_{12}^{\leq N}$ is a consequence of the following lemma:

**Lemma 8.** *Let $n \in \mathbb{Z}[1/p]$ be an element satisfying $n^2 \leq D_1 D_2$. Then the set*

$$\Pi(D_1, D_2, n) := \{(a_1, a_2) \in R[1/p]^2 \\ \textit{with } \langle a_1, a_1 \rangle = D_1, \quad \langle a_2, a_2 \rangle = D_2, \quad \langle a_1, a_2 \rangle = n\}$$

*is preserved by the conjugation action of $\Gamma_p$ and is a union of finitely many orbits for this action.*

**Proof of lemma.** Consider the definite quadratic form attached to $(a_1, a_2) \in \Pi(D_1, D_2, n)$:

$$\begin{aligned} F(X, Y) &= \mathrm{disc}(Xa_1 + Ya_2) \\ &= D_1 X^2 + 2nXY + D_2 Y^2. \end{aligned}$$

Let $B_{12}$ be the Clifford algebra of the quadratic space $(\mathbb{Q}^2, F)$. It is a quaternion algebra over $\mathbb{Q}$ with a basis $\{1, e_1, e_2, e_1 e_2\}$ satisfying

$$e_1^2 = D_1, \quad e_2^2 = D_2, \quad \mathrm{Tr}(e_1 e_2) = 2n.$$

Let $R_{12}$ be the $\mathbb{Z}[1/p]$-order in $B_{12}$ generated by $e_1$ and $e_2$. The map sending $e_i$ to $a_i$ is an embedding of $\mathbb{Z}[1/p]$-orders

$$\iota : R_{12} \hookrightarrow R[1/p],$$

and simultaneous conjugation of $(a_1, a_2)$ by $\Gamma_p = R[1/p]_1^\times$ corresponds to conjugation of the embedding. The index of any such embedding is bounded by $D_1 D_2 - n^2$, and hence there are at most finitely many $\Gamma_p$-conjugacy classes of such embeddings. In particular, there are only finitely many possible pairs $(a_1, a_2) \in \Pi(D_1, D_2, n)$ up to $\Gamma_p$-equivalence. The result follows. $\square$

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**End of proof of Theorem 7.** Lemma 8 and the discussion preceding it implies that $\Sigma_{12}^{\leq N}$ is finite. Furthermore, if $g$ belongs to the complement of $\Sigma_{12}^{\leq N}$, then $v_p(n_g) > p^N$, and hence

$$(\gamma_1\ ;\ g\gamma_2) = \frac{n_g + \sqrt{D_1 D_2}}{n_g - \sqrt{D_1 D_2}} \equiv 1 \pmod{p^N}.$$

It follows that for any given $N \geq 1$, all but finitely many factors in the infinite product (10) are congruent to 1 modulo $p^N$. This infinite product therefore converges absolutely $p$-adically, as was to be shown. $\square$

We now turn to the algebraicity properties of the quantity $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ of (10). Assume for simplicity that the discriminants $D_1$ and $D_2$ are fundamental and relatively prime, and let

$$F_1 = \mathbb{Q}(\sqrt{D_1}), \qquad F_2 = \mathbb{Q}(\sqrt{D_2}), \qquad F_{12} := \mathbb{Q}(\sqrt{D_1 D_2})$$

be the real quadratic fields of discriminants $D_1$, $D_2$, and $D_1 D_2$. Recall that $H_1$ and $H_2$ denote the narrow Hilbert class fields of $H_1$ and $H_2$, and that $H_{12}$ is their compositum. (In particular, $H_{12}$ is *not* the Hilbert class field of $F_{12}$, in spite of what the notation might misleadingly suggest!)

Let $\mathbb{Q}_{p^2}$ be the quadratic unramified extension of $\mathbb{Q}_p$, and let

$$\iota_p : H_{12} \longrightarrow \mathbb{Q}_{p^2}$$

be a $p$-adic embedding extending the $p$-adic embedding of $F_{12}$ that was made in defining $(\gamma_1 \star \gamma_2)_{\Gamma_p}$. Conjecture 2 of the introduction predicts that the $p$-arithmetic intersection belongs to $\iota_p(H_{12})$, at least in some particular settings. In general, it needs not be algebraic, but the extent to which it may fail to be is well understood, at least conjecturally. Recall that $S$ is the discriminant of the quaternion algebra that was used to define the arithmetic group $\Gamma$. Let $\mathbb{T}$ be the "good Hecke algebra" of level $Sp$, generated by operators $T_n$ with index $n$ prime to $pS$. It acts as correspondences on the formal $\mathbb{Z}$-module generated by the modular geodesics, and the symbol $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ can be extended to such formal linear combinations by multiplicativity.

**Conjecture 9.** *Let $T \in \mathbb{T}$ be any Hecke operator which annihilates the space of cusp forms of weight 2 on $\Gamma_0(Sp)$. Then the quantity $(T\gamma_1 \star \gamma_2)_{\Gamma_p}$ belongs to $\iota_p(H_{12})$.*

**Remark 10.** In particular, $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ is predicted to always be algebraic when there are no cusp forms of weight two and level $Sp$, which occurs precisely when the set $S$ (of even cardinality) is empty and $p = 2, 3, 5, 7$, or 13. This is the setting that arises in Conjecture 2 and that was studied extensively in [DV]. Conjecture 9 has been explored in more general quaternionic settings in [GMX].

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**Remark 11.** The hypothesis on the Hecke operator $T$ implies that $(T\gamma_1 \cdot \gamma_2)_\Gamma = 0$. The $p$-arithmetic intersection in $H_{12}$ might best be envisioned as a higher operation in cohomology, akin to a linking number or a Massey product.

Assuming Conjecture 9, the quantity $(T\gamma_1 \star \gamma_2)_{\Gamma_p}$ can be viewed as an element of $H_{12}$, depending on the choice of a prime of $H_{12}$ that lies above $\mathfrak{m}_p$. To suppress the dependence on this choice, it is convenient to think of $(T\gamma_1 \star \gamma_2)_{\Gamma_p}$ as an element of $H_{12}/G_{12}$, where $G_{12} := \mathrm{Gal}(H_{12}/F_1F_2)$. This is what shall be done from now on:

**Definition 12.** The $p$-arithmetic intersection of $T\gamma_1$ and $\gamma_2$ is the element

$$(T\gamma_1 \star \gamma_2)_{\Gamma_p} \in H_{12}/G_{12},$$

whose image under a suitable embedding of $H_{12}$ into $\mathbb{Q}_{p^2}$ coincides with the infinite product of (10).

An appealing variant of Conjecture 9 occurs when $\Gamma_p = \mathrm{SL}_2(\mathbb{Z}[1/p])$ arises from the globally split quaternion algebra, and $\gamma_1$ is a geodesic joining a pair of cusps rather than a pair of conjugate real irrationalities. Suppose for instance that $\gamma_1 = \gamma_w := (0, \infty)$ is the so-called "winding element": the geodesic corresponding to the imaginary axis in $\mathscr{H}$, whose endpoints are the roots of the binary quadratic form $F_1 = XY$ of discriminant 1. Assume that $\gamma_2 = \gamma = (\tau, \tau')$ is the geodesic joining the roots of a binary quadratic form $F$ whose discriminant $D > 0$ satisfies $(\frac{D}{p}) = -1$. Letting

$$\Pi_\tau := \{z \in \Gamma\tau \text{ satisfying } zz' < 0 \text{ and } 0 \leq \mathrm{ord}_p(z) < 2\},$$

one has

$$(\gamma_w \star \gamma)_{\Gamma_p} = \prod_{z \in \Pi_\tau} \frac{z}{z'} = \prod \frac{-b + \sqrt{D}}{-b - \sqrt{D}}, \tag{11}$$

where the second product ranges over the binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $D$ with coefficients in $\mathbb{Z}[1/p]$ that are $\Gamma_p$-equivalent to $F(x,y)$ and satisfy $0 \leq \mathrm{ord}_p(a) < 2$.

For instance, when $\tau = -1 + \sqrt{3}$, a real quadratic irrationality of discriminant 12 with ring class field $\mathbb{Q}(\sqrt{3}, \sqrt{-3})$, one finds experimentally that

$$\begin{aligned}(\gamma_w \star \gamma)_{\Gamma_5} &\stackrel{?}{=} -1 + 2i,\\ (\gamma_w \star \gamma)_{\Gamma_7} &\stackrel{?}{=} ((-13 + 3\sqrt{-3})/2)^{1/3}.\end{aligned}$$

Note that the latter is not contained in the ring class field of $\tau$, though its third power is. In the language of rigid meromorphic cocycles used in [DV], which we briefly discuss below, this phenomenon may be explained due to the presence of cohomological torsion.

When $\tau = (-7+\sqrt{77})/2$ of discriminant 77, we find

$$\begin{aligned}(\gamma_w \star \gamma)_{\Gamma_2} &\stackrel{?}{=} (-5983 + 2115\sqrt{-7})/2, \\ (\gamma_w \star \gamma)_{\Gamma_3} &\stackrel{?}{=} -679 + 80\sqrt{-11}, \\ (\gamma_w \star \gamma)_{\Gamma_5} &\stackrel{?}{=} 3 + 36\sqrt{-11}.\end{aligned}$$

Finally, we consider the discriminant 321 of narrow class number 6 that was also mentioned in the introduction. Here we compute that $(\gamma_w \star \gamma)_{\Gamma_7}$ satisfies the polynomial

$$7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 120922465776x + 7^{16}.$$

In conclusion, it appears that whenever the modular curve $X_0(p)$ has genus zero, the quantities $(\gamma_w \star \gamma)_{\Gamma_p}$ are algebraic, and more precisely, that up to a small power they are $p$-units in the ring class field of discriminant $D$. This somewhat degenerate variant of Conjecture 9 has been proved in certain cases in [DPV1,DPV2] by studying the diagonal restrictions of $p$-adic deformations of Hilbert modular Eisenstein series.

When $p = 11$ or $p > 13$, i.e., when the modular curve $X_0(p)$ has genus strictly greater than zero, the $p$-arithmetic intersections are arguably even more interesting, because of the following conjecture:

**Conjecture 13.** *The $p$-adic logarithm of $(\gamma_w \star \gamma)_{\Gamma_p}$ is a finite linear combination of $p$-adic logarithms of a $p$-unit of $H$ and of formal group logarithms of global points on $J_0(p)(H)$.*

For instance, when $\tau := \frac{3+\sqrt{21}}{6}$ is a real quadratic irrationality of discriminant 21, it was verified to 100 digits of 11-adic precision that:

$$\log_{11}(\gamma_w \star \gamma)_{\Gamma_{11}} \stackrel{?}{=} \frac{1}{3}\log_{11}\left(3+4\sqrt{-7}\right) - \frac{1}{5}\log_E\left(\frac{-3-\sqrt{-7}}{2}, \frac{-3-\sqrt{-7}}{2}\right) \tag{12}$$

where $\log_E$ is the 11-adic formal group logarithm on the elliptic curve $E$ of conductor 11 with Weierstraß equation

$$E \;:\; y^2 + y = x^3 - x^2 - 10x - 20.$$

## 4. Real quadratic singular moduli

We briefly describe the relation between the $p$-arithmetic intersection number of (4) and the "differences of real quadratic singular moduli" studied in [DV] and [GMX].

The real quadratic singular moduli of [DV] are obtained as the "RM values" of *rigid meromorphic cocycles* on the $p$-adic upper half plane

$$\mathscr{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p).$$

This $p$-adic symmetric space is endowed with a natural action of $\mathrm{SL}_2(\mathbb{Q}_p)$ by Möbius transformations and equipped with a canonical *reduction map* to the *Bruhat-Tits tree* $\mathscr{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The latter is a $(p+1)$-regular graph which can be described as a disjoint union $\mathscr{T} := \mathscr{T}_0 \sqcup \mathscr{T}_1$, where the set $\mathscr{T}_0$ of vertices of $\mathscr{T}$ is in bijection with homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, and the set $\mathscr{T}_1$ of edges consists of pairs of vertices admitting representative lattices which are contained one in another with index $p$. The pre-image of a vertex under the reduction map is a so-called *standard affinoid* obtained by excising from $\mathbb{P}_1(\mathbb{C}_p)$ the $(p+1)$ "mod $p$ residue discs" centered at the points of $\mathbb{P}_1(\mathbb{F}_p)$, relative to a suitable choice of coordinate on $\mathbb{P}_1(\mathbb{Q}_p)$. Likewise, the preimage of an edge under the reduction map is an *annulus* identified with a Möbius translate of the region

$$\{z \in \mathbb{C}_p \quad \text{satisfying} \quad 1 < |z| < p\} \quad \subset \quad \mathscr{H}_p.$$

A subgraph $\mathscr{G} \subset \mathscr{T}$ is said to be *closed* if it contains the endpoint vertices of any edge in $\mathscr{G} \cap \mathscr{T}_1$, and is said to be *finite* if it contains finitely many vertices and edges. An *affinoid subset* of $\mathscr{H}_p$ is the inverse image under the reduction map of a closed finite subgraph of $\mathscr{T}$. This notion of an affinoid subset, which is somewhat restrictive but suffices for our purpose, is what might be called "an affinoid subset defined over $\mathbb{Q}_p$" elsewhere in the literature.

A *discrete divisor* on $\mathscr{H}_p$ is a formal $\mathbb{Z}$-linear combination $\mathscr{D} := \sum_{x \in \mathscr{H}_p} m_x \cdot [x]$ of points $x \in \mathscr{H}_p$, for which

$$\mathscr{D} \cap \mathcal{A} := \sum_{x \in \mathcal{A}} m_x \cdot [x]$$

is an actual divisor, for all affinoid subsets $\mathcal{A} \subset \mathscr{H}_p$. (I.e., the coefficients $m_x$ are equal to zero for all but finitely many $x \in \mathcal{A}$.) Let $\mathrm{Div}^\dagger(\mathscr{H}_p)$ denote the $\mathrm{SL}_2(\mathbb{Q}_p)$-module of discrete divisors. An element $\mathscr{D} \in \mathrm{Div}^\dagger(\mathscr{H}_p)$ is said to be *of degree zero* if $\mathscr{D} \cap \mathcal{A}$ is a degree 0 divisor, for all affinoids $\mathcal{A} \subset \mathscr{H}_p$.

The *value* of a rational function $f$ on a divisor $\mathscr{D} = \sum_x m_x \cdot [x]$ it the quantity

$$f(\mathscr{D}) := \prod_x f(x)^{m_x}.$$

The *Weil symbol* of two degree zero divisors $\mathscr{D}$, $\mathscr{D}' \in \mathrm{Div}^0(\mathbb{P}_1(\mathbb{C}_p))$ with disjoint supports is the value $f_{\mathscr{D}'}(\mathscr{D})$, where $f_{\mathscr{D}'}$ is any function having $\mathscr{D}$ as divisor. The Weil symbol can be extended to the case where $\mathscr{D} \in \mathrm{Div}^\dagger(\mathscr{H}_p)$ and $\mathscr{D}' \in \mathrm{Div}(\mathscr{H}_p)$ are both of degree zero (with disjoint supports), by choosing an admissible cover

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \mathcal{A}_n \subset \cdots, \qquad \bigcup_{n \geq 1} \mathcal{A}_n = \mathscr{H}_p$$

of $\mathscr{H}_p$ by an increasing sequence of affinoids, and setting

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$$[\mathscr{D}\ \ ;\mathscr{D}'] := \lim_{n\to\infty} [\mathscr{D} \cap \mathcal{A}_n\ ;\ \mathscr{D}'].$$

Choosing a base point $\eta \in \mathscr{H}_p$, the rigid meromorphic function

$$f_{\mathscr{D}}(z) := [\mathscr{D}\ ;\ (z) - (\eta)]$$

has divisor equal to $\mathscr{D}$. Changing the base point $\eta \in \mathscr{H}_p$ has the effect of multiplying $f_{\mathscr{D}}$ by a non-zero scalar; hence $f_{\mathscr{D}}$ should only be viewed as being determined by $\mathscr{D}$ up to such a scalar.

For simplicity, we begin by placing ourselves in the setting of [DV] where $\Gamma_p = \mathrm{SL}_2(\mathbb{Z}[1/p])$. In this setting, discrete divisors on $\mathscr{H}_p$ can be obtained from a pair $r, s$ of elements of $\mathbb{P}_1(\mathbb{Q})$ and from a real quadratic irrationality $\tau \in \mathscr{H}_p$, by setting

$$\mathscr{D}_\tau(r, s) := \sum_{w\in\Gamma_p\tau} ((w, w') \cdot (r, s)) \cdot [w],$$

where the simultaneous embeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and into $\bar{\mathbb{Q}}_p$ are used to view $(w, w')$ as a geodesic in $\mathscr{H}$, and $[w]$ as a point of $\mathscr{H}_p$. Assume for simplicity that $\tau$ satisfies a quadratic equation with integer coefficients whose discriminant $D$ is non-zero (and hence, a non-square) modulo $p$.

In the setting where $\Gamma_p = \mathrm{SL}_2(\mathbb{Z}[1/p])$ and $p \nmid D$, the discrete divisor $\mathscr{D}_\tau(r, s)$ is always of degree zero, and the assignment $(r, s) \mapsto \mathscr{D}_\tau(r, s)$ gives a $\Gamma_p$-invariant *modular symbol* with values in the $\Gamma_p$-module $\mathrm{Div}^\dagger(\mathscr{H}_p)$, i.e., a function

$$\mathscr{D}_\tau : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \mathrm{Div}^\dagger(\mathscr{H}_p)$$

satisfying $\mathscr{D}_\tau(r, t) + \mathscr{D}_\tau(t, s) = \mathscr{D}_\tau(r, s)$ for all $r, s, t \in \mathbb{P}_1(\mathbb{Q})$, and

$$\gamma\mathscr{D}_\tau(r, s) = \mathscr{D}_\tau(\gamma r, \gamma s), \qquad \text{for all } \gamma \in \Gamma_p.$$

The element $\mathscr{D}_\tau$ in the module $\mathrm{MS}(\mathrm{Div}^\dagger(\mathscr{H}_p))^{\Gamma_p}$ of such $\Gamma_p$-invariant modular symbols gives an element $\bar{J}_\tau \in \mathrm{MS}(\mathcal{M}^\times/\mathbb{C}_p^\times)^{\Gamma_p}$ by setting

$$\bar{J}_\tau(r, s)(z) := f_{\mathscr{D}_\tau(r,s)}(z), \qquad \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}). \tag{13}$$

The essential triviality of $H^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$ can be used to lift the restriction of $\bar{J}_\tau(r, s)$ to $\mathrm{SL}_2(\mathbb{Z})$ to a class $J_\tau \in \mathrm{MS}(\mathcal{M}^\times)^{\mathrm{SL}_2(\mathbb{Z})}$, which is essentially unique since there are no $\mathrm{SL}_2(\mathbb{Z})$-invariant $\mathbb{C}_p$-valued modular symbols. One has

$$J_\tau(r, s)(z) = c(r, s)\ \cdot\ [\mathscr{D}_\tau(r, s)\ ;\ (z) - (\eta)]\,, \qquad \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}),$$

where $c(r, s) \in \mathbb{C}_p^\times$ is a somewhat subtle constant that depends on $(r, s)$, and on the base point $\eta$, but not on $z$ which plays the role of a variable in the equation above.

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Given two real quadratic irrationalities $\tau_1$ and $\tau_2$ of discriminant $D$ prime to $p$ (viewed as elements of both $\mathbb{P}_1(\mathbb{R})$ and $\mathscr{H}_p$ via the fixed embeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$) let $\alpha_1$ and $\alpha_2$ be generators modulo torsion for their stabilisers in $\Gamma_p$, normalised so that

$$\lim_{n\to\infty} \alpha_j^n r = \tau_j', \quad \lim_{n\to-\infty} \alpha_j^n r = \tau_j, \qquad \text{for all } r \in \mathbb{P}_1(\mathbb{R}) - \{\tau_j, \tau_j'\}. \tag{14}$$

We then set

$$J_p(\tau_1, \tau_2) := J_{\tau_1}(r, \alpha_2 r)(\tau_2),$$

an expression which does not depend on the choice of base point $r \in \mathbb{P}_1(\mathbb{Q})$. Conjectures 1 and 2 of [DV] assert that, when $p = 2, 3, 5, 7$, or 13 (i.e., when the modular curve $X_0(p)$ has genus zero), the quantities $J_p(\tau_1, \tau_2)$ are algebraic numbers in the compositum of the Hilbert class fields attached to $D_1$ and $D_2$, and that, from the point of view of their prime factorisations, they enjoy many properties analogous to the differences $J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2)$ of singular moduli, when $\tau_1$ and $\tau_2$ are CM points on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}$.

The following proposition explains why Conjecture 9 of the present work follows from the finer conjectures of [DV].

**Proposition 14.** *Let $\gamma_1 = (\tau_1, \tau_1')$ and $\gamma_2 = (\tau_2, \tau_2')$ be a pair of real quadratic geodesics whose discriminants $D_1$ and $D_2$ are non-squares modulo $p$. Then*

$$(\gamma_1 \star \gamma_2)_{\Gamma_p} = J_p(\tau_1, \tau_2) \cdot J_p(\tau_1', \tau_2) \cdot J_p(\tau_1, \tau_2') \cdot J_p(\tau_1', \tau_2').$$

**Proof.** By definition,

$$J_p(\tau_1, \tau_2) = J_{\tau_1}(r, \alpha_2 r)(\tau_2) = c(r, \alpha_2 r) \ \cdot\ [\mathscr{D}_{\tau_1}(r, \alpha_2 r)\ ;\ (\tau_2) - (\eta)]\,. \tag{15}$$

Replacing $\tau_2$ by $\tau_2'$ and $r$ by $\alpha_2 r$, we observe that the stabiliser of $\tau_2'$, normalised as in (14), is $\alpha_2^{-1}$, and hence that

$$\begin{aligned} J_p(\tau_1, \tau_2') &= J_{\tau_1}(\alpha_2 r, r)(\tau_2') = J_{\tau_1}(r, \alpha_2 r)(\tau_2')^{-1} \\ &= c(r, \alpha_2 r)^{-1} \cdot [\mathscr{D}_{\tau_1}(r, \alpha_2 r)\ ;\ (\tau_2') - (\eta)]^{-1}\,. \end{aligned} \tag{16}$$

Multiplying (15) and (16) together leads to a simpler expression where the quantities $c(r, \alpha_2 r)$ and $\eta$ have disappeared:

$$J_p(\tau_1, \tau_2) \cdot J_p(\tau_1, \tau_2') \ = \ [\mathscr{D}_{\tau_1}(r, \alpha_2 r)\ ;\ (\tau_2) - (\tau_2')]. \tag{17}$$

On the other hand,

$$\begin{aligned}\mathscr{D}_{\tau_1}(r,\alpha_2 r) &= \sum_{\gamma\in\Gamma_p/\alpha_1^{\mathbb{Z}}} ((\gamma\tau_1,\gamma\tau_1')\cdot(r,\alpha_2 r))[\gamma\tau_1] \\ &= \sum_{\gamma\in\alpha_2^{\mathbb{Z}}\backslash\Gamma_p/\alpha_1^{\mathbb{Z}}} \left\{\sum_{j=-\infty}^{\infty} ((\alpha_2^j\gamma\tau_1,\alpha_2^j\gamma\tau_1')\cdot(r,\alpha_2 r))[\alpha_2^j\gamma\tau_1]\right\}.\end{aligned}$$

Substituting this equation into (17) and using the $\Gamma_p$-equivariance of the Weil symbol and of the intersection pairing, combined with the fact that $\alpha_2$ fixes the divisor $(\tau_2)-(\tau_2')$, we obtain

$$\begin{aligned}&J_p(\tau_1,\tau_2)\cdot J_p(\tau_1,\tau_2') \\ &\quad= \left[\sum_{\gamma\in\alpha_2^{\mathbb{Z}}\backslash\Gamma_p/\alpha_1^{\mathbb{Z}}} \left\{\sum_{j=-\infty}^{\infty} ((\gamma\tau_1,\gamma\tau_1')\cdot(\alpha_2^{-j}r,\alpha_2^{-j+1}r))\right\}[\gamma\tau_1]\ ;\ (\tau_2)-(\tau_2')\right]. \quad (18)\end{aligned}$$

The sum inside the curly braces is telescoping and converges to

$$((\gamma\tau_1,\gamma\tau_1')\cdot(\tau_2,\tau_2')) = (\gamma\gamma_1\cdot\gamma_2),$$

by (14). It follows that

$$J_p(\tau_1,\tau_2)\cdot J_p(\tau_1,\tau_2') = \left[\sum_{\gamma\in\alpha_2^{\mathbb{Z}}\backslash\Gamma_p/\alpha_1^{\mathbb{Z}}} (\gamma\gamma_1\cdot\gamma_2)\cdot[\gamma\tau_1]\ ;\ (\tau_2)-(\tau_2')\right]. \qquad (19)$$

A similar argument with $\tau_1$ replaced by $\tau_1'$ (and hence, with the orientation of $\gamma_1$ reversed) shows that

$$J_p(\tau_1',\tau_2)\cdot J_p(\tau_1',\tau_2') = \left[\sum_{\gamma\in\alpha_2^{\mathbb{Z}}\backslash\Gamma_p/\alpha_1^{\mathbb{Z}}} (\gamma\gamma_1\cdot\gamma_2)\cdot[\gamma\tau_1']\ ;\ (\tau_2)-(\tau_2')\right]^{-1}. \qquad (20)$$

Multiplying (19) and (20) together yields the desired result. □

In the general case where $p = 11$ or $p > 13$, or, even more generally, where $\Gamma_p$ is the $p$-arithmetic group arising from a maximal $\mathbb{Z}[1/p]$-order $R$ in some indefinite quaternion algebra $B$, one can still associate to an RM point $\tau \in \Gamma_p\backslash\mathscr{H}_p$ a class $\mathscr{D}_{\tau,\tau'}$ in $H^1(\Gamma_p, \mathrm{Div}^\dagger(\mathscr{H}_p))$. This is done by choosing a base point $r \in \mathscr{H}$, in sufficiently general position so that it does not lie on any of the $\Gamma_p$-translates of the geodesic $(\tau,\tau')$, and setting

$$\mathscr{D}_{\tau,\tau'}(\gamma) = \sum_{w\in\Gamma_p\tau} ((w,w')\cdot(r,\gamma r))([w]-[w']).$$

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Since $w$ and $w'$ always reduce to the same vertex on $\mathscr{T}$, the discrete divisors $\mathscr{D}_{\tau,\tau'}(\gamma)$ are of degree zero for all $\gamma \in \Gamma_p$, and one obtains a class

$$\bar{J}_{\tau,\tau'} \in H^1(\Gamma_p, \mathcal{M}^\times/\mathbb{C}_p^\times)$$

as in (13) by setting

$$\bar{J}_{\tau,\tau'}(\gamma)(z) := f_{\mathscr{D}_{\tau,\tau'}(\gamma)}(z), \qquad \text{for all } \gamma \in \Gamma_p, \quad z \in \mathscr{H}_p. \tag{21}$$

Although the class $\bar{J}_{\tau,\tau'}$ need not lift to an $\mathcal{M}^\times$-valued one-cocycle, one can still lift it to an $\mathcal{M}^\times$-valued one-cochain $J_{\tau,\tau'}$. For any pair $(\tau_1, \tau_2)$ of $\Gamma_p$-inequivalent RM points, the quantity

$$J_{\tau_1,\tau_1'}[\tau_2] \times J_{\tau_1,\tau_1'}[\tau_2'] := \frac{J_{\tau_1,\tau_1'}(\alpha_2)(\tau_2)}{J_{\tau_1,\tau_1'}(\alpha_2)(\tau_2')}$$

does not depend on the chosen lift, and is equal to the $p$-arithmetic intersection $(\gamma_1 \star \gamma_2)_{\Gamma_p}$ of the associated modular geodesics.

One can then formulate the following refinement of Conjecture 9:

**Conjecture 15.** *Let $T \in \mathbb{T}$ be any Hecke operator which annihilates the space of cusp forms of weight 2 on $\Gamma_0(Sp)$, where $S = \mathrm{Disc}(B)$. Then the one-cocycle $\bar{J}_{T\tau_1,T\tau_1'}$ lifts to a class $J_{T\tau_1,T\tau_1'}$ in $H^1(\Gamma_p, \mathcal{M}^\times)$, and the RM value $J_{T\tau_1,T\tau_1'}[\tau_2]$ belongs to $\iota_p(H_{12})$.*

The work [GMX] of Guitart, Masdeu and Xarles gives an important refinement of this conjecture by showing that there are, up to torsion, well-defined classes

$$J_{T\tau_1}, J_{T\tau_1'} \ \in \ H^1(\Gamma_p, \mathcal{M}^\times)$$

satisfying

$$J_{T\tau_1,T\tau_1'} = J_{T\tau_1} \times J_{T\tau_1'},$$

and conjectures that the quantities

$$J_p(T\tau_1, \tau_2) := J_{T\tau_1}[\tau_2] \tag{22}$$

are also algebraic numbers in $H_1 H_2$. As in Proposition 14, one has

$$(T\gamma_1 \star \gamma_2)_{\Gamma_p} = J_p(T\tau_1, \tau_2) \cdot J_p(T\tau_1', \tau_2) \cdot J_p(T\tau_1, \tau_2') \cdot J_p(T\tau_1', \tau_2'). \tag{23}$$

While the finer quantities $J_p(T\tau_1, \tau_2)$ are more subtle to define and to compute, they are the ones that enjoy the most direct analogy with the differences of singular moduli in the theory of complex multiplication.

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## 5. Incoherent intersections

Let $D_1$ and $D_2$ be a pair of positive discriminants. For ease of exposition, assume that they are fundamental and coprime. As in earlier sections, write

$$F_1 := \mathbb{Q}(\sqrt{D_1}), \qquad F_2 := \mathbb{Q}(\sqrt{D_2}), \qquad F_{12} := \mathbb{Q}(\sqrt{D_1 D_2}),$$

and let $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_{12}$ denote their respective rings of integers. Let

$$\mathcal{S} := \left\{ \text{rational primes } p \text{ such that } \left(\frac{D_1}{p}\right) = \left(\frac{D_2}{p}\right) = -1 \right\}.$$

The primes in $\mathcal{S}$ are split in the intermediate field $F_{12}$. For each $p \in \mathcal{S}$, fix an ideal $\mathfrak{m}_p$ of $F_1 F_2$ above $p$. This determines, for each such $p$, an element $\delta_p \in \mathbb{F}_p$ satisfying

$$\delta_p^2 = D_1 D_2.$$

Let $S$ be a finite subset of $\mathcal{S}$.

When $S$ has *even* cardinality, it determines an arithmetic group $\Gamma(S) \subset \mathrm{SL}_2(\mathbb{R})$ consisting of the norm one elements in a maximal order

$$R(S) \subset B(S) \subset M_2(\mathbb{R})$$

of the quaternion algebra $B(S)$ ramified exactly at the primes of $S$. The group $\Gamma := \Gamma(S)$ depends only on $S$, up to conjugation in $\mathrm{SL}_2(\mathbb{R})$. Let $(\gamma_1, \gamma_2)$ be any pair of modular geodesics on $\Gamma(S)\backslash\mathscr{H}$. One may then consider the topological and arithmetic intersection numbers

$$(\gamma_1 \cdot \gamma_2)_S := (\gamma_1 \cdot \gamma_2)_{\Gamma(S)}, \qquad (\gamma_1 \star \gamma_2)_S := (\gamma_1 \star \gamma_2)_{\Gamma(S)} \in F_{12}.$$

Assume now that $S \subset \mathcal{S}$ has *odd* cardinality. For each prime $p \in S$, we may choose a pair $(\gamma_1, \gamma_2)$ of modular geodesics on $\Gamma(S - \{p\})\backslash\mathscr{H}$ satisfying the conditions

$$\mathcal{O}_{\gamma_1} = \mathcal{O}_1, \qquad \mathcal{O}_{\gamma_2} = \mathcal{O}_2, \qquad \eta_q(\gamma_1, \gamma_2) = \delta_q, \text{ for all } q \in S - \{p\}.$$

Let $T$ be a good Hecke operator that annihilates the space of cuspidal newforms of weight 2 and level $S$. Assuming Conjecture 9, one can consider the $p$-arithmetic intersection numbers

$$(T\gamma_1 \star \gamma_2)_{S,p} := (T\gamma_1 \star \gamma_2)_{\Gamma(S-\{p\})_p}$$

as global invariants in $H_{12}/G_{12}$. These quantities can be factored as

$$(T\gamma_1 \star \gamma_2)_{S,p} = J_{S,p}(T\tau_1, \tau_2) \cdot J_{S,p}(T\tau_1, \tau_2') \cdot J_{S,p}(T\tau_1', \tau_2) \cdot J_{S,p}(T\tau_1', \tau_2'),$$

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where $J_{S,p}(T\tau_1,\tau_2)$ is defined as in (22). The finer invariant $J_{S,p}(T\tau_1,\tau_2)$ appears to *depend only on* $S$ and not on the choice of $p \in S$ that was made to compute it $p$-adically:

**Conjecture 16.** *There is an element* $J_S(T\tau_1,\tau_2) \in H_{12}/G_{12}$ *such that*

$$J_S(T\tau_1,\tau_2) = J_{S,p}(T\tau_1,\tau_2), \qquad \textit{for all } p \in S.$$

**Remark 17.** It would be tempting to conjecture, based on (23) that the $p$-arithmetic intersection number $(T\gamma_1 \star \gamma_2)_{S,p}$ is likewise independent of $p$, but the authors do not believe this to be the case. Indeed, Conjecture 16 predicts that, for $p,q \in S$,

$$\begin{aligned} J_{S,p}(T\tau_1,\tau_2) &= \sigma_1 J_{S,q}(T\tau_1,\tau_2), \qquad & J_{S,p}(T\tau_1',\tau_2) &= \sigma_2 J_{S,q}(T\tau_1',\tau_2),\\ J_{S,p}(T\tau_1,\tau_2') &= \sigma_3 J_{S,q}(T\tau_1,\tau_2'), \qquad & J_{S,p}(T\tau_1',\tau_2') &= \sigma_4 J_{S,q}(T\tau_1',\tau_2'), \end{aligned}$$

where $\sigma_1,\sigma_2,\sigma_3$ and $\sigma_4$ are automorphisms in $G_{12}$ which *need not be the same*. For a numerical illustration of this phenomenon in the setting of CM points on Shimura curves, see [Gi].

**Remark 18.** The set $S$ of primes of odd cardinality can be viewed as the data for an *incoherent indefinite quaternion algebra* over $\mathbb{Q}$ in the sense of [Gr2] and [Gr3]. These references focus on incoherent *definite* data, where $S$ contains the archimedean place, and explain how such data naturally corresponds to a Shimura curve over $\mathbb{Q}$. When $\infty \notin S$, the arithmetic quotients $\Gamma(S-\{p\})\backslash\mathscr{H}$ for $p \in S$ can be envisaged as an "incoherent collection of Shimura curves". Conjecture 16 rests on the strong analogy between compatible systems of geodesics like $T\gamma_1$ and $\gamma_2$ on $\{\Gamma(S-\{p\})\backslash\mathscr{H}\}_{p\in S}$ (with $\infty \notin S$) and CM points on the Shimura curve attached to an odd cardinality set $S$ containing $\infty$.

**Definition 19.** The conjectural quantity

$$J_S(T\tau_1,\tau_2) \overset{?}{\in} H_{12}/G_{12}$$

of Conjecture 16 is called the *incoherent intersection number* attached to the pair of RM divisors $(T\tau_1,\tau_2)$.

**Remark 20.** Conjecture 16 adds nothing to Conjecture 9 when $S=\{p\}$ is a singleton, since $(\gamma_1\star\gamma_2)_p$ can then only be computed one way, as a $p$-arithmetic intersection number for $\mathrm{SL}_2(\mathbb{Z}[1/p])$. This is the case that was considered in [DV]. The recent work [GMX] of Guitart, Masdeu and Xarles gives striking experimental confirmation of (a more general version of) Conjecture 16 in the case where $S=\{2,3,5\}$, observing that, for a few pairs $(D_1,D_2)$ of discriminants in which 2, 3 and 5 are inert, and for suitable Hecke operators $T$ annihilating $S_2(\Gamma_0(30))$, one indeed has a global invariant

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$$J_{30}(T\tau_1, \tau_2) \overset{?}{\in} H_{12}/G_{12}$$

which coincides with

$$J_{\Gamma(15)_2}(T\tau_1, \tau_2) \overset{?}{=} J_{\Gamma(10)_3}(T\tau_1, \tau_2) \overset{?}{=} J_{\Gamma(6)_5}(T\tau_1, \tau_2),$$

at least up to the high level of 2-adic, 3-adic and 5-adic accuracies respectively with which these three quantities were computed numerically. The calculations in [GMX] involve the three neighbouring quaternion algebras, of discriminants 15, 10 and 6 respectively, in a way that evokes the Cerednik Drinfeld "interchange of invariants" occurring in the $p$-adic uniformisation theory of Shimura curves. What mathematical structure might underly the incoherent collections of Shimura curves and their compatible systems of modular geodesics in accounting for the global nature of $J_{30}(T\tau_1, \tau_2) \overset{?}{\in} H_{12}/G_{12}$, remains to be elucidated.

## 6. Factorisation conjectures

To each finite subset $S \subset \mathcal{S}$, we have associated quantities $(\gamma_1 \star \gamma_2)_S$ whose arithmetic nature depends crucially on the parity of the cardinality of $S$.

- When $S$ has even cardinality,

$$(\gamma_1 \star \gamma_2)_S := (\gamma_1 \star \gamma_2)_{\Gamma(S)} \in \mathbb{Q}(\sqrt{D_1 D_2})$$

  is an arithmetic intersection number on the indefinite quaternion algebra ramified at the places of $S$.
- When $S$ has odd cardinality, and $p \in S$, the $p$-arithmetic intersection number

$$(\gamma_1 \star \gamma_2)_{S,p} \overset{?}{\in} H_{12}/G_{12}$$

  lies significantly deeper. *Conjecturally*, it belongs to $H_{12}$, and is well defined up to the action of $G_{12}$, the dependency arising from the choice of an embedding $H_{12} \hookrightarrow \mathbb{Q}_{p^2}$.

For any $q \in \mathcal{S}$, the prime $\mathfrak{m}_q$ of $F_1 F_2$ splits completely in $H_{12}$, and the set of primes $\mathfrak{Q}$ of $H_{12}$ lying above $\mathfrak{m}_q$ is therefore a principal homogeneous space for $G_{12}$. Define a map

$$\mathrm{Ord}_{\mathfrak{m}_q} : H_{12}^\times/G_{12} \longrightarrow \mathbb{Z}[G_{12}]/G_{12} \tag{24}$$

by choosing a fixed $\mathfrak{Q}$ above $\mathfrak{m}_q$ and setting

$$\mathrm{Ord}_{\mathfrak{m}_q}(x) = \sum_{\sigma \in G_{12}} \mathrm{ord}_{\mathfrak{Q}}(x^\sigma)\sigma^{-1}. \tag{25}$$

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The value of $\mathrm{Ord}_{\mathfrak{m}_q}$ does not depend on the choice of prime $\mathfrak{Q}$ that was made to define it, since replacing $\mathfrak{Q}$ by another prime above $\mathfrak{m}_q$ merely has the effect of multiplying the right hand side of (25) by a group-like element in $G_{12} \subset \mathbb{Z}[G_{12}]^\times$.

It is natural to seek to relate $\mathrm{Ord}_{\mathfrak{m}_q}((\gamma_1 \star \gamma_2)_{S,p})$, when $S$ is of odd cardinality, to the $\mathfrak{m}_q$-adic valuations of certain arithmetic intersection numbers.

Given $q \in \mathcal{S}$, let

$$S_q := \begin{cases} S - \{q\} & \text{if } q \in S, \\ S \cup \{q\} & \text{if } q \notin S. \end{cases}$$

The set $S_q$ then has even cardinality, and the indefinite quaternion algebra $B(S_q)$ represents a *nearby quaternion algebra* for the incoherent datum $S$.

Recall that the narrow class groups $G_1 := \mathrm{Pic}^+(\mathcal{O}_{\gamma_1})$ and $G_2 := \mathrm{Pic}^+(\mathcal{O}_{\gamma_2})$ act simply transitively on the set of geodesics on $\Gamma(S_q)\backslash\mathscr{H}$ of discriminants $D_1$ and $D_2$ with the given orientations at $q \in S$. Since the extensions $H_1$ and $H_2$ are linearly disjoint over $F_1F_2$, one has a canonical identification of $G_1 \times G_2 = G_{12}$. For each $q \in \mathcal{S}$, let

$$\mathrm{Ord}_{\mathfrak{m}_q}((\gamma_1 \star \gamma_2)_{S_q}) := \sum_{\sigma \in G_{12}} (\gamma_1^\sigma \star \gamma_2^\sigma)_{S_q} \cdot \sigma^{-1} \in \mathbb{Z}[G_{12}]/G_{12}.$$

Viewing the target as $\mathbb{Z}[G_{12}]$ modulo the action of the group-like elements $G_{12} \subset \mathbb{Z}[G_{12}]^\times$ means that $\mathrm{Ord}_{\mathfrak{m}_q}((\gamma_1 \star \gamma_2)_{S_q})$ is independent of the choice of geodesics $\gamma_1$ and $\gamma_2$, but depends only on their discriminants and on the orientations $\eta_r(\gamma_1, \gamma_2)$ for $r \in S_q$.

**Conjecture 21.** *Let $S \subset \mathcal{S}$ be a set of odd cardinality, let $p \in S$, and let $T$ be a Hecke operator that annihilates the space of modular forms of weight two and level $S$. If $q \notin \mathcal{S}$ and $q \nmid D_1D_2$, then*

$$\mathrm{ord}_{\mathfrak{Q}}((T\gamma_1 \star \gamma_2)_{S,p}) = 0,$$

*for any prime $\mathfrak{Q}$ of $H_{12}$ above $q$.*

This conjecture and the numerical experiments documented in [DV] and [GMX] suggest the following natural question:

**Question 22.** For the primes $q \in \mathcal{S}$, what relation is there between the elements $\mathrm{Ord}_{\mathfrak{m}_q}((T\gamma_1 \star \gamma_2)_{S,p})$ and $\mathrm{Ord}_{\mathfrak{m}_q}((T\gamma_1 \star \gamma_2)_{S_q})$ of $\mathbb{Z}[G_{12}]/G_{12}$?

## References


[DPV1] H. Darmon, A. Pozzi, J. Vonk, Diagonal restrictions of $p$-adic Eisenstein families, Math. Ann. (2020), https://doi.org/10.1007/s00208-020-02086-2.

[DPV2] H. Darmon, A. Pozzi, J. Vonk, On the RM values of the Dedekind–Rademacher cocycle, preprint.

[DV] H. Darmon, J. Vonk, Singular moduli for real quadratic fields: a rigid analytic approach, Duke Math. J. 170 (2021) 23–93.


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[Gi] S. Gianpietro, A $p$-adic approach to the calculation of singular moduli on Shimura curves, preprint.

[GMX] X. Guitart, M. Masdeu, X. Xarles, A quaternionic construction of $p$-adic singular moduli, preprint.

[Gr1] B.H. Gross, Heights and the special values of $L$-series, CMS Conf. Proc. 7 (1987) 115–187.

[Gr2] B.H. Gross, Heegner points and representation theory, in: Heegner Points and Rankin $L$-Series, in: MSRI Pub., vol. 49, 2004, pp. 37–65.

[Gr3] B.H. Gross, Incoherent definite spaces and Shimura varieties, preprint.

[GZ] B.H. Gross, D.B. Zagier, On singular moduli, J. Reine Angew. Math. 355 (1984) 191–220.

[Ri] J. Rickards, Intersections of closed geodesics on Shimura curves, McGill PhD thesis, 2020.

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